# Asymptotic behavior of the approximation numbers of the Hardy-type operator from $L^p$ into $L^q$ (cases $1 < p \leq q \leq 2, 2 \leq p \leq q < \infty$ and $1 < p \leq 2 \leq q < \infty$)

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**Abstract**

We consider the Hardy-type operator

$$(Tf)(x) := v(x) \int_a^x u(t) f(t) dt, \qquad x > a.$$

and establish properties of $T$ as a map from $L^p(a,b)$ into $L^q(a,b)$ for $1 < p \leq q \leq 2$, $2 \leq p \leq q < \infty$ and $1 < p \leq 2 \leq q < \infty$. The main result is that, with appropriate assumptions on $u$ and $v$, the approximation numbers $a_n(T)$ of $T$ satisfy the inequality

$$c_1 \int_a^b |uv|^r dt \leq \liminf_{n\to\infty} n a_n^r(T) \leq \limsup_{n\to\infty} n a_n^r(T) \leq c_2 \int_a^b |uv|^r dt$$

when $1 < p \leq q \leq 2$ or $2 \leq p \leq q < \infty$, and in the case $1 < p \leq 2 \leq q < \infty$ we have

$$\limsup_{n\to\infty} n a_n^r(T) \leq c_3 \int_0^d |u(t)v(t)|^r dt$$

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and

$$c_4 \int\limits_0^d |u(t)v(t)|^r dt \le \liminf_{n\to\infty} n^{(1/2-1/q)r+1} a_n^r(T),$$

where $r = \frac{p'q}{p'+q}$ and constants $c_1, c_2, c_3, c_4$. Upper and lower estimates for the $l^s$ and $l^{s,k}$ norms of $\{a_n(T)\}$ are also given.

*Key words:* Approximation numbers, Hardy-type operators, Integral operators
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## 1 Introduction

The operator $T : L^p(a,b) \to L^q(a,b)$ ( where $0 \le a \le b \le d < \infty$ ) defined by

$$Tf(x) = v(x) \int\limits_0^x u(t)f(t)dt \tag{1}$$

was studied in [1] and [5], in the case $1 \le p \le q \le \infty$, for real–valued functions $u \in L^{p'}(0,c), v \in L^p(c,d)$, for any $c \in (0,d)$ and $p' = p/(p-1)$. In the aformentioned works, the following estimates for the approximation numbers $a_n(T)$ of $T$ were obtained:

$$a_{N(\varepsilon)+3} \le \sigma_p \varepsilon, \tag{2}$$

$$a_{N(\varepsilon)-1} \ge \nu_q (N(\varepsilon)-1)^{1/q-1/p}\varepsilon, \qquad \text{for } p < q < \infty \tag{3}$$

and

$$a_{N(\varepsilon)/2-1} \ge \varepsilon/2, \qquad \text{for } p = q, \tag{4}$$

where $\sigma_p, \nu_q$, are constants depending on $q$, and $N(\varepsilon)$ is an $\varepsilon$-depending natural number .

In the case $p = q$, these results are sharp and are used in [2] and [5] to obtain asymptotic results for the approximation numbers.

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Specifically, it was proved in [2] that for $p = q = 2$

$$\lim_{n\to\infty} na_n(T) = \frac{1}{\pi}\int_0^d |u(t)v(t)|dt \tag{5}$$

and that for $1 < p = q < \infty$,

$$\frac{1}{4}\alpha_p\int_0^d |u(t)v(t)|dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le \alpha_p\int_0^d |u(t)v(t)|dt. \tag{6}$$

The endpoint cases were studied in [5]: it was shown there that for $p = q = \infty$ (and similarly for $p = q = 1$)

$$\frac{1}{4}\int_0^d |u(t)v_s(t)|dt \le \liminf_{n\to\infty} na_n(T) \le \limsup_{n\to\infty} na_n(T) \le \int_0^d |u(t)v_s(t)|dt, \tag{7}$$

where

$$v_s(t) = \lim_{\varepsilon\to 0_+} \|v\ \chi_{(t-\varepsilon,t+\varepsilon)}\|_{L^\infty}.$$

If $p < q$, the estimates (2) and (3) are not sharp.

The estimates (2) and (3) were used in [7] to obtain the following asymptotic results for the approximation numbers in the case $1 < p < q < \infty$:

$$\limsup_{n\to\infty} na_n^r(T) \le c_{p,q}\int_0^d |u(t)v(t)|^r dt \tag{8}$$

and

$$\le d_{p,q}\int_0^d |u(t)v(t)|^r dt \le \liminf_{n\to\infty} n^{(1/p-1/q)r+1}a_n^r(T) \tag{9}$$

where $r = pq'/(q+p')$.

Since the estimates upon which they are based are not sharp, these results aren't sharp either, in contrast to (5), (6). Our research is directed toward finding alternative, refined versions of (2) and (3) in the case $p < q$, aiming to get better asymptotic results than (8) and (9). In this paper, we succeed in showing that for $1 \leq p \leq q \leq \infty$,

$$a_{N(\varepsilon)+1} \leq 2\varepsilon, \tag{10}$$

and for $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$

$$a_{N(\varepsilon)/4-1} \geq c\varepsilon, \tag{11}$$

and for $1 < p \leq 2 \leq q < \infty$

$$a_{N(\varepsilon)/4-1} \geq c\varepsilon N(\varepsilon)^{1/2-1/q}, \tag{12}$$

where $c$ is a constant independent of $\varepsilon$ and $N(\varepsilon)$. And under some condition on $u$ and $v$ we show that for $1 \leq p \leq q \leq 2$ or $2 \leq p \leq q \leq \infty$

$$c_1 \int_a^b |uv|^r \leq \liminf_{n\to\infty} n a_n^r(T) \leq \liminf_{n\to\infty} n a_n^r(T) \leq c_2 \int_a^b |uv|^r,$$

and for $1 < p \leq 2 \leq q < \infty$

$$\limsup_{n\to\infty} n a_n^r(T) \leq c_{p,q} \int_0^d |u(t)v(t)|^r dt$$

and

$$d_{p,q} \int_0^d |u(t)v(t)|^r dt \leq \liminf_{n\to\infty} n^{(1/2-1/q)r+1} a_n^r(T),$$

where $r = \frac{p'q}{p'+q}$. We also describe $l^r$ and $l^{r,s}$ norms of $\{a_n\}_{n=1}^{\infty}$.

## 2 Preliminaries

Throughout this paper we will suppose that $1 < p \leq q \leq 2$. In what follows we shall be concerned with the operator $T$ defined in (1) as a map from $L^p(0,d)$ into $L^q(0,d)$ where $0 < d \leq \infty$. The functions $u, v$ are subject to the following restrictions: for all $x \in (0,d)$

$$u \in L^{p'}(0,x), \tag{13}$$

and

$$v \in L^q(x,d). \tag{14}$$

It is well-known that these assumptions guarantee that $T$ is well defined (see (9)). Moreover, the norm of this operator is equivalent to:

$$J := \sup_{x\in(0,d)} \left(\int_0^x |u(t)|^{p'} dt\right)^{1/p'} \left(\int_x^d |v(t)|^q dt\right)^{1/q},$$

(see [4],[8] and [5]). We define the operator $T_I$ by

$$T_I f(x) := v(x)\chi_I(x)\int_0^x u(t)f(t)\chi_I(t)dt, \qquad x > 0, \tag{15}$$

where $I = (a,b) \subset (0,d)$, and the quantity

$$J(I) \equiv J(a,b) := \sup_{x\in I} \left(\int_a^x |u(t)|^{p'} dt\right)^{1/p'} \left(\int_x^d |v(t)|^q dt\right)^{1/q}. \tag{16}$$

It is obvious that $J(I) \approx \|T_I\|_{p\to q}$, where the symbol $\approx$ indicates that the quotient of the two sides is bounded above and below by positive constants.

**Proposition 1** *There are two positive constants $K_1$, $K_2$ such that for any $I = (a,b) \subset (0,d)$ the inequality*

$$K_1 J(a,b) \le \|T_I\| \le K_2 J(a,b)$$

*holds.*

We start by proving an important continuity property of $J$:

**Lemma 2.1** *Suppose that (13) and (14) are satisfied. Then the function $J(.,b)$ is continuous and non–increasing on $(0,b)$, for any $b \le \infty$.*

**Proof:** It is easy to verify that $J(.,b)$ is non-increasing on $(0,b)$. To prove the continuity of $J$, fix $x \in (0,b)$ an $\varepsilon > 0$. By (13) and (14) there exists $0 < h_0 < \min\{x, b-x\}$ such that

$$\left(\int_{x-h_0}^x |u(t)|^{p'} dt\right)^{1/p'} \|v\|_{q,(x-h_0,x)} < \varepsilon.$$

It follows that for $h$, $0 < h < h_0$,

$$\begin{aligned} J(x,b) \le J(x-h,b) &= \sup_{x-h<z<b} \left( \int_{x-h}^{z} |u(t)|^{p'} dt \right)^{1/p'} \|v\|_{q,(z,b)} \\ &= \max \Bigg\{ \sup_{x-h<z<x} \left( \int_{x-h}^{z} |u(t)|^{p'} dt \right)^{1/p'} \|v\|_{q,(z,b)}, \\ &\qquad \sup_{x<z<b} \left( \left( \int_{x-h}^{x} + \int_{x}^{z} \right) |u(t)|^{p'} dt \right)^{1/p'} \|v\|_{q,(z,b)} \Bigg\} \\ &\le \max \{\varepsilon, \varepsilon + J(x,d)\} = \varepsilon + J(x,d), \end{aligned} \tag{17}$$

which yields $0 < J(x-h,b) - J(x,b) < \varepsilon$. The inequality $0 < J(x,b) - J(x+h,b) < \varepsilon$ can be proved analogously. □

For the sake of completeness, we include the following known result (see [4] and [9]):

**Proposition 2.2** *The operator $T$ defined by (1), with $1 < p < \infty$ and $u, v$ satisfying (13), (14) and $J < \infty$ is a compact map from $L^p(0,d)$ into $L^q(0,d)$ if and only if $\lim_{c\to 0_+} J(0,c) = \lim_{c\to d_-} J(c,d) = 0$.*

In what follows $A(I)$ is a function defined on all sub-intervals $I = (a,b) \subset (0,d)$, defined by

$$A(I) = A(a,b) := \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \|Tf - \alpha v\,\|_{p,I}. \tag{18}$$

A similar function can be found in [5]. Next, we prove some basic properties of $A(I)$. Choosing $\alpha = 0$ in (18) we immediately obtain for any $I = (a,b)$, $0 \le a < b \le d$,

$$A(I) \le \|T_I\|. \tag{19}$$

**Lemma 2.3** *Let $I = (a,b)$ and $\|u\|_{p',I} < \infty$, $\|v\|_{q,I} < \infty$. Set*

$$\widetilde{A}(I) = \sup_{\|f\|_{p,I}=1} \inf_{|\alpha|\le 2\|u\|_{p',I}} \|Tf - \alpha v\,\|_{p,I}.$$

*Then $A(I) = \widetilde{A}(I)$.*

**Proof:** Hölder's inequality yields

$$\|T_I\| = \sup_{\|f\|_{p,I}=1} \int_a^b \left( \left| \int_a^x f(t)u(t)dt \right|^q dx \right)^{1/q}$$
$$\leq \sup_{\|f\|_{p,I}=1} \left( \int_a^b |v(x)|^q \left( \int_a^x |f(t)|^p dt \right)^{q/p} \left( \int_a^x |u(t)|^{p'} dt \right)^{q/p'} dx \right)^{1/q}$$
$$\leq \left( \int_a^b |v(x)|^q \left( \int_a^b |u(t)|^{p'} dt \right)^{q/p'} dx \right)^{1/q} = \|u\|_{p',I} \|v\|_{q,I}.$$

If $\|v\|_{q,I} = 0$ then $A(I) = \widetilde{A}(I) = 0$. Assume $\|v\|_{q,I} > 0$. Let $\|f\|_{p,I} = 1$ and suppose that $|\alpha| > 2\|u\|_{p',I}$. Then $|\alpha| \geq 2\frac{\|T_I\|}{\|v\|_{q,I}}$ and using the trivial inequality $|a-b|^q \geq 2^{1-q}|a|^q - |b|^q$ valid for any real numbers $a$, $b$ we obtain for each $\alpha \in \Re$

$$\int_a^b \left| \left( \alpha - \int_a^x f(t)u(t)dt \right) v(x) \right|^q dx \geq \int_a^b \left| |\alpha v(x)| - \Big| \int_a^x f(t)u(t)dt \Big| \right|^q dx$$
$$\geq 2^{1-q}|\alpha|^q \int_a^b |v(x)|^q dx - \int_a^b \Big| v(x) \int_a^x f(t)u(t)dt \Big|^q dx$$
$$> 2^{1-q}\Big(2\frac{\|T_I\|}{\|v\|_{q,I}}\Big)^q \int_a^b |v(x)|^q dx - \|T_I\|^q = \|T_I\|^q.$$

In conjuction with (19), the above yields

$$\|T_I\| \geq A(I)$$
$$= \sup_{\|f\|_{p,I}=1} \min \left\{ \inf_{|\alpha| \leq 2\|u\|_{p',I}} \left( \int_a^b \left| \left( \alpha - \int_a^x f(t)u(t)dt \right) v(x) \right|^q \right)^{1/q}, \right.$$
$$\left. \inf_{|\alpha| > 2\|u\|_{p',I}} \left( \int_a^b \left| \left( \alpha - \int_a^x f(t)u(t)dt \right) v(x) \right|^q \right)^{1/q} \right\}$$
$$= \inf_{|\alpha| \leq 2\|u\|_{p',I}} \left( \int_a^b \left| \left( \alpha - \int_a^x f(t)u(t)dt \right) v(x) \right|^q \right)^{1/q} = \widetilde{A}(I),$$

which finishes the proof. □

**Lemma 2.4** *Let $u$ and $v$ satisfy (13) and (14) respectively. Then $A(I_1) \leq A(I_2)$, provided $I_1 \subset I_2$. Moreover, given $0 < b < d$ the function $A(.,b)$ is continuous on $(0,b)$.*

**Proof:** Let $0 \le a_1 \le a_2 < b_2 \le b_1 \le d$, $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ . Then

$$\begin{aligned}
A(I_1) &= \sup_{\|f\|_{p,I_1}=1} \inf_{\alpha\in\Re} \left( \int_{a_1}^{b_1} \left| v(x)\Big( \int_{a_1}^{x} (f(t)u(t)dt - \alpha)\Big) \right|^q dx \right)^{1/q} \\
&\ge \sup_{\|f\chi_{I_2}\|_{p,I_1}=1} \inf_{\alpha\in\Re} \left( \int_{a_1}^{b_1} \left| v(x)\Big( \int_{a_1}^{x} (f(t)u(t)dt - \alpha)\Big) \right|^q dx \right)^{1/q} \\
&\ge \sup_{\|f\|_{p,I_2}=1} \inf_{\alpha\in\Re} \left( \int_{a_2}^{b_2} \left| v(x)\Big( \int_{a_2}^{x} (f(t)u(t)dt - \alpha)\Big) \right|^q dx \right)^{1/q} = A(I_2)
\end{aligned}$$

which proves the first part of lemma.

