2020 Gordon exam solutions

1. *Evaluate* $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x}\,dx.$

Solution.

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x}\,dx &= \int_0^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x}\,dx + \int_0^{\pi} \frac{\sin(-2020x)}{(1+2^{-x})\sin(-x)}\,dx \\
&= \int_0^{\pi} \frac{\sin(2020x)}{(1+2^x)\sin x}\,dx + \int_0^{\pi} \frac{2^x\sin(2020x)}{(1+2^x)\sin x}\,dx = \int_0^{\pi} \frac{\sin(2020x)}{\sin x}\,dx \\
&= \int_0^{\pi/2} \frac{\sin(2020x)}{\sin x}\,dx + \int_0^{\pi/2} \frac{\sin(2020(\pi-x))}{\sin(\pi-x)}\,dx \\
&= \int_0^{\pi/2} \frac{\sin(2020x)}{\sin x}\,dx - \int_0^{\pi/2} \frac{\sin(2020x)}{\sin x}\,dx = 0.
\end{aligned}$$

2. *Let $G = \{A_1, \dots, A_n\}$ be a finite multiplicative group of real $k \times k$ matrices, and let $S = \sum_{i=1}^n A_i$. If* $\operatorname{trace}(S) = 0$, *prove that $S = 0$.*

Solution. For every $i = 1, \dots, n$, multiplication by A_i permutes the elements of G, so $A_iS = S$, and so $S^2 = \sum_{i=1}^n A_iS = nS$. For $P = \frac{1}{n}S$, we therefore have $P^2 = P$, that is, the linear transformation defined by P is a projection. Hence, P is diagonaliziable with all eigenvalues equal to either 0 or 1; if $\operatorname{trace} S = n \operatorname{trace} P = 0$, then all eigenvalues of P are zeroes, and $P = 0$.

3. *Find all $b \in \mathbb{N}$ for which $\sqrt[3]{2+\sqrt{b}} + \sqrt[3]{2-\sqrt{b}}$ is an integer.*

Solution. Let $\alpha = \sqrt[3]{2+\sqrt{b}}$, $\beta = \sqrt[3]{2-\sqrt{b}}$, and $n = \alpha + \beta$. We have

$$n^3 = (\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3(\alpha+\beta)\alpha\beta = 4 + 3n\sqrt[3]{4-b}.$$

Hence,

$$4 - b = \tfrac{1}{27}(n^2 - 4/n)^3 = \tfrac{1}{27}\big(x^6 - 12n^3 + 48 - (4/n)^3\big).$$

If both b and n are integers, then $4/n$ is an integer, so $n = \pm1, \pm2, \pm4$. Now, if $n = -1$, 2, or -4, then $n^2 - 4/n$ is not divisible by 3 and b is fractional; if $n = 1$, then $b = 5$; if $n = -2$ then $b = -4$; if $n = 4$ then $b = -121$. Since $b > 0$, we see that the only valid solution is $b = 5$.

4. *Let $v_1, \dots, v_n \in \mathbb{R}^d$. Prove that $\sum_{i,j=1}^n e^{v_i \cdot v_j} \geq n^2$.*

Solution. By the AM-GM inequality,

$$\sum_{i,j=1}^n e^{v_i\cdot v_j} \geq n^2\Big(\prod_{i,j=1}^n e^{v_i\cdot v_j}\Big)^{1/2} = n^2\Big(e^{\Sigma_{i,j=1}^n v_i\cdot v_j}\Big)^{1/2} = n^2\Big(e^{|\Sigma_i^n v_i|^2}\Big)^{1/2} \geq n^2$$

since $\left|\sum_i^n v_i\right|^2 \geq 0$.

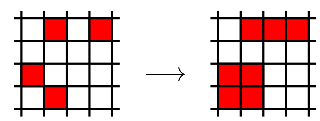
5. *Let $a, b \in \mathbb{R}$, $ab = 1$. Evaluate* $\det\begin{pmatrix} 2 & a & a^2 & \dots a^{n-1} \\ b & 2 & a & \dots a^{n-2} \\ b^2 & b & 2 & \dots a^{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots \;\; 2 \end{pmatrix}.$

Solution. Let $A = \begin{pmatrix} 2 & a & a^2 & \dots a^{n-1} \\ b & 2 & a & \dots a^{n-2} \\ b^2 & b & 2 & \dots a^{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots \;\; 2 \end{pmatrix}$, then $A = I + B$ where $B = \begin{pmatrix} 1 & a & a^2 & \dots a^{n-1} \\ b & 1 & a & \dots a^{n-2} \\ b^2 & b & 1 & \dots a^{n-3} \\ \vdots & \vdots & \vdots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots \;\; 1 \end{pmatrix}$. B has rank 1, – all its columns are multiples of $u = \begin{pmatrix} 1 \\ b \\ b^2 \\ \vdots \\ b^{n-1} \end{pmatrix}$; so, the null space of B is $(n-1)$-dimensional. We have

$Bu = nu$, so B has the eigenvalues 0, of multiplicity $n-1$, and n. The eigenvalues of $A = I + B$ are therefore 1, of multiplicity $n-1$, and $n+1$, and $\det A = n+1$.

Another solution. After multiplying the rows of the matrix successively by $1, a, a^2, \dots, a^{n-1}$ and then dividing the columns successively by $1, a, a^2, \dots, a^{n-1}$ we get the matrix $\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & 2 & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 2 \end{pmatrix}$. Next, after subtracting the first row of this matrix from all other rows we get $\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}$, and after adding to the first column all other columns we get the matrix $\begin{pmatrix} n+1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ with determinant $n+1$. Since these row-column operations don't affect the determinant, the determinant of the original matrix is also equal to $n+1$.

6. *Suppose a 10×10 board has some of its 1×1 squares colored red. After each minute passes, every non-red square that shares a side with at least two red squares also becomes red. If there are exactly 9 red squares at the start, may it happen that eventually all squares of the board become red?*



Solution. Let $N(k)$ be the number of edges between red and non-red squares after k kminutes. (We count the exterior of the board as non-red squares.) At the start, there are 9 red squares, so $N(0) \leq 36$. The perimeter of the whole board is 40, so to make the board red, the value $N(0)$ has to increase from at most 36 to 40. But $N(k)$ cannot increase as time passes. Indeed, each "new" red square has at least 2 sides in common with some already-red squares. That new square adds atmost 2 new edges to $N(k)$, but also subtracts at least 2 edges (since the previous boundary edges are no longer between red and non-red). Therefore, $N(k)$ cannot increase, and the board cannot become all red.