Analysis Qualifying Examination 1 **August 17, 2015**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[15] **1.** Let $f\colon \mathbf{R} \to \mathbf{C}$ be continuous and satisfy $f(x+1) = f(x)$ for all $x \in \mathbf{R}$. Let α be an irrational number, let x_0 be any real number, and for $j = 1, 2, 3, \dots$, let $x_j = x_0 + j\alpha$. Prove that as $n \to \infty$,

$$\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int_0^1 f(x)\,dx.$$

(Hint: First consider f of the form $f(x) = e^{2\pi i k x}$, where $k \in \mathbf{Z}$. Then use a suitable approximation theorem to deduce the general case from this special case.)

2. Let (X, ρ) be a complete metric space.

[5] **(a)** Let (F_n) be a sequence of closed subsets of X such that $X = \bigcup_n F_n$. For each n, let G_n be the interior of F_n. Let $U = \bigcup_n G_n$. Prove that U is dense in X.

[10] **(b)** Let (h_n) be a sequence of continuous functions $h_n\colon X \to \mathbf{R}$. Suppose (h_n) is pointwise-convergent. Let E be the set of all $x \in X$ such that (h_n) is equicontinuous at x. Prove that E is dense in X. (Hint: As part of your proof, show that E contains $\bigcap_m U_m$, where for each m, U_m is a suitable open subset of X which is dense in X by part (a).)

[5] **(c)** Let h be the function on $\mathbf{R}$ which is 1 on the rationals and 0 on the irrationals. Use part (b), with $X = \mathbf{R}$, to prove that h is not the pointwise limit of a sequence of continuous functions.

[15] **3.** Let m be Lebesgue outer measure on $\mathbf{R}$. Let A be a Lebesgue-measurable subset of $\mathbf{R}$. Prove in detail that for each $\varepsilon > 0$, there exist sets $F, G \subseteq \mathbf{R}$ such that F is closed, G is open, $F \subseteq A \subseteq G$, and $m(G \setminus F) < \varepsilon$. (Warning: Your proof should be based on the definition of Lebesgue outer measure and of Lebesgue-measurability and on suitable properties more elementary than the one you are asked to prove. It should not use the fact that there exist Borel sets $B, C \subseteq \mathbf{R}$ such that $B \subseteq A \subseteq C$ and $m(C \setminus B) = 0$. This fact follows from what you are asked to prove. It is not more elementary than what you are asked to prove.)

[15] **4.** Let $(Y, \mathscr{B}, \mu)$ be a measurable space. Let $f\colon Y \to \mathbf{C}$ and $g\colon Y \to \mathbf{R}$ be measurable functions. Suppose $\int_Y |f|\,d\mu < \infty$. Define $h\colon \mathbf{R} \to \mathbf{C}$ by $h(x) = \int_Y f(y) e^{ixg(y)}\,d\mu(y)$. Suppose also that $\int_Y |fg|\,d\mu < \infty$. Prove that h is differentiable on $\mathbf{R}$ and find its derivative h'. You may use the dominated convergence theorem without proof but be sure to verify its hypotheses. You may not use theorems on differentiation under the integral sign without proof.

5. Let $S = [0,1] \times [0,1]$ and let m be Lebesgue measure on $\mathbf{R}^2$. Define $f\colon S \to \mathbf{R}$ by $f(x,y) = \left(y - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)^{-3}$ if $\left|y - \frac{1}{2}\right| < \left|x - \frac{1}{2}\right|$, $f(x,y) = 0$ otherwise.

[6] **(a)** Are the iterated integrals $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ defined and equal? Justify your answer.

[9] **(b)** Is the double integral $\int_S f\,dm$ defined? Justify your answer.

6. For each $f\colon \mathbf{R}^d \to \mathbf{C}$ and each $x \in \mathbf{R}^d$, define $\tau_x f\colon \mathbf{R}^d \to \mathbf{C}$ by $(\tau_x f)(y) = f(y - x)$ and define $\tilde{f}\colon \mathbf{R}^d \to \mathbf{C}$ by $\tilde{f}(y) = f(-y)$. Let $p \in [1, \infty)$ and $q \in (1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$.

[10] **(a)** Let $f \in L^p(\mathbf{R}^d)$. Prove that $x \mapsto \tau_x f$ is continuous from $\mathbf{R}^d$ into $L^p(\mathbf{R}^d)$.

[6] **(b)** Let $f \in L^p(\mathbf{R}^d)$ and let $g \in L^q(\mathbf{R}^d)$. Prove that $f * g$ is continuous and bounded.[1]

[4] **(c)** Suppose in addition that $p > 1$. Let $f \in L^p(\mathbf{R}^d)$ and let $g \in L^q(\mathbf{R}^d)$. Prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

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[1] Reminder: $(f * g)(x) = \int_{\mathbf{R}^d} f(x - y) g(y)\,dy$ for each $x \in \mathbf{R}^d$. Your solution should include a mention of why this integral is defined for each $x \in \mathbf{R}^d$.