For the remaining statement, fix $b \in (0, d)$ and $0 < y < b$. Let $\varepsilon > 0$. By (13) and (14) there exists $0 < h_0$ such that $0 < y - h_0$ and

$$\int_{y-h_0}^{y} |u|^{p'} < \varepsilon \text{ and } \int_{y-h_0}^{y} |v|^q < \varepsilon.$$

Set $D_h = 2\|u\|_{p',(y-h,b)}$ for any $0 \le h < y$. Recall that by (13), one has $D_h < \infty$ for $0 \le h < d$. Using the trivial inequality $(a+b)^{1/q} \le a^{1/q} + b^{1/q}$, the triangle inequality and the Hölder inequality, it follows that

$$\begin{aligned}
A(y, b) &\le A(y - h, b) \\
&= \sup_{\|f\|_{p,(y-h,b)}=1} \inf_{\alpha\in\Re} \left( \int_{y-h}^{b} \left| \Big(\alpha - \int_{y-h}^{x} f(t)u(t)dt\Big) v(x) \right|^q dx \right)^{1/q} \\
&= \sup_{\|f\|_{p,(y-h,b)}=1} \inf_{|\alpha|\le D_h} \Bigg\{ \int_{y-h}^{y} \left| \Big(\alpha - \int_{y-h}^{x} f(t)u(t)dt\Big) v(x) \right|^q dx \\
&\quad + \int_{y}^{b} \left| \Big( \int_{y-h}^{y} f(t)u(t)dt + \int_{y}^{x} f(t)u(t)dt - \alpha\Big) v(x) \right|^q dx \Bigg\}^{1/q}
\end{aligned}$$

$$\begin{aligned}
&\le \sup_{\|f\|_{p,(y-h,b)}=1} \inf_{|\alpha|\le D_h} \Bigg\{ \left[ \int_{y-h}^{y} |v(x)|^q \Big( \int_{y-h}^{x} |u(t)|^{p'} dt\Big)^{q/p'} \Big( \int_{y-h}^{x} |f(t)|^p dt\Big)^{q/p} dx \right]^{1/q} \\
&\quad + \left[ |\alpha|^q \int_{y-h}^{y} |v(x)|^q dx \right]^{1/q}
\end{aligned}$$

$$+\left[\int_y^b |v(x)|^q dx\Big(\int_{y-h}^y |u(t)|^{p'}dt\Big)^{q/p'}\Big(\int_{y-h}^y |f(t)|^p dt\Big)^{q/p}\right]^{1/q}$$

$$+\left[\int_y^b \Big|v(x)\Big(\int_y^x f(t)u(t)dt-\alpha\Big)\Big|^q dx\right]^{1/q}\Bigg\}$$

$$\leq \Bigg\{\varepsilon^{1+1/p'}+D_h\varepsilon^{1/q}+\|v\|_{q,(y,b)}\varepsilon^{1/p'}$$

$$+\sup_{\|f\|_{p,(y-h,b)}=1}\inf_{|\alpha|\leq D_h}\Big(\int_y^b\Big|\Big(\int_y^x f(t)u(t)dt-\alpha\Big)v(x)\Big|^q dx\Big)^{1/q}\Bigg\}$$

Since $D_0\leq D_h\leq D_{h_0}$ we have by Lemma 2.3

$$\inf_{|\alpha|\leq D_h}\Big(\int_y^b\Big|\Big(\int_y^x f(t)u(t)dt-\alpha\Big)v(x)\Big|^q dx\Big)^{1/q}$$

$$\leq \inf_{|\alpha|\leq D_0}\Big(\int_y^b\Big|\Big(\int_y^x f(t)u(t)dt-\alpha\Big)v(x)\Big|^q dx\Big)^{1/q}=A(y,b)$$

and thus

$$A(y,b)\leq A(y-h,b)\leq 2^{q-1}(\varepsilon^{1+1/p'}+D_{h_0}\varepsilon^{1/q}+\|v\|_{q,(y,b)}\varepsilon^{1/p'}+A(y,b))$$

which proves that

$$\lim_{h\to 0_+}A(y-h,b)=A(y,b).$$

Analogously,

$$\lim_{h\to 0_+}A(y+h,b)=A(y,b).$$

which finishes the proof of our lemma. □

**Lemma 2.5** *Suppose $u,v>0$ satisfy (13) and (14) and that $T:L^p(a,b)\to L^q(a,b)$ is compact. Let $I_1=(c,d)$ and $I_2=(c',d')$ be subintervals of $(a,b)$, with $I_2\subset I_1$, $|I_2|>0$, $|I_1-I_2|>0$, $\int_a^b v^q(x)dx<\infty$. Then $0<A(I_2)<A(I_1)$.*

**Proof:** Let $0\leq f\in L^p(I_2)$, $0<\|f\|_{p,I_2}\leq\|f\|_{p,I_1}\leq 1$ with $\operatorname{supp} f\subset I_2$. Let $y\in I_2$ then

$$\|T_{(c',y)}\|_{p,I_2}>0 \qquad\text{and}\qquad \|T_{(y,d')}\|_{p,I_2}>0$$

and then by simple modification of [EHL2, Lemma 3.5] for case $p<q$ we have

$$\min\{\|T_{(c',y)}\|_{q,I_2},\|T_{(y,d')}\|_{q,I_2}\}\leq\min_{x\in J}\|T_{x,J}\|_{q,I_2}$$

which means $A(I_2) > 0$.

Next, suppose that $c = c' < d' < d$. A slight modification of [EHL2, Theorem 3.8] for $p < q$, yields $x_0 \in I_2$ and $x_1 \in I_1$ such that $A(I_2) = \|T_{x_0,I_2}\|_{q,I_2}$ and $A(I_1) = \|T_{x_1,I_1}\|_{q,I_1}$. Since $u, v > 0$ on $I_1$, it is then quite easy to see that $x_0 \in I_2^o$ and $x_1 \in I_1^o$.

If $x_0 = x_1$, then, since $u, v > 0$ on $I_1$, we get

$$A(I_1) = \|T_{x_1,I_1}\|_{q,I_1} > \|T_{x_1,I_1}\|_{q,I_2} = \|T_{x_1,I_2}\|_{q,I_2} = A(I_2).$$

On the other hand, if $x_0 \neq x_1$, then

$$A(I_1) = \|T_{x_1,I_1}\|_{q,I_1} \geq \|T_{x_1,I_1}\|_{q,I_2} \geq \|T_{x_1,I_2}\|_{q,I_2} > \|T_{x_0,I_2}\|_{q,I_2} = A(I_2).$$

The case $c < c' < d' = d$ could be proved similarly and the case $c < c' < d' < d$ follows from previous cases and the monotonicity of $A(I_1)$. □

Let $I = (a, b) \subset (0, d)$ and $I_i = (a_i, b_i) \subset I$, $i = 1, 2 \ldots, k$. Say that $\{I_i\}_{i=1}^k \in \mathcal{P}(I)$ if $\bigcup_{i=1}^k \overline{I_i} \supset I$ and assume the intervals $\{I_i\}_{i=1}^k$ to be non-overlapping.

Now, for any interval $I \subseteq (0, d)$ and $\varepsilon > 0$, we define the numbers $M$ and $N$, as follows:

$$M(I, \varepsilon) := \inf\{n : J(I_i) \leq \varepsilon, \{I_i\}_{i=1}^n \in \mathcal{P}(I)\}. \tag{20}$$

and

$$N(I, \varepsilon) := \inf\{n; A(I_i) \leq \varepsilon, \{I_i\}_{i=1}^n \in \mathcal{P}(I)\}. \tag{21}$$

Since by Proposition 1, $A(I) \leq \|T_I\| \leq K_2 J(I)$, we have

$$N(I, \varepsilon) \leq M(I, K_2\varepsilon). \tag{22}$$

Put $N(\varepsilon) = N((0, d), \varepsilon)$ and $M(\varepsilon) = M((0, d), \varepsilon)$. From Proposition 2.3 and the definition of $J(I)$ one gets the following:

**Remark 2.6** *Suppose that (13) and (14) are satisfied. Then $T : L^p(0, d) \to L^q(0, d)$ is compact if and only if $M(\varepsilon) < \infty$ for each $\varepsilon > 0$.*

**Lemma 2.7** *Let $T$ be a compact operator. Then*

$$\lim_{x\to 0_+} A(0, x) = 0 \text{ and } \lim_{x\to d_-} A(x, d) = 0.$$

**Lemma 2.8** *Suppose that $T$ is a compact operator, $\varepsilon > 0$ and $I = (a,b) \subset (0,d)$. Let $m = N(I,\varepsilon)$. Then there exists a sequence of non-overlapping intervals $\{I_i\}_{i=1}^m$ covering $I$, such that $A(I_i) = \varepsilon$ for $i \in \{2,\ldots,m-1\}$, $A(I_1) \le \varepsilon$, and $A(I_m) \le \varepsilon$.*

**Proof:** From Remark 2.7 and (22), one has $m < \infty$. Define a system $\mathcal{S} = \{I_j\}_{j\in\mathcal{J}}$, $I_j \subset I$, of intervals as follows: Set $b_1 = \inf\{x \in I; A(x,b) \le \varepsilon\}$. By Lemma 2.7 we have $a \le b_1 < b$. Put $I_1 = [b_1, b]$. Then $A(I_1) \le \varepsilon$. If $a = b_1$ write $\mathcal{S} = \{I_1\}$, otherwise set $b_2 = \inf\{x \in I; A(x,b_1) \le \varepsilon\}$ and $I_2 = [b_2, b_1]$. Observe that by Lemma 2.4 we have $A(I_2) = \varepsilon$. We can now proceed by mathematical induction to construct a (finite or infinite) system of intervals $\mathcal{S} = \{I_j\}_{j=1}^{\alpha}$. Note that we have only $A(I_\alpha) \le \varepsilon$ (not $A(I_\alpha) = \varepsilon$) provided $\alpha < \infty$ and $A(I_\beta) = \varepsilon$ for $\beta < \alpha$. Writing $b_0 = b$ we can set $I_j = [b_j, b_{j-1}]$, $1 \le j \le \alpha$.

Our next step is to show that $\alpha = m$. By the definition of $m$ one has $\alpha \ge m$ and a finite sequence of numbers $a = a_m < a_{m-1} < \ldots a_0 = b$ and intervals $J_i = [a_i, a_{i-1}]$, $i = 1, 2, \ldots, m$ such that $A(J_i) \le \varepsilon$. Notice that $b_1 \le a_1$, for if not, we can take $\lambda : 0 < \lambda < b_1$, which, from Lemma 2.4 and the definition of $I$, would yield $\varepsilon < A(\lambda, b_0) \le A(J_1) \le \varepsilon$, which is a contradiction. Assume now that for some $\alpha > 1, b_k > a_k$. If $b_{k-1} \le a_{k-1}$, then talking $a_k < \lambda < b_k$, Lemma 2.4 and the definition of $I_k$ yield $\varepsilon < A(\lambda, b_{k-1}) \le A(J_k) \le \varepsilon$, which is a contradiction, so that $a_{k-1} \le b_{k-1}$. Repeating this reasoning, one arrives at $b_1 > a_1$, which is again a contradiction. Thus, $b_k \le a_k$ for all $k = 1, 2, \ldots m$. Choosing $k = m$ we have $b_m = a$ and consequently, $\alpha = m$ and $\mathcal{S}$ covers $I$ which finishes the proof. □

For future reference (see the proof of (11) in the next section) we include the following lemmas and remarks.

Let $X$ be a Banach space and $M \subset X$. Recall the definition of the distance function $\mathrm{dist}(., M)$,

$$\mathrm{dist}(x, M) = \inf\{\|x - y\|; y \in M\}, \ x \in X.$$

**Lemma 2.9** *Let $T$ be a compact operator, $u, v > 0$, $\varepsilon > 0$, $I = (a,b) \subset (0,d)$ and $m = N(I,\varepsilon)$.*

*(i) Then there exists $0 < \varepsilon_1 < \varepsilon$ and a sequence of non-overlapping intervals $\{I_i\}_{i=1}^m$ covering $I$, such that $A(I_i) = \varepsilon_1$ for $i \in \{1,\ldots,m\}$.*

*(ii) There exists $\varepsilon_2 : 0 < \varepsilon_2 < \varepsilon$ such that $m + 1 = N(I, \varepsilon_2)$.*

**Proof:** The proof follows from the strict monotonity and the continuity of $A(I)$. □

**Lemma 2.10** *Let $H$ be an infinite dimensional separable Hilbert space. Let*

*$Y = \{u_1, \dots, u_{2n}\}$ be any orthonormal set with $2n$ vectors and let $X$ be any $m$-dimensional subspace of $H$ with $m \le n$. Then there exists an integer $j$, $1 \le j \le 2n$, such that*

$$\operatorname{dist}(u_j, X) \ge \frac{1}{\sqrt{2}}.$$

**Proof:** Denote the inner product in $H$ by $(u, v)$. Extend $Y$ to an orthonormal topological basis $\{u_i\}_{i=1}^{\infty}$ of $H$. Choose an orthonormal basis of $X$, say $v_1, \dots, v_m$. Denote by $P$ the orthogonal projection of $H$ into $X$. Then

$$Pu = \sum_{j=1}^{m} (u, v_j) v_j \qquad \text{for any } u \in H.$$

Since $P$ is a self-adjoint projection we obtain

$$\begin{aligned}
\sum_{k=1}^{2n} \|u_k - Pu_k\|^2 &= \sum_{k=1}^{2n} (1 - 2(u_k, Pu_k) + (Pu_k, Pu_k)) \\
&= 2n - \sum_{k=1}^{2n} (u_k, Pu_k) = 2n - \sum_{k=1}^{2n} \sum_{j=1}^{m} (u_k, v_j)^2 \\
&= 2n - \sum_{j=1}^{m} \sum_{k=1}^{2n} (u_k, v_j)^2.
\end{aligned}$$

The Parseval identity yields

$$\sum_{k=1}^{\infty} (u_k, v_j)^2 = \|v_j\|^2 = 1,$$

which implies

$$\sum_{k=1}^{2n} (u_k, v_j)^2 \le 1.$$

Consequently,

$$\sum_{k=1}^{2n} \|u_k - Pu_k\|^2 \ge 2n - m \ge n,$$

which guarantees the existence of an integer $j$, $1 \le j \le 2n$, with $\|u_j - Pu_j\|^2 \ge 1/2$. Then

$$\operatorname{dist}(u_j, X) = \|u_j - Pu_j\| \ge \frac{1}{\sqrt{2}},$$

which finishes the proof. □

**Lemma 2.11** *Let $1 \le p \le 2$ and $X$ be any $n$-dimensional subspace of $l_p$. Set $e_j \in l_p$, $e_j = \{\delta_{ij}\}_{i=1}^{\infty}$ where $\delta_{ij}$ is Kronecker's symbol. Then there exists an integer $j$, $1 \le j \le 2n$, such that*

$$\operatorname{dist}_p(e_j, X) \ge \frac{1}{\sqrt{2}}$$

.

