1. Evaluate \( \int_{-\pi}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x} \, dx \).

Solution.

\[
\int_{-\pi}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x} \, dx = \int_{0}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x} \, dx + \int_{0}^{\pi} \frac{-\sin(-2020x)}{(1+2^{-x})\sin(-x)} \, dx
= \int_{0}^{\pi/2} \frac{\sin(2020x)}{\sin x} \, dx + \int_{0}^{\pi/2} \frac{\sin(2020(\pi-x))}{\sin(\pi-x)} \, dx
= \int_{0}^{\pi/2} \frac{\sin(2020x)}{\sin x} \, dx - \int_{0}^{\pi/2} \frac{\sin(2020x)}{\sin x} \, dx = 0.
\]

2. Let \( G = \{A_1, \ldots, A_n\} \) be a finite multiplicative group of real \( k \times k \) matrices, and let \( S = \sum_{i=1}^{n} A_i \). If trace\((S) = 0 \), prove that \( S = 0 \).

Solution. For every \( i = 1, \ldots, n \), multiplication by \( A_i \) permutes the elements of \( G \), so \( A_i S = S \), and so \( S^2 = \sum_{i=1}^{n} A_i S = nS \). For \( P = \frac{1}{n} S \), we therefore have \( P^2 = P \), that is, the linear transformation defined by \( P \) is a projection. Hence, \( P \) is diagonalizable with all eigenvalues equal to either 0 or 1; if trace \( S = n \) trace \( P = 0 \), then all eigenvalues of \( P \) are zeroes, and \( P = 0 \).

3. Find all \( b \in \mathbb{N} \) for which \( 3\sqrt{2 + \sqrt{3}} + 3\sqrt{2 - \sqrt{3}} \) is an integer.

Solution. Let \( \alpha = \sqrt{2 + \sqrt{3}} \), \( \beta = \sqrt{2 - \sqrt{3}} \), and \( n = \alpha + \beta \). We have

\[ n^3 = (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3(\alpha + \beta)\alpha\beta = 4 + 3n\sqrt{4-b} \]

Hence,

\[ 4 - b = \frac{1}{27}(n^2 - 4/n)^3 = \frac{1}{27}(x^6 - 12n^3 + 48 - (4/n)^3) \]

If both \( b \) and \( n \) are integers, then \( 4/n \) is an integer, so \( n = \pm1, \pm2, \pm4 \). Now, if \( n = -1, 2, \) or \( -4 \), then \( n^2 - 4/n \) is not divisible by 3 and \( b \) is fractional; if \( n = 1 \), then \( b = 5 \); if \( n = -2 \) then \( b = -4 \); if \( n = 4 \) then \( b = -121 \). Since \( b > 0 \), we see that the only valid solution is \( b = 5 \).

4. Let \( v_1, \ldots, v_n \in \mathbb{R}^d \). Prove that \( \sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq n^2 \).

Solution. By the AM-GM inequality,

\[ \sum_{i,j=1}^{n} e^{v_i \cdot v_j} \geq n^2 \left( \prod_{i,j=1}^{n} e^{v_i \cdot v_j} \right)^{1/2} = n^2 \left( e^{\sum_{i,j=1}^{n} v_i \cdot v_j} \right)^{1/2} = n^2 \left( e^{\|v\|^2} \right)^{1/2} \geq n^2 \]

since \( \|v\|^2 \geq 0 \).

5. Let \( a, b \in \mathbb{R} \), \( ab = 1 \). Evaluate \( \det \begin{pmatrix} 2 & a & a^2 & \cdots & a^{n-1} \\ b & 2 & a & \cdots & a^{n-2} \\ b^2 & b & 2 & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \cdots & 2 \end{pmatrix} \).

Solution. Let \( A = \begin{pmatrix} 2 & a & a^2 & \cdots & a^{n-1} \\ b & 2 & a & \cdots & a^{n-2} \\ b^2 & b & 2 & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \cdots & 2 \end{pmatrix} \), then \( A = I + B \) where \( B = \begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ b & 1 & a & \cdots & a^{n-2} \\ b^2 & b & 1 & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \cdots & 1 \end{pmatrix} \). \( B \) has rank \( 1 \), all its columns are multiples of \( u = \begin{pmatrix} 1 \\ b \\ b^2 \\ \vdots \\ b^{n-1} \end{pmatrix} \); so, the null space of \( B \) is \((n-1)\)-dimensional. We have
Bu = nu, so B has the eigenvalues 0, of multiplicity n − 1, and n. The eigenvalues of $A = I + B$ are therefore 1, of multiplicity $n − 1$, and $n + 1$, and det $A = n + 1$.

Another solution. After multiplying the rows of the matrix successively by 1, $a$, $a^2$, . . . , $a^{n−1}$ and then dividing the columns successively by 1, $a$, $a^2$, . . . , $a^{n−1}$ we get the matrix

$$\begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 2 & 1 & \cdots & 1 \\
1 & 1 & 2 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 2
\end{pmatrix}.$$ 

Next, after subtracting the first row of this matrix from all other rows we get

$$\begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
-1 & 1 & 0 & \cdots & 0 \\
-1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & 1
\end{pmatrix},$$ 

columns we get the matrix

$$\begin{pmatrix}
n+1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}$$

with determinant $n + 1$. Since these row-column operations don’t affect the determinant, the determinant of the original matrix is also equal to $n + 1$.

6. Suppose a 10 × 10 board has some of its 1 × 1 squares colored red. After each minute passes, every non-red square that shares a side with at least two red squares also becomes red. If there are exactly 9 red squares at the start, may it happen that eventually all squares of the board become red?

Solution. Let $N(k)$ be the number of edges between red and non-red squares after $k$ kminutes. (We count the exterior of the board as non-red squares.) At the start, there are 9 red squares, so $N(0) \leq 36$. The perimeter of the whole board is 40, so to make the board red, the value $N(0)$ has to increase from at most 36 to 40. But $N(k)$ cannot increase as time passes. Indeed, each “new” red square has at least 2 sides in common with some already-red squares. That new square adds atmost 2 new edges to $N(k)$, but also subtracts at least 2 edges (since the previous boundary edges are no longer between red and non-red). Therefore, $N(k)$ cannot increase, and the board cannot become all red.