Algebra Qualifying Exam I
2020

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. If an element x in a group G of odd order is conjugate to its inverse, then $x = e$ is the identity.

2. A group G is *nilpotent* if it possesses a finite normal series

$$G = A_0 \triangleright A_1 \triangleright A_2 \triangleright \cdots \triangleright A_m = \{e\}$$

with each A_i normal in G, such that A_{i-1}/A_i is in the center of G/A_i. Such a normal series is called a *central series* for G. If G is a group, the *lower central series* (or descending central series) for G is the sequence of normal subgroups of G defined by $G^0 = G$, $G^i = (G^{i-1}, G)$, so that

$$G^0 \triangleright G^1 \triangleright \cdots \triangleright G^i \triangleright \cdots .$$

Show that the lower central series is indeed a central series and that G is nilpotent iff there exists an integer k such that $G^k = \{e\}$.

3. Show that α is a root of a monic polynomial with integer coefficients if and only if $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module.

4. Let $R = M_4(\mathbb{C})$ be the ring of 4×4 matrices over the complex numbers. Suppose $e_1, e_2, \ldots, e_t$ are an orthogonal family of idempotent elements in R, i.e., each e_i is nonzero, $e_i^2 = e_i$, and $e_i e_j = e_j e_i = 0$. Show that $t \leq 4$.

5. Let V be a finite-dimensional vector space over a field F. Let B be a non-degenerate bilinear form on V. Prove that any linear function on a subspace U of V has the form $y \mapsto B(x, y)$ for some x in V.