You may submit solutions for at most 5 out of the following 6 problems. Each question will be graded out of 10 points.

Problem 1 (Measures)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose $E_n \in \mathcal{M}$ are such that

$$(*) \qquad \sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

(1) Show that $\mu(\limsup_{n\to\infty} E_n) = 0$, where $\limsup_{n\to\infty} E_n := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$.
(2) Either prove or disprove that the conclusion of (1) follows when $(*)$ above is replaced by

$$\sum_{n=1}^{\infty} \mu(E_n)^2 < \infty.$$

Problem 2 (Integration)**.** Give an example of measure spaces (X, μ), (Y, ν), and a non-negative measurable function f on $X \times Y$ such that the values of each of the following three integrals are distinct:

$$\int_X \left(\int_Y f \, d\nu \right) d\mu, \qquad \int_Y \left(\int_X f \, d\mu \right) d\nu, \qquad \text{and} \qquad \int_{X \times Y} f \, d(\mu \times \nu).$$

Justify your answer.
Half credit will be given if only two of the above three are distinct.

Problem 3 (Signed measures and differentiation)**.** Show that $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (there is an $M > 0$ such that $|F(x) - F(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$) if and only if F is absolutely continuous and $|F'| \le M$ a.e.

Problem 4 (Topology)**.** Suppose X is a compact Hausdorff space and $\{U_1, \ldots, U_n\}$ is an open cover of X. Prove that there are continuous functions $g_i : X \to \mathbb{R}$ for $i = 1, \ldots, n$ such that $g_i = 0$ on U_i^c and $\sum_{i=1}^n g_i = 1$ everywhere on X.

Problem 5 (Functional analysis)**.** Suppose X is a Banach space. Prove that the weak topology on X agrees with the norm topology if and only if X is finite dimensional.

Problem 6 (Radon measures)**.** Suppose X is a locally compact Hausdorff space, μ is a σ-finite Radon measure on X, and $E \subset X$ is a Borel set. Prove directly from the definitions that for every $\varepsilon > 0$, there is an open set U and a closed set F with $F \subset E \subset U$ such that $\mu(U \setminus F) < \varepsilon$.