2020 Rasor-Bareis exam solutions

1. *Prove that* 17^{2020} *cannot be represented as* m^3+n^3 *for positive integers* m *and* n.

Solution. The assertion follows from the fact that 17^{2020} cannot be represented as a sum m^3+n^3 modulo 7. Indeed, by the little Fermat's theorem, $17^6 \equiv 1 \bmod 7$, and since $2020 = 636 \cdot 6 + 4$,

$$17^{2020} \equiv 17^4 \bmod 7 \equiv 3^4 \bmod 7 \equiv 4 \bmod 7.$$

On the other hand, for any integer m, $m^3 \bmod 7$ can only be equal to 0, 1, or $-1 \equiv 6$, and the sum of two cubes cannot be 4 modulo 7.

Another solution. Assume that for some $k \in \mathbb{N}$ there are $m, n \in \mathbb{N}$ such that $17^k = m^3+n^3$. Choose the smallest k with this property; then neither m nor n are divisible by 17. Indeed, if 17 is a factor of one of them, then it is a factor of the other, and we can cancel 17^3 to get an identity with a smaller k; but this contradicts the choice of k if $k \geq 1$ and is impossible if $k \leq 0$.

Since $m^3+n^3 = (m+n)(m^2-mn+n^2)$ and 17 is prime, the Unique Factorization theorem gives that both $m+n$ and m^2-mn+n^2 are powers of 17. Then 17 divides $(m+n)^2 = (m^2-mn+n^2)+3mn$, so, divides mn, and so divides m or n, contradiction.

2. *Prove that for any* $x, y, z \in [0,1]$, $\dfrac{x}{7+y^3+z^3} + \dfrac{y}{7+z^3+x^3} + \dfrac{z}{7+x^3+y^3} \leq \dfrac{1}{3}$.

Solution. Since $x^3, y^3, z^3 \leq 1$, we have

$$\frac{x}{7+y^3+z^3} + \frac{y}{7+z^3+x^3} + \frac{z}{7+x^3+y^3} \leq \frac{x}{6+x^3+y^3+z^3} + \frac{y}{6+x^3+y^3+z^3} + \frac{z}{6+x^3+y^3+z^3} = \frac{x+y+z}{6+x^3+y^3+z^3}.$$

To prove that this quotient is $\leq \frac{1}{3}$, we need to show that $6+x^3+y^3+z^3 \geq 3x+3y+3z$. We are done if we have $2+x^3 \geq 3x$ for all $x \in [0,1]$. But this is so since the polynomial x^3-3x+2 is decreasing on $[0,1]$ and vanishes at 1.

3. *Prove that* $\displaystyle\int_0^{\pi/2} \cos(2020x)(\cos x)^{2018}\,dx = 0$.

Solution. For any $n \in \mathbb{N}$, by a trigonometric formula, we have

$$I = \int_0^{\pi/2} \cos\big((n+2)x\big)(\cos x)^n\,dx = \int_0^{\pi/2} \cos\big((n+1)x\big)(\cos x)^{n+1}\,dx - \int_0^{\pi/2} \sin\big((n+1)x\big)\sin x\,(\cos x)^n\,dx.$$

Integrating the second term by parts, with $u = \sin\big((n+1)x\big)$ and $dv = (\cos x)^n \sin x\,dx$, we get

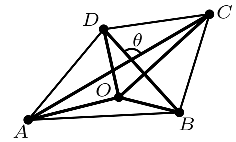
$$\int_0^{\pi/2} \sin\big((n+1)x\big)\sin x\,(\cos x)^n\,dx = -\frac{1}{n+1}\sin\big((n+1)x\big)(\cos x)^{n+1}\Big|_0^{\pi/2} + \int_0^{\pi/2} (\cos x)^{n+1}\cos\big((n+1)x\big)\,dx = \int_0^{\pi/2} \cos\big((n+1)x\big)(\cos x)^{n+1}\,dx.$$

Subtracting this from the first term, we get $I = 0$.