**Proof:** Denote by $\|.\|_p$ the norm and by $\operatorname{dist}_p$ the distance function in $l_p$. Since $\|.\|_{l_2} \le \|.\|_{l_p}$ we can consider $X$ as an $n$-dimensional subspace of $l_2$. Thus, using the previous lemma there is $j$, $1 \le j \le 2n$ with $\operatorname{dist}_2(e_j, X) \ge \frac{1}{\sqrt{2}}$ from which immediately follows that

$$\operatorname{dist}_p(e_j, X) = \inf\{\|e_j - x\|_p; x \in X\} \ge \inf\{\|e_j - x\|_2; x \in X\} = \operatorname{dist}_2(e_j, X) \ge \frac{1}{\sqrt{2}}.$$

□

**Lemma 2.12** *Let $2 < p \le \infty$, $n \in \mathbb{N}$ and $X$ be any $n$-dimensional subspace of $l^p$. Set $e_j = \{\delta_{ij}\}_{i=1}^{\infty} \in l_p$ where $\delta_{ij}$ is the Kronecker's symbol. Then there is $j$, $1 \le j \le 2n$ such that*

$$\operatorname{dist}_p(e_j, X) \ge 2^{1/p-1} n^{1/p-1/2}. \tag{23}$$

**Proof:** Let $R : l^p \to l^p$ be the restriction operator given by

$$R(a) = (a_1, a_2, \ldots, a_{2n}, 0, 0, \ldots)$$

where $a = (a_1, a_2, \ldots) \in l^p$. Chose $u_i \in X$ such that $\operatorname{dist}_p(e_i, X) = \|e_i - u_i\|$. Using the well-known inequality

$$\|R(a)\|_2 \le (2n)^{1/2-1/p} \|R(a)\|_p \ \text{ for all } \ a \in l^p$$

it follows that for each $1 \le i \le 2n$,

$$\begin{aligned} \operatorname{dist}_p(e_i, X) &= \|e_i - u_i\|_p \ge \|R(e_i) - R(u_i)\|_p \\ &\ge (2n)^{1/2-1/p} \|R(e_i) - R(u_i)\|_2 \ge (2n)^{1/2-1/p} \operatorname{dist}_2(e_i, R(X)). \end{aligned}$$

Since $R(X)$ is a linear subspace of $l^2$, by Lemma 2.10 there exists $j$ with

$$\text{dist}_2(e_j, X) \geq \frac{1}{\sqrt{2}},$$

which finishes the proof of the lemma. □

It is shown in the appendix that the power of $n$ in (23) is the best possible if $2 < p \leq \infty$.

With the aid of the last lemmas we can use get a modified version Lemma 2.10 with $H$ replaced with $L^p(0, d)$ .

We start by recalling some lemmas referring to the properties of the map taking $x \in X$ to its nearest element $M_A(x) \in A \subset X$.

**Lemma 2.13** *Assume that $X$ is a strictly convex Banach space, $V \subset X$ be a finite dimensional subspace of $X$ and $x_0 \in X$. Set $A = \{x_0 + v; v \in V\}$ . Then for any $x \in X$ there exists a unique element $v$ such that*

$$\|x - v\| = \inf\{\|x - y\|; y \in A\}.$$

*Denote by $M_A$ the mapping which assigns to $x \in X$ the nearest element of $A$.*

**Lemma 2.14** *For any $\alpha \in \mathbb{R}$, $x \in X$ and $v \in V$, one has*

$$M_V(\alpha x) = \alpha M_V(x), \tag{24}$$

$$M_V(x + v) = M_V(x) + v \tag{25}$$

*and*

$$\|x - v\| \geq \frac{1}{2}\|M_V(x) - v\|. \tag{26}$$

The proof of these last two lemas can be found in [10].

Recall that $P : X \to X$ is called a projection if $P$ is linear, $P^2 = P$ and $\|P\| < \infty$.

**Lemma 2.15** *Let $X$ is a strictly convex Banach space and $V \subset X$ be a subspace, $\dim(V) = \sqrt{n}$ is finite. Then there exists a projection $P : X \to V$ which is onto such that $\|P\| \leq \sqrt{n}$.*

For proof see [11,III.B, Theorem 10].

The following lemma, whose proof is included for the sake of completions , plays a critical role in the sequel, since it provides an approximation to the map $M_A$ above by an linear operator of at most one dimensional range. The proof can also be found in [5].

**Lemma 2.16** *Let* $I \subset (0,d)$,$1 \le q \le \infty$ *and let* $\int_I |g(t)v(t)|^q dt < \infty$. *Set*

$$\omega_I(g) = \begin{cases} 0 & \text{if} \quad \int_I |v(t)|^q dt = 0 \\ (\int_I g(t)|v(t)|^q dt)/\int_I |v(t)|^q dt & \text{if} \quad 0 < \int_I |v(t)|^q dt < \infty \\ 0 & \text{if} \quad \int_I |v(t)|^q dt = \infty. \end{cases}$$

*Then*

$$\inf_{\alpha \in \Re} \|(g-\alpha)v\|_{q,I} \le \|(g-\omega_I(g))v\|_{q,I} \le 2 \inf_{\alpha\in\Re} \|(g-\alpha)v\|_{q,I} \tag{27}$$

**Proof:** It suffices to prove the second inequality. Fix $g$ such that $\int_I g(t)|v(t)|^q dt < \infty$.

Assume first that $\int_I |v(t)|^q dt = 0$. Then $v(t) = 0$ almost everywhere in $I$ and all members in (27) are equal zero.

Let $\int_I |v(t)|^q dt = \infty$. We claim that $\|\alpha v\|_{q,I} \le \|(\alpha - g)v\|_{q,I}$. If $\alpha = 0$ the inequality is clear. Let $\alpha \neq 0$, otherwise $\|\alpha v\|_{q,I} = \infty$ and by the triangle inequality, it follows that $\|(\alpha - g)v\|_{q,I} \ge \|\alpha v\|_{q,I} - \|gv\|_{q,I} = \infty$ and hence the claim. Thus, for each $\alpha \in \mathbb{R}$

$$\|(g-\omega_I(g))v\|_{q,I} = \|(g-\alpha+\alpha)v\|_{q,I} \le 2\|(g-\alpha)v\|_{q,I}$$

which gives

$$\|(g-\omega_I(g))v\|_{q,I} \le 2 \inf_{\alpha\in\Re} \|(g-\alpha)v\|_{q,I}.$$

Assume now $0 < \int_I |v(t)|^q dt < \infty$. By the Hölder's inequality, we obtain, for any $\alpha \in \Re$

$$\begin{aligned} \|(\alpha - w_I(g))v\|_{q,I}^q &= \int_I \left| \Big(\alpha - \frac{\int_I g(t)|v(t)|^q dt}{\int_I |v(t)|^q dt}\Big) v(x) \right|^q dx \\ &= \int_I |v(x)|^q \left| \Big( \frac{\int_I (\alpha - g(t))|v(t)|^q dt}{\int_I |v(t)|^q dt} \Big) \right|^q dx \\ &= \int_I \frac{|v(t)|^q}{(\int_I |v(t)|^q dt)^q} \left| \int_I (\alpha - g(t))|v(t)|^q dt \right|^q dx \end{aligned}$$

$$= \left(\int_I |v(t)|^q dx\right)^{1-q} \left|\int_I (\alpha - g(t))|v(t)||v(t)|^{q-1} dt\right|^q$$

$$\leq \left(\int_I |v(x)|^q dx\right)^{1-q} \int_I |(\alpha - g(t))v(t)|^q dt \left(\int_I |v(t)|^{q'(q-1)} dt\right)^{q/q'}$$

$$= \int_I |(\alpha - g(t))v(t)|^q dt = \|(\alpha - g)v\|_{q,I}^q$$

which proves $\|(\alpha - w_I(g))v\|_{q,I} \leq \|(\alpha - g)v\|_{q,I}$.

Now, using this inequality, for any real $\alpha$ one has:

$$\|(g - w_I(g))v\|_{q,I} \leq \|(g - \alpha)v\|_{q,I} + \|(\alpha - w_I(g))v\|_{q,I} \leq 2\|(\alpha - w_I(g))v\|.$$

The lemma follows by taking the infimum over $\alpha$ on the right hand side . □

**Lemma 2.17** *Let $X = L^p(0,d)$, $p > 1$. Let $v_1, v_2, \ldots, v_n$ be functions in $X$ with pairwise disjoint supports with $\|v_i\|_p = 1$ for $i = 1, 2, \ldots, n$. Set $V = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$. Then there is a projection $P_V$ with $\operatorname{rank} P_v \leq n$, such that*

$$\|f - M_V(f)\|_{p,(0,d)} \leq \|f - P_V(f)\|_{p,(0,d)} \leq 2\|f - M_V(f)\|_{p,(0,d)}$$

*where $M_V$ is defined on Lemma 2.13.*

**Proof:** Denote $S_i = \operatorname{supp} v_i$, $V_i = \operatorname{span}\{v_i\}$. Given any $f \in X$, with $\operatorname{supp} f \subset S_i$, let $M_i(f) = M_{v_i}(f)$. Put $P_i f = \omega_i(f\chi_{S_i})\chi_{S_i}$, and $Pf = \sum_{i=1}^n P_i(f\chi_{S_i})\chi_{S_i}$.

From the definition of $M_v$ and $P_v$ we have $\|f - M_V(f)\|_{p,(0,d)} \leq \|f - P_V(f)\|_{p,(0,d)}$, which is the first inequality. Also

$$\|f - M_V(f)\|_p^p = \sum_{i=1}^n \|f\chi_{S_i} + M_v(f)\chi_{S_i}\|_{p,S_i}^p \geq \sum_{i=1}^n \|f\chi_{S_i} - M_i(f\chi_{S_i})\chi_{S_i}\|_{p,S_i}^p$$

$$\leq 2^{-1/p} \sum_{i=1}^n \|f\chi_{S_i} - P_i(f\chi_{S_i})\chi_{S_i}\|_{p,S_i}^p = 2^{-1/p}\|f - \sum_{i=1}^n P_i(f\chi_{S_i})\chi_{S_i}\|_p^p$$

$$\leq 2^{-1/p}\|f - P(f)\chi_{S_i}\|_p^p,$$

which gives the second inequality and finishes the proof. □

**Lemma 2.18** *Let $1 < p \leq 2$ and let $u_1, ..., u_{2n}$ be a system of functions from $L^p(0,d)$ with disjoint supports. Let $X \subset L^p(0,d)$ be a subspace, $\dim X \leq n$.*

*Then there exists an integer $j$, $1 \leq j \leq 2n$, such that*

$$\mathrm{dist}_p(u_j, X) \geq \frac{1}{3\sqrt{2}}\|u_j\|_p.$$

**Proof:** If $\|u_i\|_p = 0$ for some $i$, it suffices to choose $j = i$. Let $\|u_i\|_p > 0$ for all $1 \leq i \leq 2n$. Set $v_i = \frac{u_i}{\|u_i\|_p}$. Let $V = \mathrm{span}\{v_1, v_2, \ldots, v_{2n}\}$ and let $P_V$ be the projection from the previous lemma. Let $Y = P_V(X)$. Then $Y \subset V$, $\dim Y \leq n$. Denote by $Z$ the subspace of $l^p$ consisting of all sequences $\{a_i\}_{i=1}^{\infty}$ such that $a_k = 0$ for all $k > 2n$. Let $e_i$ be the canonical basis of $Z$. Define a linear mapping $I : Y \to Z$ by

$$I(\sum_{i=1}^{2n} \alpha_i v_i) = \sum_{i=1}^{2n} \alpha_i e_i.$$

Since $\|v_i\| = 1$ and the functions $v_i$ have pairwise disjoint supports, it follows that $I$ is an isometry between $Y$ and $Z$. According to Lemma 2.11 there exists $1 \leq j \leq 2n$ such that

$$\mathrm{dist}_p(e_j, I(Y)) \geq \frac{1}{\sqrt{2}}, \tag{28}$$

and from Lemma 2.13 there is a unique $x \in X$ with

$$\mathrm{dist}_p(v_j, X) = \|v_j - x\|_p. \tag{29}$$

By the definition of $P_V$ and $M_V$, we have

$$\frac{1}{2}\|x - M(x)\|_p \leq \frac{1}{2}\|x - P_V(x)\|_p \leq \|x - M_V(x)\|_p \leq \|v_j - x\|_p$$

which yields, with the triangle inequality,

$$\|P_V(x)-v_j\|_p \leq \|P_V(x)-x\|_p+\|x-v_j\|_p \leq 2\|x-v_j\|_p \leq 2\|x-v_j\|_p+\|x-v_j\|_p \leq 3\|x-v_j\|_p.$$

This together with (28) and (29), gives

$$\begin{aligned} &\mathrm{dist}_p(v_j, X) = \|v_j - x\|_p \geq \frac{1}{3}\|v_j - P_V(x)\|_p \\ &\geq \frac{1}{3}\,\mathrm{dist}_p(v_j, Y) = \frac{1}{3}\,\mathrm{dist}_p(e_j, I(Y)) \geq \frac{1}{3\sqrt{2}}. \end{aligned}$$

Denoting by $M_1$ the mapping which assigns to any $f \in L^p(0, d)$ the element of $X$ nearest to $f$ and using (24) we can rewrite the previous inequality as

$$\begin{aligned}\mathrm{dist}_p(u_j, X) &= \|u_j - M_1(u_j)\|_p = \|u_j\|_p \ \|v_j - M_1(v_j)\|_p \\ &= \|u_j\|_p \ \mathrm{dist}_p(v_j, X) \geq \frac{1}{3\sqrt{2}}\|u_j\|_p\end{aligned}$$

which yields the claim. □

**Lemma 2.19** *Let $2 < p \leq \infty$ and let $u_1, ..., u_{2n}$ be a system of functions from $L^p(0, d)$ with disjoint supports. Let $X \subset L^p(0, d)$ be a subspace,* $\dim X \leq n$. *Then there exists an integer $j$, $1 \leq j \leq 2n$, such that*

$$\mathrm{dist}_p(u_j, X) \geq \frac{1}{2\sqrt{2}}\|u_j\|_p n^{1/p-1/2}.$$

**Proof:** Let $V, M_V, P_V, Y, Z$ and $I$ have the same meanins as in Lemma 2.18. Proceeding as before, Lemma 2.12 yields $j: \quad 1 \leq j \leq 2n$ such that

$$\mathrm{dist}_p(e_j, I(Y)) \geq \frac{1}{2}n^{1/p-1/2}.$$

Let $x \in X$ be the element given by Lemma 2.13 so that

$$\mathrm{dist}(v_j, X) = \|v_j - x\|_p.$$

In exactly the same way as in Lemma 2.18, one gets

$$\mathrm{dist}_p(v_j, X) \geq \frac{1}{3}n^{1/p-1/2},$$

which can be written as

$$\mathrm{dist}_p(u_j, X) \geq \frac{1}{3}\|u_j\|_p n^{1/p-1/2},$$

and the proof is complete. □

## 3 Bounds for the approximation numbers

We recall that, given any $m \in \mathbf{N}$, the $m^{th}$ approximation number $a_M(S)$ of a bounded operator $S$ from $L^p$ into $L^q$, is defined by

$$a_m(S) := \inf_F \|S - F\|_{p \to q},$$

where the infimum is taken over all bounded linear maps $F : L^p(0,d) \to L^q(0,d)$ with rank less than $m$. Futher discussions on approximation numbers may be found in [3]. An operator $S$ is compact if and only if $a_m(S) \to 0$ as $m \to \infty$. The first two lemmas of this section provide estimates for $a_m(T)$ for $T$ as in (1), which are the analogous of those obtained in [1] and [5]. Hereafter, we shall always assume (13) and (14).

