Title **Von Neumann Regular Separative Rings**

Speaker: **S.K. Jain**

Abstract

The question whether every regular ring is separative has been open for a quarter of a century and has drawn the attention of several authors because several open quetions in regular rings have positive answer (c.f. K.Goodearl, von Neumann Regular rings, Krieger Publishing co, 1991 alongwith updates on the open questions by Goodearl)

I will be speaking on a preliminary work that has been going on with André Leroy.

A regular ring R is called separative if for all finitely generated projective R-modules $A$and B,

$A \oplus A \cong B \oplus B \cong A \oplus B$ implies $A \cong B$. Equivalently, R is separative (as observed by O'Meara), $A^n \cong B^n$ for all n implies $A \cong B$. Selfinjective regular rings, unit regular rings, regular rings with a polynomial identity, are separative.

Goodearl et.al ,showed if A is an ideal in a regular ring such that both R and R/A are separative rings then R is separative.

O' Meara (JAA 2013) showed that a regular ring R is separative iff for every nonunit $a \in R$ the following two conditions are equivalent:

1. a is a product of idempotents
2. for every a in R , $Rr.ann(a) = l.anna(a)R = R(1-a)R$.

We remark that (1) always implies (2) in a regular ring (Hanna-O'Meara, (J. Algebra, 1989)).

O'Meara also showed that for selfinjective regular rings R, the above two conditions are equivalent. (J. Algebra 1999).

We provide a new proof which gives a direct argument for CS rings to be separative and are able to add new classes of separative regular rings. Below is the list of rings that we consider.

1 CS rings

2. Pseudo-injective(Auto-invariant) rings

3. Rings R having closure extension property, that is, if any two right ideals in R are isomorphic then their essential closures in R are also isomorphic. This is equivalent to saying that if the closures are subisomorphic to each other then they are isomorphic.

4. O'Meara believes that if there exists an example of a regular ring which is not separative, it is probably the subring of Q where Q is the infinite direct product of full matrix rings $M_{n_i}(F)$ of strictly increasing orders over a fixed countable field F. Let e be the identity of Q. Note that Q is selfinjective with socle $S = \oplus(M_{n_i}(F))$ and it is the right as well the left max quotient ring of S.

Also, Q and $S + eF$ are separative regular rings. The question is whether the intermediate rings are separative. It is easy to note that an intermediate ring $S + eF + A$ where A is a subalgebra generated by a single element is separative regular ring.

It will be interesting to know whether the subalgebra $S + \ eF + A$ when A is a finitely generated subalgebra containg S and e in Q, is separative?

Since this talk is using Zoom, I am not sure how much can I write on my white board. So let me give a taste of some results.

We show that a regular ring which is pseudo-injective (also known as auto-invariant) is not necessarily CS but separative. In this connection, we consider a Boolean ring.This is a surprizing to us that the max quotient ring of a Boolean ring is Boolean. We have not seen this in any book , including in the classical book of Lambek which discusses max quotient rings of commutative rings and Boolean rings (Chapter 2)) and in the book of B. Stenstrorm (Chapter 9). This fact gives an example of pseudo-injectve (auto-invariant) module which is not CS, and hence provides a new class of separative rings. There are a couple of examples of pseudo-injectivewhich are not quasi-injective but none as simple as this.

Lemma.The max quotient ring of a Boolean ring is Boolean.

Let Q be the max quotient ring of a Boolean ring B. Let $q \in Q = Q$. Then there exists an essential ideal such that $qE \subset B$. This implies $qx \in B$ for all $x \in E$. Therefore, $(qx)^2 = qx$.Thus $q^2x^2 = qx$.This implies $q^2x = qx$.So $(q^2 - q)E = 0$. Since B is nonsingular, $(q^2 - q) = 0$, completing the proof.

The following Lemmas are our main tools to prove separativity:

Lemma 1 *Let R be a regular ring and $Q = Q(R)$ be its right maximal quotient ring. If for any finite set of idempotents, $e_1, \ldots, e_n, f_1, \ldots, f_l$ in R, $\prod_{i=1}^n e_iQ \cong \prod_{j=1}^l f_jQ$ implies $\prod_{i=1}^n e_iR \cong \prod_{j=1}^l f_jR$, then the ring R is separative.*

Lemma 2 *Let M be a right pseudo injective non singular module with injective hull $E(M)$. If N is a closed submodule of $E(M)$ and σ is an injective morphism from N into $E(M)$, then $\sigma(N \cap M) = \sigma(N) \cap M$.*