Ph.D. Algebra Qualifying Exam
6112

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Directions

1. Initial the roster sheet and enter a code name for yourself that is different from any code name that has already been entered.
2. Answer each question on a separate sheet or sheets of paper, and write your *code name* and the *problem number* on each sheet of paper that you submit for grading. Do not put your real name on any sheet of paper that you submit for grading.
3. Answer as many questions as you can. Do not use theorems which make the solution to the problem trivial. Always clearly display your reasoning. The judgment you use in this respect is an important part of the exam.
4. This is a closed book, closed notes exam.

In problems 1–3, R is a ring with 1. Modules are taken to be left R-modules. In problem 1, in addition R is commutative.

1. Let R be a commutative ring with 1 and let $S \subset R$ be a multiplicatively closed set. Prove that $M \mapsto S^{-1}M$ is an exact functor from R–mod to $S^{-1}R$–mod.

2. Prove that an R-module Q is injective iff any homomorphism of a left ideal $\mathfrak{a}$ of R into Q can be extended to a homomorphism of R into Q.

3. Recall the definition of $\mathrm{Ext}^k_R(M, N)$ for two R–modules M and N. Take a projective resolution of M:

   $$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

   and then consider the chain complex:

   $$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d_0} \mathrm{Hom}(P_1, N) \xrightarrow{d_1} \mathrm{Hom}_R(P_2, N) \to \cdots$$

   Then $\mathrm{Ext}^k_R(M, N) := \mathrm{Ker}(d_k)/\,\mathrm{Im}(d_{k-1})$.

   (a) Prove that in the category of $\mathbb{Z}$-modules (or abelian groups) $\mathrm{Ext}^k_{\mathbb{Z}}(M, N) = 0$ for $k > 1$ and all finitely generated $\mathbb{Z}$–modues M and N. [Note: This is still true without the finite generation assumption.]

   (b) Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

4. Consider the extension $E_m = \mathbb{Q}(\zeta_m)$ of $\mathbb{Q}$, where $\zeta_m = e^{2\pi i/m}$.

   (a) Prove that $E_m/\mathbb{Q}$ is abelian and the order of the Galois group $Gal(E_m/\mathbb{Q})$ is $\varphi(m)$ = number of $1 \le k < m$ coprime to m.

   (b) Let $m \geq 1$ be a positive integer. Prove that $\cos\left(\frac{2\pi}{m}\right) \in \mathbb{Q}$ if and only if $m = 1, 2, 3, 4, 6$. (What is $\varphi(m)$ in these cases?)

5. E/F be a finite extension of finite fields. Show that the norm map $N_{E/F} : E^\times \to F^\times$ is surjective.