**Lemma 3.1** *Let $1 \le p \le q \le \infty$ and suppose that $T : L^p(0,d) \to L^q(0,d)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbf{N}$ and numbers $c_k, k = 0,1,\ldots,N$, with $0 = c_0 < c_1 < \ldots < c_N = d$, such that $A(I_k) \le \varepsilon$ for $k = 0,1,\ldots,N-1$, where $I_k = (c_k, c_{k+1})$. Then $a_{N+1}(T) \le 2\varepsilon$.*

**Proof:** Consider for $f \in L^p(a,b)$ and $0 \le k \le N-1$ one-dimensional linear operators given by

$$P_{I_k} f(x) := \chi_{I_k}(x) v(x) \left( \int_{c_k}^{x} uf dt + \omega_{I_k} \left( \int_{c_k}^{x} uf dt \right) \right).$$

where $\omega_{I_k}$ is the functional from Lemma 2.16. We claim that $P_k$ is bounded from $L^p(0,d)$ into $L^q(0,d)$ for each $k$.

Assume first that either $0 = \|v\|_{q,I_k}$ or $\|v\|_{q,I_k} = \infty$. Then $P_k = 0$ and consequently, it is bounded.

Assume now $0 < \|v\|_{q,I_k} < \infty$ and fix $f, \|f\|_{p,(0,d)} = 1$. Then using Hölder's inequality, we obtain

$$\begin{aligned}
\left| \omega_{I_k} \left( \int_{c_k}^{x} u(t) f(t) dt \right) \right| &= \left| \frac{\int_{I_k} \int_{c_k}^{x} u(t) f(t) dt |v(x)|^q dx}{\int_{I_k} |v(x)|^q dx} \right| \\
&\le \frac{\int_{I_k} |v(x) \int_{c_k}^{x} u(t) f(t) dt| \, |v(x)|^{q-1} dx}{\int_{I_k} |v(x)|^q dx} \\
&\le \frac{\left( \int_{I_k} |v(x) \int_{c_k}^{x} u(t) f(t) dt|^q dx \right)^{1/q} \left( \int_{I_k} |v(x)|^{(q-1)q'} dx \right)^{1/q'}}{\int_{I_k} |v(x)|^q dx} \\
&\le \frac{\|T_{I_k} f\|_q}{\|v\|_{q,I_k}} \le \frac{\|T\|}{\|v\|_{q,I_k}}
\end{aligned}$$

and consequently,

$$\int_0^d |(P_k f)(x)|^q dx = \int_{I_k} \left| v(x) \left( \int_{c_k}^{x} uf dt + \omega_{I_k} \left( \int_{c_k}^{x} uf dt \right) \right) \right|^q dx$$

$$\leq 2^{q-1}\left(\int_{I_k}\left|v(x)\int_{c_k}^{x} ufdt\right|^q + \omega_{I_k}^q\left(\int_{c_k}^{x} ufdtdx\right)\right)$$

$$\leq 2^{q-1}(\|T_k f\|_q + \frac{\|T\|}{\|v\|_{q,I_k}}) \leq \|T\|(1 + \frac{1}{\|v\|_{q,I_k}}).$$

Set $P = \sum_{k=0}^{N-1} P_k$. Then $P$ is a linear bounded operator from $L^p(0,d)$ into $L^q(0,d)$. Moreover, we have by Lemma 2.16 and the well-known inequality $(\sum_{k=1}^{\infty}|a_k|^q)^{1/q} \leq (\sum_{k=1}^{\infty}|a_k|^p)^{1/p}$

$$\|Tf - Pf\|_q^q = \sum_{k=0}^{N-1}\|Tf - P_{I_k}f\|_{q,I_k}^q$$

$$= \sum_{k=0}^{N-1}\|v(x)\left[\int_{c_k}^{x} ufdt - \omega_{I_k}\left(\int_{c_k}^{x} ufdt\right)\right]\|_{q,I_k,\mu}^q$$

$$\leq 2^{q-1}\sum_{k=0}^{N-1}\inf_{\alpha\in\Re}\|T_{I_k}f - \alpha f\|_{q,I_k}^q \leq 2^q\sum_{k=0}^{N-1}A^q(I_k)\|f\|_{p,I_k}^q$$

$$\leq (2\varepsilon)^q\sum_{k=0}^{N-1}\|f\|_{p,I_k}^q \leq (2\varepsilon)^q\left(\sum_{k=0}^{N-1}\|f\|_{p,I_k}^p\right)^{q/p} \leq (2\varepsilon)^q$$

by Lemma 2.5. Since $\operatorname{rank} P \leq N$, the proof of the lemma is complete . □

**Lemma 3.2** *Let $1 < p \leq q < \infty$, $T$ be bounded from $L^p(0,d)$ to $L^q(0,d)$, $0 \leq a < b < c < d$ and denote $I = [a,b]$, and $J = [b,c]$. Further, let $f, g \in L^p(0,d)$ with $\operatorname{supp} f \subset I$, $\operatorname{supp} g \subset J$, $\|f\|_p = \|g\|_p = 1$.*

*Let $r$, $s$ be real numbers and set*

$$h(x) = v(x)\int_0^d u(t)(rf(t) + sg(t))dt.$$

*Assume $\int_a^c u(t)h(x) = 0$. Then*

$$\|h\|_q \geq (|r|^q\inf_{\alpha\in\Re}\|T_I f - \alpha v\|^q + |s|^q\inf_{\alpha\in\Re}\|T_J g - \alpha v\|^q)^{1/q}.$$

**Proof:** Since $\operatorname{supp} f \subset I$ and $\operatorname{supp} g \subset J$ we have

$$\int_0^a\left|v(x)\int_0^x u(t)(rf(t) + sg(t))dt\right|^q dx = 0. \tag{30}$$

If $x \in (c,d)$ we have (recall that $\int_a^c u(t)h(x) = 0$) that

$$\int_c^d \left| v(x) \int_0^x u(t)(rf(t) + sg(t))dt \right|^q dx = \int_c^d \left| v(x) \int_a^c u(t)h(t)dt \right|^q dx = 0. \quad (31)$$

Assume first $s \neq 0$. Then it follows from (30) and (31) that

$$\begin{aligned}
\|h\|_q^q &= \int_0^d \left| v(x) \int_0^x u(t)(rf(t) + sg(t))dt \right|^q dx = \int_0^a + \int_a^b + \int_b^c + \int_c^d \\
&= \int_I \left| v(x) \int_0^x u(t)(rf(t) + sg(t))dt \right|^q dx \\
&\quad + \int_J \left| v(x) \int_0^x u(t)(rf(t) + sg(t))dt \right|^q dx \\
&= \int_I \left| v(x) \int_0^x u(t)rf(t)dt \right|^q dx \\
&\quad + \int_J \left| v(x) \left( \int_0^b u(t)rf(t)dt + \int_b^x u(t)sg(t))dt \right) \right|^q dx \\
&= |r|^q \int_I \left| v(x) \int_a^x u(t)f(t)dt \right|^q dx \\
&\quad + |s|^q \int_J \left| v(x) \left( \int_I u(t)\frac{r}{s}f(t)dt + \int_b^x u(t)g(t))dt \right) \right|^q dx \\
&\geq |r|^q \inf_{\alpha \in \Re} \int_I \left| v(x) \left( \int_a^x u(t)f(t)dt - \alpha \right) \right|^q dx \\
&\quad + |s|^q \inf_{\alpha \in \Re} \int_J \left| v(x) \left( \int_b^x u(t)g(t)dt - \alpha \right) \right|^q dx \\
&= |r|^q \inf_{\alpha \in \Re} \|T_I f - \alpha v\|_{q,I}^q + |s|^q \inf_{\alpha \in \Re} \|T_J g - \alpha v\|_{q,J}^q.
\end{aligned}$$

Assume now $s = 0$. Then

$$\|h\|_q^q = \int_0^d \left| v(x) \int_0^x u(t)rf(t)dt \right|^q dx$$

$$= |r|^q \int_I \left| v(x) \int_a^x u(t)f(t)dt \right|^q dx$$

$$\geq |r|^q \inf_{\alpha\in\Re} \int_I \left| v(x) \left( \int_a^x u(t)f(t)dt - \alpha \right) \right|^q dx =$$

$$|r|^q \inf_{\alpha\in\Re} \|T_I f - \alpha v\|_{q,I}^q$$

which finishes the proof of the lemma. □

**Lemma 3.3** *Let* $1 < p \leq q \leq 2$, *$T$ be bounded from* $L^p(0,d)$ *to* $L^q(0,d)$, $\varepsilon > 0$, $N \in \mathbb{N}$ *and* $0 \leq d_0 < d_1 < \ldots < d_{4N} < d$. *Set* $I_k = (d_k, d_{k+1})$ *and assume that* $A(I_k) \geq \varepsilon$ *for* $k = 0, 1, \ldots, 4N-1$. *Then* $a_N(T) \geq 2^{1/q-1/p-3/2}\varepsilon$.

**Proof:** Let $0 < \gamma < 1$. Then there exist functions $f_k \in L^p(I_k)$ such that $\|f_k\|_{p,I_k} = 1$ and

$$\inf_{\alpha\in\Re} \|Tf_k - \alpha v\|_{q,I_k} \geq \gamma A(I_k) \geq \gamma\varepsilon. \tag{32}$$

By definition of the approximation numbers, there is a bounded linear mapping with rank $P \leq N$ such that

$$a_{N+1}(T) \geq \gamma \|T - P\|_{p\to q}.$$

Then $P = \sum_{i=1}^N P_i$, where $P_i$ are one-dimensional operators from $L^p(0,d)$ into $L^q(0,d)$. Thus, we can write $(P_i f)(x) = \phi_i(x) R_i(f)$ where $\phi_i \in L^q(0,d)$ and $R_i \in (L^p(0,d))^*$. Since $(L^p(0,d))^* = L^{p'}(0,d)$, it follows that $R_i f(x) = \int_0^d \psi_i(t)f(t)dt$ and that there are functions $\psi_i \in L^{p'}(0,d)$ such that

$$(Pf)(x) = \sum_{i=1}^N \phi_i(x) \int_0^d \psi_i(t)f(t)dt.$$

Denote by $X$ the range of $P$. Notice that $\dim(X) \leq N$.

Define $J_i := I_{2i} \cup I_{2i+1}$ for $i = 0, 1, \ldots, 2N-1$. For each $i \in \{0, 1, \ldots, 2N-1\}$. Let $(r_i, s_i)$ be orthogonal to the 2-dim vector. So that

$$|r_i|^p + |s_i|^p > 0 \text{ and } r_i \int_{I_{2i}} uf_{2i} + s_i \int_{I_{2i+1}} uf_{2i+1} = 0. \tag{33}$$

Set $g_i(t) = r_i f_{2i} + s_i f_{2i+1}$ and $h_i(x) = v(x)\int_0^x u(t)g_i(t)dt$. From $\|f_i\| = 1$ for each $i$: $0 \le i \le 2N-1$ and (3.2), one has

$$\|g_i\|_p = \left(|r_i|^p \int\limits_{I_{2i}} |f_{2i}(t)|^p dt + |s_i|^p \int\limits_{I_{2i+1}} |f_{2i+1}(t)|^p dt\right)^{1/p} = (|r_i|^p + |s_i|^p)^{1/p}.$$

Consequently, $\|h_i\|_q = \|Tg_i\|_q < \infty$. Moreover, $\int_0^d h_i(t)dt = \int_{J_i} h_i(t)dt = 0$ whence

$$\operatorname{supp} h_i \subset J_i \text{ for all } i = 0, 1, \ldots, 2N-1.$$

Thus, using Lemma 2.18 one finds that there exists an integer $k$, $0 \le k \le 2N-1$, such that

$$\operatorname{dist}_q(h_k, X) \ge \frac{1}{2\sqrt{2}}\|h_k\|_q,$$

from which it follows that

$$\begin{aligned} a_{N+1}(T) &\ge \gamma\|T-P\|_{p\to q} \\ &\ge \sup_{f\in L^p, \operatorname{supp} f\subset J_k} \frac{\gamma\|Tf - Pf\|_q}{\|f\|_p} \\ &\ge \frac{\gamma\|Tg_k - Pg_k\|_q}{\|g_k\|_p} = \frac{\gamma\|h_k - Pg_k\|_q}{\|g_k\|_p} \\ &\ge \gamma\frac{\operatorname{dist}_q(h_k, X)}{\|g_k\|_p} \ge \frac{\gamma}{2\sqrt{2}}\frac{\|h_k\|_q}{\|g_k\|_p}. \end{aligned}$$

Using Lemma 3.2, (34) and the inequality

$$(|r_k|^p + |s_k|^p)^{1/p} \le 2^{1/p-1/q}(|r_k|^p + |s_k|^p)^{1/p}$$

we obtain

$$\begin{aligned} \frac{\|h_k\|_q}{\|g_k\|_p} &\ge \frac{(|r_k|^q \inf_{\alpha\in\Re}\|T_{I_{2k}}f - \alpha v\|_q^q + |s_k|^q \inf_{\alpha\in\Re}\|T_{I_{2k+1}} - \alpha v\|_q^q)^{1/q}}{(|r_k|^p + |s_k|^p)^{1/p}} \\ &\ge \gamma\varepsilon\frac{(|r_k|^q + |s_k|^q)^{1/q}}{(|r_k|^p + |s_k|^p)^{1/p}} \ge \gamma\ \varepsilon\ 2^{1/q-1/p} \end{aligned}$$

which together with the previous estimate gives

$$a_{N+1}(T) \ge \gamma^2\ 2^{1/q-1/p-3/2}.$$

The proof is complete.□

Using the properties of approximation numbers on dual operators we can extend the previous result

**Lemma 3.4** *Let $2 \le p \le q \le \infty$ and suppose that $T : L^p(0,d) \to L^q(0,d)$ is bounded. Let $\varepsilon > 0$ and suppose that there exist $N \in \mathbf{N}$ and numbers $d_k, k = 0,1,\ldots,4N$ with $0 = d_0 < d_1 < \ldots < d_{4N} < d$ such that $A(I_k) \ge \varepsilon$ for $k = 0,1,\ldots,N-1$, where $I_k = (d_k, d_{k+1})$. Then $a_N(T) \ge c\varepsilon$ where $c$ is positive and depends only on $p, d$.*

**Proof:** The adjoint of $T$, $T'$, is bounded from $L^{q'}$ into $L^{p'}$. It is easy to see that Lemma 3.2 holds for $T$ replaced by $T'$. Then the proof follows immediately from Proposition 2.5 and Remark 2.6 in [3]. □

