OSU Analysis Qualifying Examination 1 **August 14, 2017**

You may submit solutions for at most 5 out of the following 7 problems. Each question will be graded out of 10 points.

(1) Suppose that $\mathcal{C}$ is a non-empty collection of open balls in $\mathbb{R}^n$, and let $U = \bigcup_{B\in\mathcal{C}} B$. Show that if $c < m(U)$ (where m is Lebesgue measure), then there are *disjoint* $B_1, \ldots, B_k \in \mathcal{C}$ such that $\sum_{i=1}^k m(B_i) > 3^{-n}c$.
*Note: You may **not** use the Vitali Covering Lemma without proof.*

(2) Give examples of sequences $(f_n)_{n=1}^\infty$ of measurable functions on some measure space such that:
- (f_n) converges almost uniformly to a limit function f, but not everywhere to f.
- (f_n) converges everywhere to a limit function f, but not in measure to f.
- (f_n) converges in L^1 to a limit function f, but not almost everywhere to f.
- (f_n) converges uniformly to a limit function f, but not in L^1 to f.

For each example, you must indicate the measure space and the sequence of functions, together with a justification of the first mode of convergence, and not the second.

(3) (a) State the Monotone Convergence Theorem.
(b) State Fatou's Lemma.
(c) Assuming the Monotone Convergence Theorem, prove Fatou's Lemma.
(d) Assuming Fatou's Lemma, prove the Monotone Convergence Theorem.

(4) Suppose μ is a positive measure on $(X, \mathcal{M})$ and $f \in L^1(X)$.
(a) Prove that if $E \subset X$ with $\mu(E) = 0$, then $\int_E f\, d\mu = 0$.
(b) Prove that if $\int_E f\, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ μ-a.e.

(5) Let $L^2(\mathbb{T})$ denote the space of complex-valued square-integrable 1-periodic functions on $\mathbb{R}$, and let $C(\mathbb{T}) \subset L^2(\mathbb{T})$ denote the subspace of continuous 1-periodic functions.
(a) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
(b) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n\rangle_{L^2(\mathbb{T})} = \int_0^1 f(x)\exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

(6) Suppose $(X, \mathcal{M}, \mu)$ is a finite measure space and $f \in L^\infty(X)$. Show that $f \in L^p(X)$ for all $p > 0$, and that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

(7) Suppose f is continuous and g is locally integrable on the reals with $\int f\phi' = -\int g\phi$ for every smooth (infinitely differentiable) function ϕ with compact support. Prove that f is absolutely continuous and $f' = g$ a.e.