Analysis Qualifying Examination 1 **August 15, 2016**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

1. Recall that $L^2(\mathbf{T})$ denotes the vector space of Lebesgue-measurable complex-valued 1-periodic functions[1] f on $\mathbf{R}$ such that $\int_0^1 |f(x)|^2\,dx < \infty$ and that for all $f, g \in L^2(\mathbf{T})$, $\langle f \,|\, g\rangle = \int_0^1 \overline{f(x)} g(x)\,dx$ (note that $\langle f \,|\, g\rangle$ is linear in g and antilinear in f) and $\|g\|_2 = \langle g \,|\, g\rangle^{1/2}$. Also, for each $k \in \mathbf{Z}$, e_k is the function on $\mathbf{R}$ defined by $e_k(x) = e^{2\pi i k x}$. Let $f \in L^2(\mathbf{T})$. For each finite set $B \subseteq \mathbf{Z}$, let E_B be the linear subspace of $L^2(\mathbf{T})$ consisting of all linear combinations of the vectors e_k, $k \in B$, and let $S_B f = \sum_{k\in B} \hat{f}(k) e_k$, where $\hat{f}(k) = \langle e_k \,|\, f\rangle$.

[10] **(a)** Let B be a finite subset of $\mathbf{Z}$ and let $h \in E_B$. Prove that $\|f - S_B f\|_2 \le \|f - h\|_2$, with equality if and only if $h = S_B f$ a.e. To get full marks, your proof should be be reasonably self-contained and should be simple, clear, and efficient. For instance, the algebraic steps involved in the proof should correspond clearly to the geometry that the algebra is meant to justify.

[10] **(b)** Deduce from part (a) that

$$f = \sum_{k\in\mathbf{Z}} \hat{f}(k) e_k,$$

in the sense that for each $\varepsilon > 0$, there is a finite set $A \subseteq \mathbf{Z}$ such that for each finite set $B \subseteq \mathbf{Z}$, if $A \subseteq B$, then $\|f - S_B f\|_2 < \varepsilon$. You may use the following results without proof: **(i)** $C(\mathbf{T})$ is dense in $L^2(\mathbf{T})$ with respect to the norm $\|\cdot\|_2$; **(ii)** Fejér's theorem.

[15] **2.** Let (X, ρ) be a complete metric space. Let Φ be a non-empty set of continuous real-valued functions on X. Define g on X by

$$g(x) = \sup\{\, f(x) : f \in \Phi \,\}.$$

Suppose that $g(x) < \infty$ for each $x \in X$. Let G be the set of all $x \in X$ such that there is a neighborhood V of x in X such that g is bounded above on V. Prove that G is a dense open subset of X. (Hint: The Baire category theorem may help.)

[15] **3.** Let (f_n) be a sequence in $L^2[0,1]$ such that $\|f_n\|_2 \le 2$ for each n and $f_n \to 0$ in measure (with respect to Lebesgue measure) as $n \to \infty$. Prove that $\|f_n\|_1 \to 0$ as $n \to \infty$.

[1] Of course, strictly speaking, the elements of $L^2(\mathbf{T})$ are really equivalence classes of such functions, where two such functions are equivalent if and only if they are equal almost everywhere with respect to Lebesgue measure.

4. Let $\mathscr{B}$ be the Borel σ-field[2] on $\mathbf{R}$. Let $\mathscr{H} = \{\, (a, b] : -\infty < a \leq b < \infty \,\}$.

[10] **(a)** Prove that $\mathscr{B}$ is also the σ-field on $\mathbf{R}$ generated by $\mathscr{H}$.

[10] **(b)** Let $\mathscr{A}$ be a collection of subsets of $\mathbf{R}$ such that $\mathscr{A}$ is closed under countable intersections and under countable disjoint unions. Suppose $\mathscr{H} \subseteq \mathscr{A}$. Prove that $\mathscr{B} \subseteq \mathscr{A}$.

[10] **5. (a)** Let $(X, \mathscr{A}, \mu)$ be a measure space, let $p \in [1, \infty)$, and let $L^p = L^p(\mu)$. (For definiteness, assume complex scalars.) The Riesz-Fischer theorem states that L^p is complete as a metric space. Give the following portion of the proof of this result. Let (f_n) be a sequence in L^p and let $M = \sum_{n=1}^{\infty} \|f_n - f_{n-1}\|_p$, where $f_0 = 0$. Suppose that $M < \infty$. Prove that there exists $f \in L^p$ such that $\|f - f_n\|_p \to 0$ as $n \to \infty$.

[10] **(b)** It follows easily from problem 1(b) that the map[3] $f \mapsto \hat{f}$ is a linear isometry from $L^2(\mathbf{T})$ into $\ell^2(\mathbf{Z})$. (You may use this without proof.) First show that the range of this map is dense in $\ell^2(\mathbf{Z})$. Then use part (a) to prove that in fact, the range of this map is equal to $\ell^2(\mathbf{Z})$.

[10] **6.** For each $p \in [1, \infty]$, let $L^p = L^p(\mathbf{R}^d)$. Suppose that $f \in L^1 \cap L^p$ and $g \in L^1 \cap L^q$, where $p, q \in [1, \infty]$ and $p^{-1} + q^{-1} = 1$. Let $h \in L^1 \cap C$, where C denotes the set of continuous functions from $\mathbf{R}^d$ into $\mathbf{C}$. Suppose that $\hat{f}\hat{g} = \hat{h}$. Prove that for each $x \in \mathbf{R}^d$, $(f * g)(x)$ is defined and is equal to $h(x)$. You may use any facts about convolutions and Fourier transforms which are proved in Folland.

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[2] A σ-field is the same thing as a σ-algebra. By definition, the Borel σ-field on $\mathbf{R}$ is the σ-field on $\mathbf{R}$ generated by the open subsets of $\mathbf{R}$.

[3] Here $\hat{f}$ is as in problem 1.