**Lemma 3.5** *Let $1 \le p \le 2 \le q \le \infty$ and suppose that $T : L^p(0,d) \to L^q(0,d)$ is bounded. Let $\varepsilon > 0$ and suppose that there exists $N \in \mathbf{N}$ and numbers $d_k, k = 0,1,\ldots,4N$ with $0 = d_0 < d_1 < \ldots < d_{4N} < d$ such that $A(I_k) \ge \varepsilon$ for $k = 0,1,\ldots,N-1$, where $I_k = (d_k, d_{k+1})$. Then $a_N(T) \ge c\varepsilon n^{1/q-1/2}$ where $c$ is positive and depends only on $p, d$.*

**Proof:** Let $0 < \gamma < 1$. Then there exist functions $f_k \in L^p(I_k)$ such that $\|f_k\|_{p,I_k} = 1$ and

$$\inf_{\alpha\in\Re} \|Tf_k - \alpha v\|_{q,I_k} \ge \gamma A(I_k) \ge \gamma\varepsilon. \tag{34}$$

By definition of the approximation numbers there is a bounded linear mapping with rank $P \le N$ such that

$$a_{N+1}(T) \ge \gamma\|T - P\|_{p\to q}.$$

Write $P = \sum_{i=1}^N P_i$ and let $J_i$ be as in the proof of Lemma 3.3. In the notation of Lemma 3.3, in this case we also have $\|h_i\|_q = \|Tg_i\|_q < \infty$ and $\int_0^d h_i(t)dt = \int_{J_i} h_(t)dt$, so that

$$\operatorname{supp} h_i \subset J_i \text{ for all } i = 0,1,\ldots,2N-1,$$

whence, by Lemma 2.18, there exists an integer $k$, $0 \le k \le 2N-1$, such that

$$\operatorname{dist}_q(h_k, X) \ge \frac{1}{3\sqrt{2}} n^{1/q-1/2}\|h_k\|_q,$$

which gives

$$a_{N+1}(T) \qquad\qquad \ge \gamma\|T - P\|_{p\to q}$$

$$\geq \sup_{f\in L^p, \operatorname{supp} f\subset J_k} \frac{\gamma\|Tf-Pf\|_q}{\|f\|_p}$$
$$\geq \frac{\gamma\|Tg_k-Pg_k\|_q}{\|g_k\|_p} = \frac{\gamma\|h_k-Pg_k\|_q}{\|g_k\|_p}$$
$$\geq \gamma\frac{\operatorname{dist}_q(h_k,X)}{\|g_k\|_p} \geq \frac{\gamma}{3\sqrt{2}}\frac{\|h_k\|_q}{\|g_k\|_p}n^{1/q-1/2}.$$

Using Lemma 3.2, (34) and the inequality

$$(|r_k|^p+|s_k|^p)^{1/p} \leq 2^{1/p-1/q}(|r_k|^p+|s_k|^p)^{1/p}$$

we obtain

$$\frac{\|h_k\|_q}{\|g_k\|_p} \geq \frac{(|r_k|^q \inf_{\alpha\in\Re}\|T_{I_{2k}}f-\alpha v\|_q^q + |s_k|^q \inf_{\alpha\in\Re}\|T_{I_{2k+1}}-\alpha v\|_q^q)^{1/q}}{(|r_k|^p+|s_k|^p)^{1/p}}$$
$$\geq \gamma\varepsilon\frac{(|r_k|^q+|s_k|^q)^{1/q}}{(|r_k|^p+|s_k|^p)^{1/p}} \geq \gamma\ \varepsilon\ 2^{1/q-1/p}$$

which gives with the previous estimate

$$a_{N+1}(T) \geq \gamma^2\ c\varepsilon n^{1/q-1/2}$$

for fixed $c>0$ and finishes the proof.□

The following theorem follows immediately from the previous lemmas. It improves results from [1] and [5].

**Theorem 3.6** *Suppose that $T$ is compact (see Proposition 2.2 and Remark 2.3). Then, for small $\varepsilon>0$, $1\leq p\leq q\leq\infty$*

$$a_{N(\varepsilon)+1}(T) \leq 2\varepsilon,$$

*for $1\leq p\leq q\leq 2$ or $2\leq p\leq q\leq\infty$*

$$a_{[\frac{N(\varepsilon)}{4}]-1}(T) > c\varepsilon,$$

*and for $1\leq p\leq 2\leq q\leq\infty$*

$$a_{[\frac{N(\varepsilon)}{4}]-1}(T) > c\varepsilon N(\varepsilon)^{1/q-1/2}.$$

*Here $N(\varepsilon)\equiv N((0,d),\varepsilon)$ is defined in (21) and $[x]$ denotes the integer part of $x$.*

**Proof:** The first inequality is an immediate consequence of Lemma 3.1 and definition of $N(\varepsilon)$. The second inequality follows from Lemmas 2.4, 3.1 and 3.2. □

## 4 Local asymptotic result

The first part of this section is devoted to proving lemmas that will be needed in the proof of our local asymptotic results, which we present in the second part.

**Lemma 4.1** *Let $u$ and $v$ be constant functions on the interval $I = (a, b) \subset (0, d)$ and let $1 \le p \le q \le \infty$. Then $A(I) := A(I, u, v) = |u||v||I|^{1/p'+1/q} A((0, 1), 1, 1)$.*

**Proof:** If $u = 0$ then $A(I, u, v) = 0$ and the assertion is trivial. Assume that $u \neq 0$. Using the substitutions $y = \frac{x-a}{b-a}$ and $t = a + s(b - a)$, we obtain

$$A(I,u,v) = \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \|v \left( \int_a^x u\ f(t)dt - \alpha \right) \|_{q,I}$$

$$= |v||u| \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\Re} \| \int_a^x f(t)dt - \alpha\|_{q,I}$$

$$= \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} (b-a)^{1-1/q} \| \int_0^y f(a + s(b-a))ds - \alpha\|_{q,(0,1)}.$$

Writing $g(s) = f(a + s(b - a))$ we have $\|g\|_{p,(0,1)} = (b - a)^{-1/p}\|f\|_{p,(a,b)}$ and thus

$$A(I,u,v) = |v||u||I|^{1+1/q} \sup_{\|g\|_{p,(0,1)}=(b-a)^{-1/p}} \| \int_a^x g(t)dt - \alpha\|_{q,(0,1)}$$

$$= |v||u||I|^{1/p'+1/q} \sup_{\|g\|_{p,(0,1)}=1} \| \int_a^x g(t)dt - \alpha\|_{q,(0,1)}$$

$$= |v||u||I|^{1/p'+1/q} A((0,1),1,1).$$

The proof is complete. □

**Lemma 4.2** *Let $I = (a, b) \subset (0, d)$, $1 \le p \le q \le \infty$, $u_1, u_2 \in L^{p'}(I)$ and $v \in L^q(I)$. Then*

$$|A(I, u_1, v) - A(I, u_2, v)| \le \|v\|_{q,I}\|u_1 - u_2\|_{p',I}$$

**Proof:** Suppose first that $A(I, u_1, v) \geq A(I, u_2, v)$. Then

$$
\begin{aligned}
&A(I,u_1,v) - A(I,u_2,v) = \\
&\quad = \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \|v(x)\left(\int_a^x (u_1(t)-u_2(t)+u_2(t))f(t)dt-\alpha\right)\|_{q,I} - A(I,u_2,v) \\
&\quad \leq \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \left[\|v(x)\int_a^x (u_1(t)-u_2(t))f(t)dt\|_{q,I}\right. \\
&\qquad \left. + \|v(x)\left(\int_a^x u_2(t)f(t)dt-\alpha\right)\|_{q,I}\right] - A(I,u_2,v) \\
&\quad \leq \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \left[\|v\|_{q,I}\|u_1-u_2\|_{p',I} + \|v(x)\left(\int_a^x u_2(t)f(t)dt-\alpha\right)\|_{q,I}\right] \\
&\qquad -A(I,u_2,v) \\
&\quad \leq \|v\|_{q,I}\|u_1-u_2\|_{p',I} + A(I,u_2,v) - A(I,u_2,v).
\end{aligned}
$$

The remaining case can be proved analogously. □

**Lemma 4.3** *Let $I = (a,b) \subset (0,d)$, $1 \leq p \leq q \leq \infty$, $u \in L^{p'}(I)$, and $v_1, v_2 \in L^q(I)$. Then*

$$|A(I,u,v_1) - A(I,u,v_2)| \leq 3\|v_1-v_2\|_{q,I}\|u\|_{p',I}$$

**Proof:** If $A(I,u,v_1) \geq A(I,u,v_2)$ then by Lemma 2.3 we have

$$
\begin{aligned}
&A(I,u,v_1) - A(I,u,v_2) = \\
&= \sup_{\|f\|_{p,I}=1} \inf_{\alpha\in\Re} \|v_1(x)\left[\int_a^x u(t)f(t)dt-\alpha\right]\|_{q,I} - A(I,u,v_2) \\
&= \sup_{\|f\|_{p,I}=1} \inf_{|\alpha|\leq 2\|u\|_{p',I}} \|v_1(x)\left[\int_a^x u(t)f(t)dt-\alpha\right]\|_{q,I} - A(I,u,v_2) \\
&\leq \sup_{\|f\|_{p,I}=1} \inf_{|\alpha|\leq 2\|u\|_{p',I}} \left[\|(v_1(x)-v_2(x))\left(\int_a^x u(t)f(t)dt-\alpha\right)\|_{q,I}\right. \\
&\qquad \left. +\|v_2(x)\left(\int_a^x u(t)f(t)dt-\alpha\right)\|_{q,I}\right] - A(I,u,v_2) \\
&\leq \sup_{\|f\|_{p,I}=1} \inf_{|\alpha|\leq 2\|u\|_{p',I}} \left[\|(v_1(x)-v_2(x))\|_{q,I}\|u\|_{p',I}\|f\|_{p,I} + \|(v_1-v_2)\alpha\|_{q,I}\right. \\
&\qquad \left. +\|v_2\left(\int_a^x u(t)f(t)dt-\alpha\right)\|_{q,I}\right] - A(I,u,v_2)
\end{aligned}
$$

$$
\begin{aligned}
&\le 3\|v_1 - v_2\|_{q,I}\|u\|_{p',I} \\
&+ \sup_{\|f\|_{p,I}=1} \inf_{|\alpha|\le\|u\|_{p',I}} \|v_2(x)\left[\int_a^x u(t)f(t)dt - \alpha\right]\|_{q,I} - A(I,u,v_2) \\
&= 3\|v_1 - v_2\|_{q,I}\|u\|_{p',I}.
\end{aligned}
$$

□

Now we prove a local asymptotic result which in some sense extends those in [2] and [5]:

**Lemma 4.4** *Let* $I = (a,b) \subset (0,d)$, $|I| < \infty$ *and* $1 < p \le q \le \infty$. *Assume that* $u \in L^{p'}(I)$ *and* $v \in L^q(I)$. *Set* $r = \frac{p'q}{p'+q}$. *Then*

$$
c_1\alpha_{p,q}\int_I |uv|^r \le \liminf_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon, I) \le \limsup_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon, I) \le c_2\alpha_{p,q}\int_I |uv|^r,
$$

*where* $\alpha_{p,q} = A((0,1),1,1)$.

**Proof:** Set $s = \frac{p'}{q} + 1$. Clearly,

$$
rs = p', \; rs' = q. \tag{35}
$$

Let $l \in \mathbf{N}$ be fixed. Then by the absolute convergence of the Lebesque integral and the Luzin Theorem there exists $m := m(l) \in \mathbf{N}$, $\{W_j\}_{j=1}^m \in \mathcal{P}$ and real numbers $\xi_j$, $\eta_j$ such that setting

$$
u_l = \sum_{j=1}^m \xi_j \chi_{W_j}, \qquad v_l = \sum_{j=1}^m \eta_j \chi_{W_j},
$$

we have

$$
\|u - u_l\|_{p',I} < 1/l, \qquad \|v - v_l\|_{q,I} < 1/l.
$$

and

$$
\| \, |u|^r - |u_l|^r\|_{s,I} < 1/l, \qquad \| \, |v|^r - |v_l|^r\|_{s',I} < 1/l.
$$

Consequently,

$$\left|\int_I |u|^r|v|^r - \int_I |u_l|^r|v_l|^r\right|$$
$$\leq \int_I |u|^r \left| |v_l|^r - |v|^r\right| + \int_I |v_l|^r \left| |u_l|^r - |u|^r\right|$$
$$\leq (\|u\|_{p',I} \left\| |v_l|^r - |v|^r\right\|_{s',I} + \left\| |u_l|^r - |u|^r\right\|_{s,I} \| |v_l|^r\|_{q,I})$$
$$\leq \frac{1}{l}(\|u\|_{p'} + \|v_l\|_q) \leq \frac{1}{l}(\|u\|_{p'} + \|v - v_l\|_q + \|v_l\|_q)$$
$$\leq \frac{1}{l}(\frac{1}{l} + \|u\|_{p',I} + \|v\|_q).$$

Let $\varepsilon > 0$. Put $N(\varepsilon) = N(\varepsilon, I)$. According to Lemma 2.8 there is a system of intervals $\{I_j\}_{j=1}^{N(\varepsilon)} \in \mathcal{P}$ such that

$$A(I_1) \leq \varepsilon,\ A(I_{N(\varepsilon)}) \leq \varepsilon \ \text{ and } \ A(I_i) = \varepsilon \ \text{ for } \ 2 \leq i \leq N(\varepsilon).$$

Define,

$$J_i = I_{2i} \cup I_{2i+1},\ i = 1, 2, \ldots, N(\varepsilon)/2, \text{ for even } N(\varepsilon)$$

and

$$J_i = I_{2i} \cup I_{2i+1},\ i = 1, 2, \ldots, (N(\varepsilon) - 3)/2,$$
$$J_{(N(\varepsilon)-1)/2} = J_{N(\varepsilon)-2} \cup J_{N(\varepsilon)-1} \cup J_{N(\varepsilon)} \text{ for odd } N(\varepsilon).$$

In both cases $\{J_i\}_{j=1}^{[\frac{N(\varepsilon)}{2}]} \in \mathcal{P}$ and according to the definition of $N(\varepsilon)$, $A(J_i) > \varepsilon$ for all $1 \leq i \leq [\frac{N(\varepsilon)}{2}]$. Let $W_i = [d_{i-1}, d_i]$, where $a = d_0 < d_1 < d_2 < \ldots < d_m = b$. Set

$$\mathcal{K} = \{J_i; 1 \leq i \leq [\frac{n(\varepsilon)}{2}] \text{ and there exists } j \in \{1, 2, \ldots, m\} \text{ such that } J_i \subset W_j\}.$$

