Algebra Qualifying Exam II
2020

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let $N \in \mathbb{Z}$, $N \geq 2$. Prove that $\mathbb{Z}/N\mathbb{Z}$ is injective as a $\mathbb{Z}/N\mathbb{Z}$-module.

2. Let I be an index set and let $\{N_\alpha \mid \alpha \in I\}$ be a set of left R-modules. Let $N = \bigoplus_{\alpha \in I} N_\alpha$ their direct sum. Show that N is flat if and only if each N_α is flat.

3. Let R be a ring with identity and $N \in \mathrm{Ob}(R\text{-mod})$. Show that the (covariant) Hom functor $\mathrm{Hom}(N, -)$ from R-mod to $\mathbb{Z}$-mod is left exact but not right exact.

4. Let $p(x) = x^4 - 2x^2 - 1 \in \mathbb{Q}[x]$. Denote the splitting field of $p(x)$ over $\mathbb{Q}$ by L.

   (i) Prove that $L/\mathbb{Q}$ is Galois and compute its degree.

   (ii) Find the isomorphism type of $\mathrm{Gal}(L/\mathbb{Q})$.

   (iii) Find all intermediate fields $\mathbb{Q} \subset K \subset L$ such that K is Galois over $\mathbb{Q}$.

5. Let $\mathbb{F}_p$ denote the field with p elements, p a prime, and $\overline{\mathbb{F}}_p$ be its algebraic closure. Verify that the map $\varphi_p : x \mapsto x^p$ is an automorphism of $\overline{\mathbb{F}}_p$ that pointwise fixes $\mathbb{F}_p$, that is, that $\varphi_p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Show that $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is topologically generated by φ_p, that is, $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \overline{\langle \varphi_p \rangle}$.