OSU Analysis Qualifying Examination 2 **August 15, 2017**

You may submit solutions for at most 5 out of the following 7 problems. Each question will be graded out of 10 points.

(1) Suppose X is a Banach space and $Y \subseteq X$ is a closed subspace. Show that
$$\|x+Y\|_{X/Y} := \inf\{\|x+y\|_X \mid y \in Y\}$$
is a well-defined norm on X/Y under which X/Y is a Banach space.

(2) Suppose X is a Banach space and F is a finite dimensional vector space. Show that a linear map $\varphi : X \to F$ is bounded if and only if $\ker(\varphi)$ is closed.

(3) State and prove the Baire Category Theorem for complete metric spaces.

(4) Suppose X and Y are Banach spaces and $T : X \to Y$ is a continuous linear map. Show that the following are equivalent.
(a) There exists a constant $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$.
(b) T is injective and has closed range.

(5) (a) Find a Banach space whose dual space is (isometrically isomorphic to) ℓ^1.
(b) Show that $L^1[0,1]$ with respect to Lebesgue measure is not a dual space.

(6) Let X be a Banach space. Prove that every weakly convergent sequence in X and every weak* convergent sequence in X^* are bounded (with respect to the respective norms).

(7) Let X be a Banach space and B its closed unit ball. Let $i : X \to X^{**}$ be the canonical inclusion. Prove that $i(B)$ is weak*-dense in B^{**}, the closed unit ball of X^{**}.