You may submit solutions for at most 5 out of the following 6 problems. Each question will be graded out of 10 points.

Problem 1 (L^p spaces)**.** Let (X, μ) be a σ-finite measure space, and let $(f_n : X \to \mathbb{R})$ be a sequence of measurable functions in L^p for all $p \geq 1$. Suppose $f_n \to 0$ in L^2 and L^4.

(1) Does $f_n \to 0$ in L^1? Give a proof or a counterexample.
(2) Does $f_n \to 0$ in L^3? Give a proof or a counterexample.
(3) Does $f_n \to 0$ in L^5? Give a proof or a counterexample.

Problem 2 (L^p spaces)**.** Suppose that $f \in L^p(\mathbb{R})$ for all $p \in (1, 2)$ and that $\sup_{1<p<2} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that

$$\|f\|_2 = \lim_{p \to 2-} \|f\|_p.$$

Problem 3 (Fourier analysis)**.** Suppose $f, g \in L^1(\mathbb{R}, \lambda)$ where λ is Lebesgue measure.

(1) Show that $y \mapsto f(x - y)g(y)$ is measurable for all $x \in \mathbb{R}$ and in $L^1(\mathbb{R}, \lambda)$ for a.e. $x \in \mathbb{R}$.
(2) Define the *convolution* of f and g by

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y)\, d\lambda(y).$$

Show that $f * g \in L^1(\mathbb{R}, \lambda)$.
(3) Show that $f * g = g * f$ for all $f, g \in L^1(\mathbb{R}, \lambda)$.

Problem 4 (Fourier analysis)**.** Prove that the Fourier transform is injective on $L^1(\mathbb{T}^n)$. That is, show that if $f, g \in L^1(\mathbb{T}^n)$ with $\widehat{f}(k) = \widehat{g}(k)$ for all $k \in \mathbb{Z}^n$, then $f = g$ a.e.

Problem 5 (Distributions)**.** Suppose $U \subset \mathbb{R}^n$ is open and $f, g \in L^1_{\text{loc}}(U)$. Show that the distributions associated to f, g are equal if and only if $f = g$ a.e.

Problem 6 (Distributions)**.** Let $s \in \mathbb{R}$. Construct an invertible linear isometry from the Sobolev space H_{-s} to the dual space $(H_s)^*$.