

2026 Algebra 6111 Qualifying Exam

Instructions

Enter your name.# on the roster sheet together with a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment that you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let G be a finite group acting transitively on a finite set X with $|X| > 1$. Show that there exists at least one $g \in G$ having no fixed point on X , i.e. such that $gx \neq x$ for all $x \in X$.
2. Show that the alternating group A_4 has no subgroup of order 6.
3. Let R be a commutative ring. Prove that a polynomial $a_0 + a_1x + \cdots + a_nx^n$ is invertible in $R[x]$ if and only if a_0 is invertible in R and all a_i for $i > 0$ are nilpotent.
4. Let A be a commutative (non-zero) ring. Consider the following sequence of A -linear maps, between A -modules

$$0 \rightarrow M' \longrightarrow M \longrightarrow M'' \rightarrow 0$$

Prove that this sequence is exact if, and only if, for every maximal ideal $\mathfrak{m} \subsetneq A$, the following sequence of $A_{\mathfrak{m}}$ -modules is exact:

$$0 \rightarrow M'_{\mathfrak{m}} \longrightarrow M_{\mathfrak{m}} \longrightarrow M''_{\mathfrak{m}} \rightarrow 0$$

(*Recall:* for a maximal ideal (or more generally a prime ideal) $\mathfrak{m} \subsetneq A$, $A_{\mathfrak{m}}$ is the ring of fractions obtained by formally inverting elements not in $\mathfrak{m}$. That is, $A_{\mathfrak{m}} = S^{-1}A$, where $S = A \setminus \mathfrak{m}$.)

5. Let V be a finite-dimensional vector space over a field K , and let $n = \dim(V)$. Let $T : V \rightarrow V$ be a K -linear map and let $A = (a_{ij})_{1 \leq i, j \leq n}$ be the matrix representing T relative to a base $\{v_1, \dots, v_n\}$ of V . Let $1 \leq r \leq n$ and let $\wedge^r V$ be the r -th exterior power of V .
 - (i) Determine a matrix for $\wedge^n T : \wedge^n V \rightarrow \wedge^n V$ and prove your answer.
 - (ii) Determine a matrix for $\wedge^{n-1} T : \wedge^{n-1} V \rightarrow \wedge^{n-1} V$ and prove your answer.

Be sure to provide sufficient detail and to justify the computational steps in your solution!