Algebra Qualifying Exam II
2018

Directions

Put your name and the last four digits of your Social Security Number on the roster sheet when you receive it and enter a code name for yourself that is different from any code name that has already been entered.

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.

Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a closed book, closed notes exam.

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1. Let R be a ring with identity, and consider the following two short exact sequences of left R-modules, where P_1 and P_2 are projective:

$$0 \longrightarrow N_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

Prove that $P_1 \oplus N_2$ is isomorphic to $P_2 \oplus N_1$.

2. Let R be a ring with identity and N a (left) R-module. Show that the tensor functor $- \otimes_R N$ from **mod**-R (the category of right R-modules) to $\mathbb{Z}$-**mod** (abelian groups) is right exact but not left exact.

3. Consider $M = \mathbb{Z}/3\mathbb{Z}$ as a module over the ring $R = \mathbb{Z}/9\mathbb{Z}$. Compute $\operatorname{Ext}^n_R(M, M)$ for all n.

4. Consider the extension $\mathbb{Q}[\alpha]$ over $\mathbb{Q}$, where $\alpha = \sqrt{2 + \sqrt{3}}$. Show that this extension is Galois, and compute its Galois group.

5. Let E/F be a finite Galois extension and $G = Gal(E/F)$. Let $\sigma \mapsto \alpha_\sigma$ be a map $G \to E^\times$ such that

$$\alpha_{\tau\sigma} = \tau(\alpha_\sigma)\,\alpha_\tau.$$

Show that there exists $\beta \in E^\times$ such that $\alpha_\sigma = \dfrac{\beta}{\sigma(\beta)}$.