**OSU Analysis Qualifying Examination 2** **August 2019**

Answer each question on a separate sheet or sheets of paper, and write your code name and the problem number on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading. Solutions to five problems constitute a complete exam.

Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.

This is a two hour, closed book, closed notes exam.

(1) (20pt) Show that every non-empty, closed, convex set $K$ in a Hilbert space $H$ has a unique element of minimal norm.

(2) (20pt) Let the map $D : C^1([0,1]) \to C^0([0,1])$ be given by

$$Df(x) = f'(x) \quad \text{for } x \in [0,1].$$

(a) Show that $D$ has closed graph.
(b) Show that $D$ is not continuous.
(Here both $C^1([0,1])$ and $C^0([0,1])$ are endowed with the supremum norm $\|f\| = \sup_{[0,1]} |f|$.)

(3) (20pt) Let $X$ and $Y$ be nonzero, closed linear subspaces of a Banach space $Z$. Let

$$\delta = \inf\{\|x+y\| : x \in X,\, y \in Y, \text{ and } \|x\| + \|y\| = 1\}.$$

Prove that the following statements are equivalent.
(a) $\delta > 0$;
(b) $X + Y$ is closed and $X \cap Y = \{0\}$.
You may use, without proof, that $X \times Y = \{(x,y) : x \in X,\, y \in Y\}$ is a Banach space when equipped with the norm $\|(x,y)\| := \|x\| + \|y\|$ and the vector space operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $c(x,y) = (cx, cy)$.

(4) (20pt) Let $\mathbb{X}$ be the dual space of $C([0,1])$. Construct a sequence of measures $\{\mu_n\} \subset \mathbb{X}$ which converges in the weak$^*$ topology but not in the weak topology. (Recall that $C([0,1])$ is the space of continuous functions on $[0,1]$. And that the space of measures can be identified with a subspace of $\mathbb{X}$. A sequence of measures $\mu_n$ is said to converge in the weak$^*$ topology to $\mu$ if $\mu_n(f) \to \mu(f)$ as $n \to \infty$ for all $f \in C([0,1])$. )

(5) (20pt) Let $f \in C(\mathbb{T})$ and let $\hat{f} : \mathbb{Z} \to \mathbb{C}$ be its Fourier transform. If $f$ has continuous derivatives up to order $n$, show that $\sum_{k \in \mathbb{Z}} |k|^{n-1} |\hat{f}(k)| < +\infty$.

(6) (20pt) Consider the convolution

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy.$$

(a) Show that $f * g$ is uniformly continuous when $f$ is integrable and $g$ bounded.
(b) If, in addition, $g$ is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.