If $J_i \notin \mathcal{K}$, there exists $k \in \{1, 2, \ldots, m-1\}$ such that $d_k \in \operatorname{int}(J_i)$. The number of such intervals $J_i$ can be estimate by $m - 1$. Then $\#\mathcal{K} \geq [\frac{N(\varepsilon)}{2}] - m + 1$. Using Lemmas 4.1, 4.2 and 4.3 one sees that

$$([N(\varepsilon)/2] - m - 1)\,\varepsilon^r \leq \sum_{k\in\mathbf{K}} A^r(I_k; u, v)$$
$$\leq \sum_{k\in\mathbf{K}} [A(I_k; u_l, v_l) + (A(I_k; u, v) - A(I_k; u_l, v)) + (A(I_k; u_l, v) - A(I_k; u_l, v_l))]^r$$
$$\leq \max(1, 3^{r-1}) \sum_{k\in\mathbf{K}} \Big(A^r(I_k; u_l, v_l) + |A(I_k; u, v) - A(I_k; u_l, v)|^r$$
$$+ |A(I_k; u_l, v) - A(I_k; u_l, v_l)|^r\Big)$$
$$\leq \max(1, 3^{r-1}) \Big[\alpha^r_{p,q} \sum_{j=1}^m |\xi_j|^r |\eta_j|^r |W(j)| + \sum_j^m \|u - u_l\|^r_{p',W(j)} \|v\|^r_{q,W(j)}$$

$$+\sum_{j=1}^{m}\|v-v_l\|^r_{q,W(j)}\|u\|^r_{p',W(j)}\Big].$$

Using the discrete version of Hölder's inequality

$$\sum_{i=1}^{m}a_ib_i\le(\sum_{i=1}^{m}a_i^s)^{1/s}(\sum_{i=1}^{m}b_i^{s'})^{1/s'}$$

and (35) we obtain

$$\begin{aligned}
([N(\varepsilon)/2]-m+1)\varepsilon^r &\le \max(1,3^{r-1})\Bigg(\alpha^r_{p,q}\sum_{j=1}^{m}|\xi_j|^r\ |\eta_j|^r\ |W_j| \\
&+\Big(\sum_{j=1}^{m}\|u-u_l\|^{p'}_{p',W_j}\Big)^{1/s}\Big(\sum_{j=1}^{m}\|v\|^q_{q,W_j}\Big)^{1/s'} \\
&+\Big(\sum_{j=1}^{m}\|v-v_l\|^q_{q,W_j}\Big)^{1/s'}\Big(\sum_{j=1}^{m}\|u\|^{p'}_{p',W_j}\Big)^{1/s}\Bigg) \\
&\le \max(1,3^{r-1})\Big(\alpha^r_{p,q}\int_I|uv|^r+\frac{1}{l}(\frac{1}{l}+\|u\|_{p',I}+\|v\|_{q,I})+\frac{1}{l^r}(\|u\|^r_{p',I}+\|v\|^r_{q,I})\Big) \\
&\le \max(1,3^{r-1})\Big(\alpha^r_{p,q}\int_I|uv|^r+\frac{1}{l}(\frac{1}{l}+\|u\|_{p',I}+\|v\|_{q,I})+\frac{1}{l^r}(\|u\|^r_{p',I}+\|v\|^r_{q,I})\Big).
\end{aligned}$$

Thus, there is a constant $c_1>0$ independent of $\varepsilon$ and $l$ such that

$$([N(\varepsilon)/2]-m+1)\varepsilon^r\le c_1\left(\int_I|uv|^r+\frac{1}{l}+\frac{1}{l^r}\right) \tag{36}$$

Let $I_i=[c_{i-1},c_i]$, $i=1,2,\dots,N(\varepsilon)$. Thus, $a=c_0<c_1<\dots<c_{N(\varepsilon)}=b$. Let $\mathcal{D}=\{e_k:1\le k\le M\}$ stand for the set-theoretic union of $\{c_i:1\le i\le N(\varepsilon)\}$ and $\{d_j:1\le j\le m\}$, so that $a=e_1<e_2<\dots<e_M=b$ and write $L_k=[e_{k-1},e_k]$. Then $\{L_k\}_{k=1}^M\in\mathcal{P}$ and for each $1\le k\le M$ there exists $i$, $1\le i\le N(\varepsilon)$ such that $L_k\subset I_i$ and, consequently, by Lemma 2.4 it is $A(L_k)\le A(I_i)\le\varepsilon$. Thus,

$$\begin{aligned}
\alpha^r_{p,q}\int_I|uv|^r &\le \max(1,3^{r-1})\alpha^r_{p,q}\left(\int_I|u_lv_l|^r+\int_I|u-u_l|^r\ |v|^r+\int_I|u_l|^r|v-v_l|^r\right) \\
&\le \max(1,3^{r-1})\alpha^r_{p,q}\left(\sum_{j=1}^{m}|\xi_j|^r\ |\eta_j|^r\ |W_j|\right. \\
&+(\sum_{j=1}^{m}\|u-u_l\|^{p'}_{p',W_j})^{1/s}(\sum_{j=1}^{m}\|v\|^q_{q,W_j})^{1/s'}
\end{aligned}$$

$$+ \Big(\sum_{j=1}^{m} \|v - v_l\|_{q,W_j}^{q}\Big)^{1/s'} \Big(\sum_{j=1}^{m} \|u\|_{p',W_j}^{p'}\Big)^{1/s}\Bigg)$$

$$\leq \max(1, 3^{r-1})\alpha_{p,q}^r \Big(\sum_{j=1}^{m} \sum_{\{k; L_k \subset W_j\}} |\xi_j|^r \ |\eta_j|^r \ |L_k| + \frac{1}{l^r}(\|u\|_{p',I}^r + \|v\|_{q,I}^r)\Big)$$

$$\leq \max(1, 3^{r-1})\Big(\sum_{j=1}^{m} \sum_{\{k; L_k \subset W_j\}} A^r(L_k, \xi_j, \eta_j) + \frac{\alpha_{p,q}^r}{l^r}(\|u\|_{p',I}^r + \|v\|_{q,I}^r)\Big)$$

$$\leq \max(1, 3^{r-1})\Big((N(\varepsilon) + m)\varepsilon^r + \frac{\alpha_{p,q}^r}{l^r}(\|u\|_{p',I}^r + \|v\|_{q,I}^r)\Big).$$

Thus, there exists $c_2 > 0$, independent of $\varepsilon$ and $l$ such that

$$\int_I |uv|^r \leq c_2\Big((N(\varepsilon) + m)\varepsilon^r + \frac{1}{l^r}\Big).$$

Letting $\varepsilon \to 0_+$ here and in (36) we obtain for each $l$

$$\limsup_{\varepsilon \to 0_+} \varepsilon^r N(\varepsilon) \leq 2c_1 \left(\int_I |uv|^r + \frac{1}{l} + \frac{1}{l^r}\right)$$

and

$$\int_I |uv|^r \leq c_2 \liminf_{\varepsilon \to 0_+} \Big(\varepsilon^r N(\varepsilon) + \frac{1}{l^r}\Big).$$

The lemma follows letting $l \to \infty$. □

The latter lemma coupled with Theorem 3.4 yields the following theorem:

**Theorem 4.5** *Let* $1 < p \leq q \leq 2$ *or* $2 < p \leq q < \infty$, $\|v\|_q < \infty$, $\|u\|_{p'} < \infty$ *and* $u, v > 0$. *Then*

$$c_1 \int_0^d |uv|^r \leq \liminf_{n \to \infty} n a_n^r(T) \leq \limsup_{n \to \infty} n a_n^r(T) \leq c_2 \int_0^d |uv|^r.$$

*Let* $1 < p < 2 < q < \infty$ $\|v\|_q < \infty$, $\|u\|_{p'} < \infty$ *and* $u, v > 0$. *Then*

$$c_3 \int_0^d |uv|^r \leq \liminf_{n \to \infty} n^{(1/2-1/q)r+1} a_n^r(T) \leq \limsup_{n \to \infty} n a_n^r(T) \leq c_4 \int_0^d |uv|^r.$$

*where* $r = \frac{p'q}{p'+q}$.

## 5 The main result

For the remainder of this section we assume that $\int_0^d |u(t)|^{p'} dt = \infty$. Furthermore, we set $U(x) := \int_0^x |u(t)|^{p'} dt$. Let $\{\xi_k\}_{k=-\infty}^{\infty}$, be a sequence satisfiyng

$$U(\xi_k) = 2^{\frac{kp'}{q}}, \tag{37}$$

and

$$\sigma_k := 2^{k/q} \|v\|_{q,Z_k}, \qquad Z_k = [\xi_k, \xi_{k+1}]. \tag{38}$$

The sequence $\{\sigma_k\}$ is the analogue of the sequence defined in [2] and [5], which in turn, was motivated by a similar sequence introduced in [8].

The following technical lemmas play a central role in this section.

**Lemma 5.1** *Let $r > 0$, $k_0, k_1 \in \mathbf{Z}$ with $k_0 \leq k_1$. Let $I = (a,b) \subset \cup_{k=k_0}^{k_1} Z_k$. Then*

$$J^r(I) \leq 4^{r/q} \max_{k_0 \leq k \leq k_1} \sigma_k^r.$$

**Proof:** Let $x \in (a,b)$. Then there exists $n \in \mathbf{Z}$, $k_0 \leq n \leq k_1$ such that $x \in Z_n$. Clearly,

$$\left( \int_a^x |u|^{p'} \right)^{r/p'} \|v\chi_{(x,b)}\|_q^r \leq \left( \int_0^{\xi_{n+1}} |u|^{p'} \right)^{r/p'} \|v\chi_{(\xi_n, \xi_{k_1}+1)}\|_q^r$$
$$\leq 2^{(n+1)r/q} \left( \sum_{i=n}^{k_1} \|v\chi_{(\xi_i,\xi_{i+1})}\|_q^q \right)^{r/q} = 2^{(n+1)r/q} \left( \sum_{i=n}^{k_1} \frac{\sigma_i^q}{2^i} \right)^{r/q}$$
$$\leq 2^{(n+1)r/q} ( \max_{i=n,\ldots,k_1} \sigma_i^q)^{r/q} \, 2^{(1-n)r/q} = 4^{r/q} \max_{i=n,\ldots,k_1} \sigma_i^r.$$

so that

$$J^r(I) \leq 4^{r/q} \max_{k_0 \leq k \leq n_1} \sigma_k^r.$$

□

**Lemma 5.2** *Let $r \geq \frac{p'q}{p'+q}$, $I_i = (a_i, b_i)$, $1 \leq i \leq l$ and $b_i \leq a_{i+1}$, $1 \leq l-1$.*

*Let $k \in \mathbb{Z}$ be such that $\cup_{i=1}^{l} I_i \subset Z_k$. Then*

$$\sum_{i=1}^{l} J^r(I_i) \le (2^{p'/q} - 1)^{r/p'} \sigma_k^r.$$

**Proof:** Set $s = (p' + q)/p'$. Thus $s > 1$ and $p'/s' = q/s = p'q/(p' + q)$. Fix $x_i \in (a_i, b_i)$. According to the assumption $r \ge \frac{p'q}{p'+q}$ we have $r \ge p'/s'$, $r \ge q/s$ and

$$\begin{aligned}
\sum_{i=1}^{l} \left( \int_{a_i}^{x_i} |u|^{p'} \right)^{r/p'} \|v\chi_{(x_i,b_i)}\|_q^r &\le \sum_{i=1}^{l} \|u\chi_{I_i}\|_{p'}^r \|v\chi_{I_i}\|_q^r \\
&\le \left( \sum_{i=1}^{l} \|u\chi_{I_i}\|_{p'}^{rs'} \right)^{1/s'} \left( \sum_{i=1}^{l} \|v\chi_{I_i}\|_q^{rs} \right)^{1/s} \\
&\le \left( \sum_{i=1}^{l} \|u\chi_{I_i}\|_{p'}^{p'} \right)^{r/p'} \left( \sum_{i=1}^{l} \|v\chi_{I_i}\|_q^{q} \right)^{r/q} \\
&\le \|u\chi_{Z_k}\|_{p'}^r \|v\chi_{Z_k}\|_q^r = (2^{p'/q} - 1)^{r/p'} \sigma_k^r.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{l} J^r(I_i) = \sum_{i=1}^{l} \sup_{x_i \in I_i} \left( \int_{a_i}^{x_i} |u|^{p'} \right)^{r/p'} \|v\chi_{(x_i,b_i)}\|_q^r \le (2^{p'/q} - 1)^{r/p'} \sigma_k^r.$$

□

**Lemma 5.3** *Let $\cup_{i=1}^{l} I_i \subset \cup_{k=k_0}^{k_1} Z_k$ and $r \ge \frac{p'q}{p'+q}$. Then*

$$\sum_{i=1}^{l} J^r(I_i) \le ((2^{p'/q} - 1)^{r/p'} + 2^{1+2r/q}) \sum_{k=k_0}^{k_1} \sigma_k^r.$$

**Proof:** Let

$$\begin{aligned}
A &= \{i \in \{1, 2, \dots, l\} : \text{ there exists } k \in \mathbb{Z} \text{ such that } \xi_k \in \text{int} I_i\}, \\
B &= \{i \in \{1, 2, \dots, l\} : \text{ there exists } k \in \mathbb{Z} \text{ such that } I_i \subset Z_k\}.
\end{aligned}$$

Clearly, $A \cap B = \emptyset$, $A \cup B = \{1, 2, \dots, l\}$. By Lemma 5.2 we obtain

$$\sum_{i \in B} J^r(I_i) \le (2^{p'/q} - 1)^{r/p'} \sum_{k=k_0}^{k_1} \sigma_k^r. \tag{39}$$

Set $A_i = \{k \in \mathbb{Z}; \mathrm{int}(I_i \cap Z_k) \neq \emptyset\}$ for $i \in A$. Let $\mathcal{A} = \{A_i; i \in A\}$. Since each $k$ belongs at most to two elements of $\mathcal{A}$, Lemma 5.1 yields

$$\sum_{i \in A} J^r(I_i) \leq 4^{r/q} \sum_{i \in A} \max_{k \in A_i} \sigma_k^r \leq 4^{r/q}\ 2 \sum_{k=k_0}^{k_1} \sigma_k^r.$$

which coupled, with (39) yields the assertion of this lemma. □

**Lemma 5.4** *Let $K_1$, $K_2$ be the constants from Proposition 1. Then*

$$K_1 \sup_{k \in \mathbb{Z}} \sigma_k \leq \|T\| \leq 4^{1/q} K_2 \sup_{k \in \mathbb{Z}} \sigma_k.$$

*Moreover, $T$ is compact if and only if*

$$\lim_{n \to \infty} \sup_{k \geq n} \sigma_k = \lim_{n \to -\infty} \sup_{k \leq n} \sigma_k = 0.$$

**Proof:** Let $(a, b) \subset (0, d)$. Set

$$a(\varepsilon) = a + \varepsilon,\ b(\varepsilon) = \begin{cases} b - \varepsilon & \text{if } \ b < \infty, \\ \frac{1}{\varepsilon} & \text{if } \ b = \infty. \end{cases}$$

Define a function $f(\varepsilon, x)$ by

$$f(\varepsilon, x) = \left( \int_{a(\varepsilon)}^{x} |u|^{p'} \right)^{1/p'} \left( \int_{x}^{b(\varepsilon)} |v|^q \right)^{1/q}.$$

Since $f(\varepsilon, x) \nearrow f(0, x)$ for $\varepsilon \to 0_+$ and any fixed $x$ we have

$$J(a(\varepsilon), b(\varepsilon)) = \sup_{a(\varepsilon) \leq x \leq b(\varepsilon)} f(\varepsilon, x) \nearrow \sup_{a \leq x \leq b} f(0, x) = J(a, b).$$

Choosing $a = 0$, $b = d$ we have by Lemma 5.1

$$J(a(\varepsilon), b(\varepsilon)) \leq 4^{1/q} \sup_{k \in \mathbb{Z}} \sigma_k$$

and consequently,

$$J(a, b) \leq 4^{1/q} \sup_{k \in \mathbb{Z}} \sigma_k.$$

By the definition of $\sigma_k$ it is easy to see that $\sigma_k \leq J(0, d)$ for each $k \in \mathbb{Z}$ which implies

$$\sup_{k \in \mathbb{Z}} \sigma_k \leq J(a, b).$$

Now, the first part of our lemma follows by applying Lemma 1.

