2019 Gordon examination problems

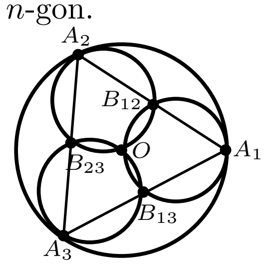
1. Prove that there are infinitely many primes p such that for some $n \in \mathbb{N}$ the integer $n^2 + n + 1$ is divisible by p.

2. Let f be the function $(0, \infty) \longrightarrow \mathbb{R}$ defined by: $f(x) = 0$ if $x \notin \mathbb{Q}$, and $f(x) = 1/n^3$ if $x = m/n$ is rational in lowest terms. If $k \in \mathbb{N}$ is not a perfect square, prove that f is differentiable at $\sqrt{k}$.

3. Find the maximum of the integral $\int_0^1 \left(x^{2020} f(x) - x^{2019} f^2(x)\right) dx$ over all continuous functions $f\colon [0,1] \longrightarrow \mathbb{R}$.

4. Let $S = \{z \in \mathbb{C} : |z| = 1\}$. Suppose $z_1, \ldots, z_n \in S$ satisfy $\left|(z - z_1) \cdots (z - z_n)\right| \le 2$ for every $z \in S$. Prove that $z_1, \ldots, z_n$ are the vertices of a regular n-gon.

5. Suppose C_1, C_2 and C_3 are cirlces of equal radius inscribed in a circle C and having a common intersection point O. For every $1 \le i \le 3$ let A_i be the tangency point of C_i and C, and for every $1 \le i < j \le 3$ let B_{ij} be the intersection point of C_i and C_j other than O. Prove that for each $1 \le i < j \le 3$, the points A_i, B_{ij}, and A_j are collinear.



6. Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$ be an $n \times n$ real matrix with zero trace, i.e. $\sum_{i=1}^n a_{i,i} = 0$. Prove that A is conjugate to a matrix with zero main diagonal. (That is, prove there exists an invertible $n \times n$ matrix P such that $PAP^{-1} = \begin{pmatrix} 0 & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & 0 & \cdots & b_{2,n} \\ \vdots & \vdots & & \vdots \\ b_{n,1} & b_{n,2} & \cdots & 0 \end{pmatrix}$ for some real numbers $b_{i,j}$.)