Analysis Qualifying Examination 2 **August 16, 2016**

Please start each problem on a new page and remember to write your code on each page of your answers.

You should exercise good judgement in deciding what constitutes an adequate solution. In particular, you should not try to solve a problem by just quoting a theorem that reduces what you are asked to prove to a triviality. If you are not sure whether you may use a particular theorem, ask the proctor.

[15] **1.** Let $\mathscr{F}$ be a field of subsets of a set X. Let $\nu\colon \mathscr{F} \to \mathbf{C}$ be σ-additive.[1] Let $|\nu|$ be the variation[2] of ν. Prove that $|\nu|$ is σ-additive.

2. Let A be the set of all functions $f\colon [0,\infty) \to \mathbf{C}$ such that for each $b \in (0,\infty)$, f is absolutely continuous on $[0,b]$. For each $p \in (0,\infty)$, let L^p denote the set of all Lebesgue-measurable functions $f\colon [0,\infty) \to \mathbf{C}$ such that

$$\int_0^\infty |f(x)|^p\,dx < \infty.$$

[7] **(a)** Give an example of a real-valued function $f \in \bigcap_{0<p<\infty} L^p \cap A$ such that

$$\limsup_{b\to\infty} f(b) = \infty.$$

(Don't work hard. There is a simple example. It would not be hard to give a formula for f but you don't have to give a formula. A suitably labelled sketch of the graph and/or a suitable description in words would be fine.)

[8] **(b)** Let $p \in (0,\infty)$, let $f \in L^p \cap A$, and suppose $f' \in L^1$. Prove that

$$f(b) \to 0 \qquad \text{as } b \to \infty.$$

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[1] To say that ν *is σ-additive* means that for each finite or countable disjoint sequence (F_n) in $\mathscr{F}$, if $\bigcup_n F_n \in \mathscr{F}$, then $\nu(\bigcup_n F_n) = \sum_n \nu(F_n)$. Incidentally, this definition implies that if ν is σ-additive, then $\nu(\varnothing) = 0$, because $\varnothing$ is the union of the empty sequence of elements of $\mathscr{F}$ and a sum with no terms has the value 0.

[2] Reminder: The variation of ν is the function $|\nu|\colon \mathscr{F} \to [0,\infty]$ defined by

$$|\nu|(E) = \sup \sum_m |\nu(E_m)|,$$

where (E_m) ranges over all finite disjoint sequences of elements of $\mathscr{F}$ such that $\bigcup_m E_m \subseteq E$. (We would get the same result if we required $\bigcup_m E_m = E$. This is easy to see and you may take it for granted.)

3. Let X be a topological space. By definition, $\mathrm{Baire}(X)$ is the σ-field on X generated by $\mathscr{H}$, where $\mathscr{H} = \{ f^{-1}[B] : f \in C(X, \mathbf{R}) \text{ and } B \in \mathrm{Borel}(\mathbf{R}) \}$. The elements of $\mathrm{Baire}(X)$ are called *Baire subsets of* X.

[10] **(a)** Let $A \in \mathrm{Baire}(X)$. Prove that there is a continuous function $f \colon X \to \mathbf{R}^{\mathbf{N}}$ such that $A \in \mathscr{E}$, where $\mathscr{E} = \{ f^{-1}[E] : E \in \mathrm{Borel}(\mathbf{R}^{\mathbf{N}}) \}$. (Reminder: $\mathbf{R}^{\mathbf{N}}$ denotes the space of all infinite sequences of real numbers, with its usual product topology. In other words, $\mathbf{R}^{\mathbf{N}}$ denotes the Cartesian product of a countably infinite number of copies of the real line $\mathbf{R}$.)

[8] **(b)** Let K be a compact Baire subset of X. Prove that K is closed in X and that K is a countable intersection of open subsets of X. (Warning: X need not be Hausdorff.)

4. Let E be a vector space and let M and N be linear subspaces of E such that $M \cap N = \{0\}$. Define P and Q on $M+N$ by $P(x+y) = x$ and $Q(x+y) = y$ for all $x \in M$ and all $y \in \mathbf{N}$. Then P and Q are well-defined linear operators on $M+N$. (You need not prove this. It is elementary.)

[2] **(a)** Suppose in addition that E is a normed linear space. Prove that P is continuous if and only if Q is continuous.

[15] **(b)** Now suppose in addition that E is a Banach space and that M and N are closed. Prove that P and Q are continuous if and only if $M+N$ is closed.

5. Let $\mathbf{K}$ be $\mathbf{R}$ or $\mathbf{C}$. Let E be a normed linear space over $\mathbf{K}$, let E^* be the Banach space[3] of continuous linear functionals on E, and let E^{**} be the Banach space of continuous linear functionals on E^*.

[10] **(a)** For each $x \in E$, define $L_x \colon E^* \to \mathbf{K}$ by $L_x(\varphi) = \varphi(x)$ and prove that $L_x \in E^{**}$ and $\|L_x\| = \|x\|$.

[10] **(b)** Let $A \subseteq E$. Suppose that for each $\varphi \in E^*$, the set $\varphi[A]$ is bounded in $\mathbf{K}$. Prove that A is norm-bounded in E.

[15] **6.** Let X be a locally compact Hausdorff space and let $C_0(X)$ be the space of complex-valued continuous functions f on X such that f tends to zero at infinity. Let f be an element of $C_0(X)$ and let (f_n) be a sequence in $C_0(X)$. Prove that $f_n \to f$ weakly in $C_0(X)$ if and only if (f_n) is uniformly bounded and $f_n \to f$ pointwise. (For part of the forward implication, you may use the result of one of the parts of an earlier problem.)

[3] The fact that E^* is a Banach space and not just a normed linear space follows from the fact that the scalar field $\mathbf{K}$ is complete as a metric space. You may use the fact that E^* is a Banach space without proof.