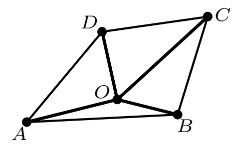
4. *Find all real polynomials* $f(x) = x^{2020} + a_{2019}x^{2019} + \cdots + a_1x + a_0$ *all of whose roots are real, and such that* $|f(i)| = 1$.

Solution. Let f be such a polynomial, and let $x_1, \ldots, x_{2020}$ be the roots of f (listed with their multiplicities), so that $f(x) = \prod_{k=1}^{2020}(x-x_k)$. For every k, since $x_k \in \mathbb{R}$, we have $|i - x_k| = \sqrt{x_k^2+1^2} \geq 1$, with equality only if $x_k = 0$. Hence, $|f(i))| = \prod_{k=1}^{2020}|i-x_k| \geq 1$ with equality only if $x_k = 0$ for all k. Thus, all roots of f are equal to 0, and $f(x) = x^{2020}$.

5. *Let $ABCD$ be a convex quadrilateral of area 1, and let O be a point inside it. Prove that $|AO|+|BO|+|CO|+|DO| \ge 2\sqrt{2}$.*



Solution. By the triangle inequality, $|AO|+|CO| \ge |AC|$ and $|BO|+|DO| \ge |BD|$, so $|AO|+|BO|+|CO|+|DO| \ge |AC|+|BD|$. The area S of $ABCD$ equals $\frac{1}{2}|AC| \cdot |BD| \sin\theta$, where θ is the angle between AC and BD. Hence $|AC| \cdot |BD| \ge 2S = 2$. By the AM-GM inequality, $|AC|+|BD| \ge 2\sqrt{|AC| \cdot |BD|} \ge 2\sqrt{2}$.

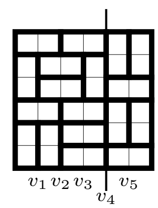


Another solution. By the formula for the area of trinagle, the areas of the triangles AOB, BOC, COD, and DOA do not exceed $\frac{1}{2}|AO|\cdot|BO|$, $\frac{1}{2}|BO|\cdot|CO|$, $\frac{1}{2}|CO|\cdot|DO|$, and $\frac{1}{2}|DO|\cdot|AO|$ respectfully. Since the sum of these areas is 1, we have

$$2 \le |AO|\cdot|BO|+|BO|\cdot|CO|+|CO|\cdot|DO|+|DO|\cdot|AO| = (|AO|+|CO|)(|BO|+|DO|).$$

By the AM-GM inequality, $\sqrt{(|AO|+|CO|)(|BO|+|DO|)} \le \frac{1}{2}\big((|AO|+|CO|)+(|BO|+|DO|)\big)$, so $|AO|+|CO|+|BO|+|DO| \ge 2\sqrt{2}$.

6. *A 6×6 board is covered with eighteen 2×1 tiles, without gaps or overlaps. No matter how those tiles are arranged, prove that there always is a straight line that cuts across the whole board without cutting any tile.*



Solution. Naturally, we only consider the lines of the grid subdividing the board into its 1×1 squares; there are 10 such lines, 5 “vertical” and 5 “horizontal”, and every tile can be cut, into halves, by exactly one of these lines. We claim that each line cuts an even (possibly, zero) number of tiles. Indeed, if the i-th vertical line v_i cuts m tiles, then the number of 1×1 squares of the board that are on the left of v_i (which is $6i$) equals $m+2k$ where k is the number of tiles on the left of v_i. Hence, $m = 6i - 2k$ and is even. The same applies to the horizontal lines. It follows that if a line cuts a tile, then it cuts ≥ 2 tiles. So if each of our ten lines cuts a tile, then those lines cut at least $10 \cdot 2 \ge 20$ tiles. This is impossible since there are only 18 tiles.