The second part can be proved analogously by using Proposition 2.2. □

**Lemma 5.5** *Let $I' = [a', b'] \subset I = [a, b] \subset [0, d]$ and let $\varepsilon > 0$. Let $\{I_i\}_{i=1}^{N(I,\varepsilon)} \in \mathcal{P}(I)$ and $A(I_i) \le \varepsilon$. Set $\mathcal{K} = \{i; I_i \subset I'\}$, $K = \#\mathcal{K}$. Then*

$$K - 2 \le N(I, \varepsilon) \le K + 2.$$

**Proof:** Let $\{I'_i\}_{i=1}^{N(I',\varepsilon)} \in \mathcal{P}(I')$, $A(I'_i) \le \varepsilon$. Let $I_i = [a_i, a_{i+1}]$, $i = 1, 2, \dots, N(I, \varepsilon)$, and $I'_j = [a'_j, a'_{j+1}]$, $j = 1, 2, \dots, N(I', \varepsilon)$ and put $k_0 = \min \mathcal{K}$ and $k_1 = \max \mathcal{K}$. Write

$$S_1 = \begin{cases} \{[a', a_{k_0}]\} & \text{if } \ a' < a_{k_0}, \\ \emptyset & \text{if } \ a' = a_{k_0}, \end{cases} \qquad S_2 = \begin{cases} \{[a_{k_1+1}, b']\} & \text{if } \ a_{k_1+1} < b', \\ \emptyset & \text{if } \ a_{k_1+1} = b'. \end{cases}$$

Remark that by Lemma 2.4, $A(\widetilde{I}) \le \varepsilon$ for each $\widetilde{I} \in S_1 \cup S_2$. Take a system of intervals $\mathcal{L} = S_1 \cup S_2 \cup \{I_i; i \in \mathcal{K}\}$ so that $\mathcal{L} \in \mathcal{P}(I')$ and $A(\widetilde{I}) \le \varepsilon$ for $\widetilde{I} \in \mathcal{L}$. Thus, by the definition of $N(I', \varepsilon)$ one has

$$N(I', \varepsilon) \le \#\mathcal{L} \le \#\mathcal{K} + 2 = K + 2.$$

To prove the inequality $K - 2 \le N(I', \varepsilon)$ set

$$S'_1 = \begin{cases} \{[a_{k_0-1}, a']\} & \text{if } \ a_{k_0-1} < a', \\ \emptyset & \text{if } \ a_{k_0-1} = a', \end{cases} \qquad S'_2 = \begin{cases} \{[b', a_{k_1+2}]\} & \text{if } \ b' < a_{k_1+2}, \\ \emptyset & \text{if } \ b' = a_{k_1+2}. \end{cases}$$

Clearly, $A(\widetilde{I}) \le \varepsilon$ for $\widetilde{I} \in S'_1 \cup S'_2$. Denote $\mathcal{N}_0 = \{I_i; I_i \subset [a, a']\}$, $\mathcal{N}_1 = \{I_i; I_i \subset [b', b]\}$ and set $n_0 = \#\mathcal{N}_0$, $n_1 = \#\mathcal{N}_1$. Take a system of intervals

$$\mathcal{L}' = S'_1 \cup S'_2 \cup \mathcal{N}_0 \cup \mathcal{N}_1 \cup \{I'_j; j = 1, 2, \dots, N(I', \varepsilon)\}.$$

Since, $A(\widetilde{I}) \le \varepsilon$ for any $\widetilde{I} \in \mathcal{L}'$ and by definition of $N(I, \varepsilon)$, $N(I, \varepsilon) \le \#\mathcal{L}'$. Moreover, since

$$n_0 + n_1 + K \le N(I, \varepsilon) \le n_0 + n_1 + K + 2$$

and

$$n_0 + n_1 + N(I', \varepsilon) \le \#\mathcal{L}' \le n_0 + n_1 + N(I', \varepsilon) + 2$$

we obtain

$$n_0 + n_1 + K \le n_0 + n_1 + N(I', \varepsilon) + 2$$

which finishes the proof. □

**Lemma 5.6** *Let $1 < p \le q < \infty$, $r = \frac{p'q}{p'+q}$. Let $\sum_{i \in \mathbb{Z}} \sigma_i^r < \infty$. Then $T$ is* *compact,* $\int_0^d |uv|^r < \infty$ *and there are positive constants* $c_1, c_2$ *such that*

$$c_1 \int_0^d |uv|^r \le \liminf_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon) \le \limsup_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon) \le c_2 \int_0^d |uv|^r.$$

**Proof:** By Lemma 5.4, $T$ is compact. Let $k \in \mathbb{Z}$ and set $s = p'/q+1$. It follows that $rs = p'$, $rs' = q$ and using Hölder's inequality, we obtain

$$\int_{Z_k} |uv|^r \le \left(\int_{\xi_k}^{\xi_{k+1}} |u|^{p'}\right)^{r/p'} \left(\int_{\xi_k}^{\xi_{k+1}} |v|^q\right)^{r/q}.$$

Moreover by the definition of $\xi_k$ one has

$$(2^{p'/q}-1)^{1/p'} \left(\int_0^{\xi_k} |u|^{p'}\right)^{1/p'} = \left(\int_{\xi_k}^{\xi_{k+1}} |u|^{p'}\right)^{1/p'}$$

and consequently,

$$\int_{Z_k} |uv|^r \le (2^{p'/q}-1)^{r/p'} \sigma_k^r. \tag{40}$$

This proves $\int_0^d |uv|^r < \infty$.

Fix $\delta > 0$. Take $k_0$, $k_1 \in \mathbb{Z}$ such that

$$\sum_{i\le k_0-1} \sigma_i^r + \sum_{i\ge k_1} \sigma_i^r \le ((2^{p'/q}-1)^{r/p'} + 2^{1+2r/q})^{-1}\delta.$$

Let $\varepsilon > 0$. Let $\{I_j\}_{j=1}^{N(\varepsilon)} \in \mathcal{P}(0,d)$, $A(I_j) \le \varepsilon$. Remark that according to the definition of $N(\varepsilon)$, $A(I_j \cup I_{j+1}) > \varepsilon$ for $j = 1, 2, \ldots, N(\varepsilon)-1$. Set $I = [\xi_{k_0}, \xi_{k_1}]$ and

$$\begin{aligned} \mathcal{N}_0 &= \{I_j; I_j \subset [0,\xi_{k_0}]\}, & n_0(\varepsilon) &= \#\mathcal{N}_0,\\ \mathcal{N}_1 &= \{I_j; I_j \subset [\xi_{k_1}, d]\}, & n_1(\varepsilon) &= \#\mathcal{N}_1,\\ \widetilde{\mathcal{N}} &= \{I_j; I_j \subset I\}, & \widetilde{n}(\varepsilon) &= \#\widetilde{\mathcal{N}}. \end{aligned}$$

Then $N(\varepsilon) \le \widetilde{n}(\varepsilon) + n_0(\varepsilon) + n_1(\varepsilon) + 2$. By Lemma 5.5, $\widetilde{n}(\varepsilon) - 2 \le N(I,\varepsilon) \le \widetilde{n}(\varepsilon) + 2$. Since $n \le 2([\frac{n}{2}]+1)$ for any positive integer $n$, we obtain

$$\varepsilon^r(N(\varepsilon) - N(I,\varepsilon)) \qquad \le \varepsilon^r(N(\varepsilon) - \widetilde{n}(\varepsilon) + 2)$$

$$\leq \varepsilon^r(n_0(\varepsilon)+n_1(\varepsilon)+4) \leq 2\varepsilon^r(\left[\frac{n_0(\varepsilon)}{2}\right]+\left[\frac{n_1(\varepsilon)}{2}\right]+3).$$

For $j_0 = \min\{j; I_j \in \mathcal{N}_1(\varepsilon)\}$, one has

$$\frac{1}{2}\varepsilon^r(N(\varepsilon)-N(I,\varepsilon)-6) \leq \sum_{j=1}^{\left[\frac{n_0(\varepsilon)}{2}\right]} \varepsilon^r + \sum_{j=j_0}^{j_0+\left[\frac{n_1(\varepsilon)}{2}\right]} \varepsilon^r$$
$$\leq \sum_{j=1}^{\left[\frac{n_0(\varepsilon)}{2}\right]} A^r(I_j \cup I_{j+1}) + \sum_{j=j_0}^{j_0+\left[\frac{n_1(\varepsilon)}{2}\right]} A^r(I_j \cup I_{j+1}).$$

Since $A(I,\varepsilon) \leq J(I,\varepsilon)$ for $I \subset J$ and according to Lemma 5.5 we have

$$\frac{1}{2}\varepsilon^r(N(\varepsilon)-N(I,\varepsilon)-6) \leq \sum_{j=1}^{\left[\frac{n_0(\varepsilon)}{2}\right]} J^r(I_j \cup I_{j+1}) + \sum_{j=j_0}^{j_0+\left[\frac{n_1(\varepsilon)}{2}\right]} J^r(I_j \cup I_{j+1})$$
$$\leq ((2^{p'/q}-1)^{r/q}+2^{1+2r/q})(\sum_{i\leq k_0-1} \sigma_i^r + \sum_{i\geq k_1} \sigma_i^r) \leq \delta$$

which gives

$$\varepsilon^r N(\varepsilon) \leq 2\delta + \varepsilon^r N(I,\varepsilon) + 6\varepsilon^r$$

and consequently,

$$\limsup_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon) \leq 2\delta + \limsup_{\varepsilon\to 0_+} \varepsilon^r N(I,\varepsilon). \tag{41}$$

Again Lemma 5.5, gives $N(I,\varepsilon) \leq \tilde{n}+2 \leq N(\varepsilon)+2$ and thus

$$\limsup_{\varepsilon\to 0_+} \varepsilon^r N(I,\varepsilon) \leq \limsup_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon). \tag{42}$$

By (40) we have

$$\left|\int_0^d |uv|^r - \int_I |uv|^r\right| \leq (2^{p',q}-1)^{r/p'}\delta. \tag{43}$$

Using Lemma 4.4 one easily sees that

$$c_1\alpha_{p,q}\int_I |uv|^r \leq \liminf_{\varepsilon\to 0_+} \varepsilon^r N(I,\varepsilon) \leq \limsup_{\varepsilon\to 0_+} \varepsilon^r N(I,\varepsilon) \leq c_2\alpha_{p,q}\int_I |uv|^r$$

which yields with (41), (42) and (43) that for any $\delta > 0$,

$$c_1\alpha_{p,q}\left(\int_0^d |uv|^r - (2^{p',q}-1)^{r/p'}\delta\right) \le \liminf_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon)$$

$$\le \limsup_{\varepsilon\to 0_+} \varepsilon^r N(\varepsilon) \le c_2\alpha_{p,q}\left(\int_0^d |uv|^r\right) + 2\delta.$$

Letting $\delta \to 0_+$ we obtain our lemma. □

**Theorem 5.7** *Suppose that (13) and (14) are satisfied and let* $r = \frac{p'q}{p'+q}$ *and* $\sum_{i=-\infty}^{\infty} \sigma_i^r < \infty$.

*Let* $1 < p \le q \le 2$ *or* $2 \le p \le q < \infty$. *Then*

$$c_1 \int_0^d |u(t)v(t)|^r dt \le \liminf_{n\to\infty} n a_n^r(T) \le \limsup_{n\to\infty} n a_n^r(T) \le c_2 \int_0^d |u(t)v(t)|^r dt. \quad (44)$$

*Let* $1 < p \le 2 \le q < \infty$. *Then*

$$c_3 \int_0^d |u(t)v(t)|^r dt \le \liminf_{n\to\infty} n^{(1/2-1/q)r+1} a_n^r(T) \le \limsup_{n\to\infty} n a_n^r(T) \le c_4 \int_0^d |u(t)v(t)|^r dt. \quad (45)$$

## 6 $l^r$ and weak–$l^r$ estimates

In this section we show that the $L^r$ ($L^{r,\infty}$)-norms of $\{a_n(T)\}_{n\in\mathbf{N}}$, and $\{\sigma_n\}_{n\in\mathbf{Z}}$ are equivalent for $r \ge \min_{s\ge 1}\max(p'/s', q/s)$.

**Lemma 6.1** *Let* $I = [a,b]$ *and* $\varepsilon > 0$. *Set*

$$\sigma(\varepsilon) := \{k \in \mathbf{Z} : Z_k \subset I, \sigma_k > \varepsilon\}.$$

*Suppose that* $\sigma_k$ *contains at least four elements. Then*

$$A(I) > \frac{\varepsilon}{4^{1/q}}.$$

**Proof:** Let $Z_{k_i}, i = 1,2,3,4,\ k_1 < k_2 < k_3 < k_4$, be 4 distinct members of $\sigma(\varepsilon)$, and set $I_1 = (\xi_{k_1}, \xi_{k_2})$, $I_2 = (\xi_{k_2+1}, \xi_{k_4})$. Then, with $f_0 = \chi_{I_1} + \chi_{I_2}$,

$$
\begin{aligned}
A(I) &\geq \inf_{\alpha} \|v(x)\left(\int_{c}^{x} |u(t)|f_0(t)dt - \alpha\right)\|_{q,I} \\
&\geq \inf_{\alpha} \max\left\{\|v\|_{q,Z_{k_2}}\left|\int_{I_1} |u(t)f(t)|dt - \alpha\right|; \|v\|_{q,Z_{k_4}}\left|\int_{I_1\cup I_2} |u(t)f(t)|dt - \alpha\right|\right\} \\
&= \inf_{\alpha} \max\left\{\|v\|_{q,Z_{k_2}}|2^{k_2/q} - 2^{k_1/q} - \alpha|; \|v\|_{q,Z_{k_4}}|2^{k_2/q} - 2^{k_1/q} + 2^{k_4/q} - 2^{(k_2+1)/q} - \alpha|\right\} \\
&\geq \inf_{\alpha} \max\left\{\frac{\varepsilon}{2^{(k_2+1)/q}}|2^{k_2/q} - 2^{k_1/q} - \alpha|; \frac{\varepsilon}{2^{(k_4+1)/q}}|2^{k_2/q} - 2^{k_1/q} + 2^{k_4/q} - 2^{(k_2+1)/q} - \alpha|\right\} \\
&\geq \frac{\varepsilon}{2^{k_4/q}+1}\frac{1}{2^{1/q}}\left(2^{k_4} - 2^{k_2+1}\right) \geq \frac{\varepsilon}{4^{1/q}}.
\end{aligned}
$$

□

**Lemma 6.2** *Let $\varepsilon > 0$. Let $\mathbf{K} = \{k \in \mathbf{Z}; \sigma_k > 2^{1/q}\varepsilon\}$. Then*

$$\#\mathbf{K} \leq 4N(\varepsilon) - 1.$$

**Proof:** Let $I_i = [c_{i-1}, c_i]$ and $i = 1, \ldots, N(\varepsilon)$. Divide $\mathbf{K}$ into two disjoint sets $\mathbf{Z}_1$ and $\mathbf{Z}_2$ by

$$\mathbf{Z}_1 = \{k \in \mathbf{K};\ \text{there exists } j \in \{1, \ldots, N(\varepsilon)\} \text{ such that } c_j \in Z_k\},$$

$$\mathbf{Z}_2 = \{k \in \mathbf{K};\ \text{there exists } j \in \{1, \ldots, N(\varepsilon)\} \text{ such that } Z_k \in I_j\},$$

Clearly, $\#\mathbf{Z}_1 \leq N(\varepsilon) - 1$.

