Each question will be graded out of 10 points.

Problem 1. Suppose X is a Banach space. Prove that the weak topology on X agrees with the norm topology if and only if X is finite dimensional.

Problem 2. Define $D : C^1[0,1] \to C^0[0,1]$ by $Df = f'$. Equip both these spaces with the supremum norm

$$\|f\| = \sup_{x\in[0,1]} |f(x)|.$$

Prove that D has closed graph, but D is not continuous. Why does this not contradict the Closed Graph Theorem?

Problem 3. Suppose X is a locally compact Hausdorff space, μ is a σ-finite Radon measure on X, and $E \subset X$ is a Borel set. Prove directly from the definitions that for every $\varepsilon > 0$, there is an open set U and a closed set F with $F \subset E \subset U$ such that $\mu(U \setminus F) < \varepsilon$.

Problem 4. Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and $f \in L^\infty(X)$. Show that $f \in L^p(X)$ for all $p > 0$, and that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$

Problem 5. Let $L^2(\mathbb{T})$ denote the space of complex-valued square-integrable 1-periodic functions on $\mathbb{R}$, and let $C(\mathbb{T}) \subset L^2(\mathbb{T})$ denote the subspace of continuous 1-periodic functions.

(1) Prove that $\{e_n(x) := \exp(2\pi inx) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.

(2) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n\rangle_{L^2(\mathbb{T})} = \int_0^1 f(x)\exp(-2\pi inx)\,dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

Problem 6. Suppose $U \subset \mathbb{R}^n$ is open and $f, g \in L^1_{\text{loc}}(U)$. Show that the distributions associated to f, g are equal if and only if $f = g$ a.e.