ANALYSIS PROFICIENCY EXAM

AUGUST 2014

To pass this exam, it is sufficient to completely solve four problems.

1. Suppose $E_n \subset X$, $n = 1, 2, \ldots$, are measurable subsets of a a measure space (X, μ) and $\lim_{n\to\infty} \mu(E_n) = \mu(X)$. If $\mu(X) < \infty$, show there is a subsequence $n_1 < n_2 < \ldots$ such that

$$\mu\left(\cap_{k=1}^{\infty} E_{n_k}\right) > 0.$$

Does it matter whether $\mu(X) < \infty$ or not? Is it necessarily true that $\mu\left(\cap_{n=1}^{\infty} E_n\right) > 0$?

2. Let $f \in L^1(A)$ be a nonnegative function, where $A \subset \mathbb{R}^n$ is a Lebesgue measurable set. Let μ denote Lebesgue measure. Show the following: given $\epsilon > 0$, there exists $\delta > 0$ such that for every subset $B \subset A$ with $\mu(B) < \delta$, it holds that

$$\int_B f < \epsilon.$$

3. Let $f \in C_0^{\infty}(\mathbb{R})$. Define

$$I_f(x) = \int_{\mathbb{R}} e^{itx} f(t)\, dt,$$

and show that I_f has faster than polynomial decrease as $x \to \infty$. That is, for each $N \in \mathbb{Z}^+$, show that there exists a constant $C > 0$ such that for $x \in \mathbb{R}^+$

$$|I_f(x)| \le C x^{-N}.$$

4. Let $N_1 = C^1([0,1])$ and $N_2 = C([0,1])$, both equipped with the sup norm. Let $D : N_1 \longrightarrow N_2$ be the linear map defined $D(f) = f'$. Show the following:
   (a) D is not bounded.
   (b) The graph of D is closed.
   (c) Conclusion (b) does not violate the Closed Graph Theorem.

5. Let H be a Hilbert space and $S \subset H$ a closed, linear subspace. Show that S is a Hilbert space with operations inherited from H.

Let $\{e_k\}_{k\in I}$ be an orthonormal basis for S. If $h \in H$, show that the unique element of S nearest h is the element

$$\tilde{h} =: \sum_{k\in I} (h, e_k)\, e_k.$$

(First show that $\tilde{h}$ is well-defined.)

6. Let (X, μ) be a measure space and suppose $\mu(X) < \infty$. Show that $g \in L^1(X, \mu)$ if and only if

$$\sum_{k=1}^{\infty} 3^k \; \mu\left\{x : |g(x)| > 3^k\right\} < \infty.$$