Say that $k_1, k_2 \in \mathbf{Z}_2$ are equivalent if there exists $j$ such that $Z_{k_1} \cup Z_{k_2} \subset I_j$. Denote the equivalence classes in $\mathbf{Z}_2$ by $Y_1$ and $Y_2$. Assume $\#Y_i \geq 4$ for some $i$. Then there are $k_1, k_2, k_3, k_4$ and $j$ such that $Z_{k_1} \cup Z_{k_2} \cup Z_{k_3} \cup Z_{k_4} \subset I_j$. Using Lemma 6.1 with $2^{1/q}\varepsilon$ instead of $\varepsilon$, we have $A(I) > \varepsilon$ which contradicts the definition of $A(I)$. Then $\#Y_i \leq 3$ for any $i \in \mathbf{Z}_2$. Consequently, the mapping $P$ defined by

$$P(i) = j \text{ if } Z_i \subset I_j \text{ for any } i \in \mathbf{Z}_2$$

is an injection and, therefore,

$$\#\mathbf{Z}_2 \leq 3N(\varepsilon).$$

Thus,

$$\#\mathbf{K} = \#\mathbf{Z}_1 + \#\mathbf{Z}_2 \leq 4N(\varepsilon) - 1$$

which completes the proof. □

**Lemma 6.3** *Let $1 < p \leq q \leq 2$ or $2 \leq p \leq q < \infty$. Then there are positive constants $c_1, c_2, c_3$ depending on $p$ and $q$ such that the inequality*

$$\#\{k; \sigma_k > t\} \leq c_1 \#\{k; a_k(T) \geq c_2 t\} + c_3$$

*holds for all $t > 0$.*

**Proof:** According to Lemma 3.4 there are two positive constants $c_1, c_2$ depending on $p, q$ such that

$$a_{[c_1 N(\varepsilon)]-1}(T) > c_2 \varepsilon.$$

Then

$$\#\{k; a_k(T) > c_2 \varepsilon\} \geq c_1 N(\varepsilon) - 2$$

and, according to Lemma 6.2, we have

$$\begin{aligned}\#\{k; \sigma_k > t\} &\leq 4N(\frac{t}{2^{1/q}}) - 1 = \frac{4}{c_1}\left(c_1 N(\frac{t}{2^{1/q}}) - 2\right) + \frac{4}{c_1} - 1 \\ &\leq \frac{4}{c_1} \# \left\{k; a_k(T) > \frac{c_2}{2^{1/q}} t\right\}.\end{aligned}$$

The lemma follows by writing $c_1, c_2$ and $c_3$ instead of $\frac{2}{c_1}$, $\frac{c_1}{2^{1/q}}$ and $\frac{4}{c_1} - 1$.□

We recall the following well-know fact: given a countable set $\mathcal{S}$ we have for any $p, 1 \leq p < \infty$

$$\sum_{k \in \mathcal{S}} |a_k|^p = p \int_0^\infty t^{p-1} \#\{k \in \mathcal{S}; |a_k| > t\} dt.$$

It is easy to see that also

$$\sum_{k \in \mathcal{S}} |a_k|^p = p \int_0^\infty t^{p-1} \#\{k \in \mathcal{S}; |a_k| \geq t\} dt.$$

**Lemma 6.4** *Let $r > 0$. There are constants $c_1 \geq 0$ and $c_2 \geq 0$ such that*

$$\|\{\sigma\}\|^r_{l^r(\mathbf{Z})} \leq c_1 \|\{a_k(T)\}\|^r_{l^r(\mathbf{N})} + c_2 \|\{\sigma\}\|^r_{l^\infty(\mathbf{Z})}$$

**Proof:** Set $\lambda = \|\{\sigma\}\|_{l^\infty(\mathbf{Z})}$. By Lemma 6.3 we have,

$$
\begin{aligned}
\|\{\sigma_k\}\|^r_{l^r(\mathbf{Z})} &= r\int\limits_0^\lambda t^{r-1}\#\{k\in\mathbf{Z};\sigma_k>t\}dt \\
&\le r\int\limits_0^\lambda t^{r-1}(c_1\#\{k;a_k(T)>c_2t\}+c_3dt \\
&= \frac{c_1}{c_2^{r+1}}q\int\limits_0^\lambda t^{r-1}\#\{k;a_k(T)>t\}dt+c_3\lambda^r \\
&= \frac{c_1}{c_2^{r+1}}\|\{a_k(T)\}\|^r_{l^r(\mathbf{N})}+c_3\lambda^r,
\end{aligned}
$$

and hence the proof is complete. □

**Lemma 6.5** *Let $r>0$. Then there is a positive constant $c$ such that*

$$\|\{\sigma\}\|_{l^r(\mathbf{Z})}\le c\|\{a_k(T)\}\|_{l^r(\mathbf{N})}.$$

**Proof:** By Remark 5.5,

$$
\begin{aligned}
\|\,\{\sigma_k\}\,\|_{l^\infty(\mathbf{Z})} &\le C\|T\| = Ca_1(T) \\
&\le C\|\,\{a_k(T)\}\,\|_{l^r(\mathbf{N})}.
\end{aligned}
$$

The result then follows from Lemma 6.4. □

Now, we tackle the remaining inequality:

**Lemma 6.6** *Let $1<p\le q\le 2$ or $2\le p\le q<\infty$ and $s>r=\frac{p'q}{p'+q}$. Then*

$$\|\{a_n(T)\}\|_{l^s}\le c\|\{\sigma_k\}\|_{l^s}.$$

**Proof:** Let $I_i, i=1,2,\ldots,N(\varepsilon)$, be the collection of intervals given by (20) with $I=(a,b)$ and $N(\varepsilon)\equiv N((a,b),\varepsilon)$: note that in view of Lemma 2.1, we have $J(I_i)=\varepsilon$ for $1\le i<N(\varepsilon)$. We group the intervals $I_i$ into families $\mathbf{F}_j, j=1,2,\ldots$ such that each $\mathbf{F}_j$ consists of the maximal number of those intervals $I_{K-1}$ in the collection, which satisfy the hypothesis of Lemma 5.1 and Lemma 5.2 : $I_{k_1}\subset(\xi_{k_0},\xi_{k_2+1})$, for some $k_0,k_2$,, and the next interval $I_k$ intersects $Z_{k_2+1}$ (This construction is based on our construction from [2], for more see Lemma 5.1. and Section 6 in [2]). Hence, by Lemma 5.1 and Lemma 5.2, there is a positive constant $c$ such that

$$\varepsilon^r\#\mathbf{F}_j\le c\max_{k_0\le n\le k_2}\sigma_n^r=c\sigma_{k_j}^r$$

It follows that, with $n_j=[c\sigma_{k_j}^r/\varepsilon^r]$,

$$
\begin{aligned}
N(\varepsilon) &= \sum_j \#\mathcal{F}_j \\
&\leq \sum_j \sum_{n=1}^{n_j} 1 = \sum_{n=1}^{\infty} \sum_{j:n_j \geq n} 1 \\
&= \sum_{n=1}^{\infty} \# \left\{ j : \frac{c\sigma_{k_j}^r}{\varepsilon^r} \geq n \right\} \\
&\leq \sum_{n=1}^{\infty} \# \left\{ k : \sigma_k^r \geq \frac{n\varepsilon^r}{c} \right\}. \qquad (46)
\end{aligned}
$$

Thus, if $\{\sigma_k\} \in l^s(\mathbf{Z})$ for some $s \in (r, \infty)$,

$$
\begin{aligned}
s \int_0^{\infty} t^{s-1} N(t) dt &\leq s \int_0^{\infty} \sum_{n=1}^{\infty} t^{s-1} \# \left\{ k : \sigma_k^r > \frac{nt^r}{c} \right\} dt \\
&= sc^{s/r} \int_0^{\infty} \sum_{n=1}^{\infty} n^{-s/r} z^{s-1} \# \{ k : \sigma_k > z \} dz \\
&\preceq \| \{\sigma_k\} \|_{l^s(\mathbf{Z})}^s \qquad (47)
\end{aligned}
$$

where $\preceq$ stands for less than or equal to a positive constant multiple of the right hand side. From the inequality $N(\varepsilon) \leq M(\varepsilon)$ and Theorem 3.4, $a_{N(\varepsilon)+1}(T) \leq 2\varepsilon$ and therefore

$$
\begin{aligned}
\# \{ k \in \mathbf{N} : a_k(T) > t \} &\leq N(t/2) + 1 \\
&\leq M(t/2) + 1.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\| \{a_k(T)\} \|_{l^s(\mathbf{N})}^s &= s \int_0^{\infty} t^{s-1} \# \{ k \in \mathbf{N} : a_k(T) > t \} dt \\
&\leq s \int_0^{\|T\|} t^{s-1} \left[ N(\frac{t}{2}) + 1 \right] dt \\
&\preceq \| \{\sigma_k\} \|_{l^s(\mathbf{Z})}^s + \|T\|^s \\
&\preceq \| \{\sigma_k\} \|_{l^s(\mathbf{Z})}^s
\end{aligned}
$$

by (47) and then, in virtue of Lemma 5.1 and Remark 5.5, $\|T\| \preceq \| \{\sigma_k(T)\} \|_{l^\infty(\mathbf{Z})} \leq \| \{\sigma_k\} \|_{l^q(\mathbf{Z})}$. □

Lemmas 6.4 and 6.5 imply the following theorem:

**Theorem 6.7** *Let* $1 < p \leq q \leq 2$ *and* $2 \leq p \leq q < \infty$, $r = \frac{p'q}{p'+q}$ *and* $k > 0$.

*(i) Then there exists a positive constant* $c_1$ *such that*

$$\|\{\sigma_k\}\|_{l^k(\mathbf{Z})} \leq c_1 \|\{a_k(T)\}\|_{l^k(\mathbf{N})}.$$

*(ii) Let* $s > r$*. Then there is a positive constant* $c_2$ *such that*

$$\|\{a_k\}\|_{l^s(\mathbf{N})} \leq c_2 \|\{\sigma_k\}\|_{l^s(\mathbf{Z})}.$$

*(iii) Let* $1 \leq j \leq \infty$*. Then there exists a positive constant* $c_1$ *such that*

$$\|\{\sigma_k\}\|_{l^{k,j}(\mathbf{Z})} \leq c_1 \|\{a_k(T)\}\|_{l^{k,j}(\mathbf{N})}.$$

*(iv) Let* $s > r$ *and* $1 \leq j \leq \infty$*. Then there is a positive constant* $c_2$ *such that*

$$\|\{a_k\}\|_{l^{s,j}(\mathbf{N})} \leq c_2 \|\{\sigma_k\}\|_{l^{s,j}(\mathbf{Z})}.$$

**Proof:** Claims (i) and (ii) follow from Lemma 6.4 and Lemma 6.5. The assertions (iii) and (iv) can be obtained from (i) and (ii), by using real interpolation on the scale $l^{p,q}$. □

## 7 Appendix

In this section we show that the power of $n$ in (23) is the best possible for $2 < p \leq \infty$. Given a square matrix of a dimension $L$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1L} \\ a_{21} & a_{22} & \dots & a_{2L} \\ \vdots & \vdots & \ddots & \\ a_{L1} & a_{L2} & \dots & a_{LL} \end{pmatrix} \tag{48}$$

we will denote, for $1 \leq I \leq L$, the $i$-th column of $A$ by $u_i(A)$ and the $i$-th row of $A$ by $v_i(A)$ , i.e.

$$c_i(A) = (a_{1i}, a_{2i}, \dots, a_{Li})$$
$$r_i(A) = (a_{i1}, a_{i2}, \dots, a_{iL}).$$

By $h(A)$ denote the rank of $A$ and by $u.v$ the canonical scalar product of vectors $u$ and $v$, i. e.

$$u.v = \sum_{i=1}^{L} u_i \; v_i$$

where $u = (u_1, u_2, \ldots, u_L)$ and $v = (v_1, v_2, \ldots, v_L)$.

**Lemma 7.1** *Let $m \in \mathbb{N}$ and $L = 2^m$. Then there exists a square matrix $A$ given by (48) such that*

$$|a_{ij}| = 1 \quad for \quad \leq i, j \leq L \tag{49}$$

*and*

$$u_i(A).u_j(A) = 0 \quad for \quad \leq i, j \leq L,\ i \neq j. \tag{50}$$

**Proof:** We use mathematical induction with respect to $m$. If $m = 1$ it suffices to take

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Assume that the matrix $A$ given by (48) with $L = 2^m$ satisfies (49) and (50). Let $B$ be a square matrix of a dimension $2L = 2^{m+1}$ given by

$$B = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1L} & a_{11} & a_{12} & \ldots & a_{1L} \\ a_{21} & a_{22} & \ldots & a_{2L} & a_{21} & a_{22} & \ldots & a_{2L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{L1} & a_{L2} & \ldots & a_{LL} & a_{L1} & a_{L2} & \ldots & a_{LL} \\ & & & & & & & \\ a_{11} & a_{12} & \ldots & a_{1L} & -a_{11} & -a_{12} & \ldots & -a_{1L} \\ a_{21} & a_{22} & \ldots & a_{2L} & -a_{21} & -a_{22} & \ldots & -a_{2L} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{L1} & a_{L2} & \ldots & a_{LL} & -a_{L1} & -a_{L2} & \ldots & -a_{LL} \end{pmatrix} := \begin{pmatrix} A & A \\ A & -A \end{pmatrix}.$$

It is easy to se that $B$ satisfies (49) and (50). □

**Lemma 7.2** *Let $n \in \mathbb{N}$ and set $K = 2^n$, $L = K^2$. Then there exists a square* *matrix of a dimension* $2L$*,*

$$M = \begin{pmatrix} m_{11} & m_{12} & \dots & m_{1L} \\ m_{21} & m_{22} & \dots & m_{2L} \\ \vdots & \vdots & \ddots & \\ m_{L1} & m_{L2} & \dots & m_{LL} \end{pmatrix},$$

*such that*

$$h(M) \leq L, \tag{51}$$

$$m_{ii} = K \quad for \quad 1 \leq i, j \leq 2L, \tag{52}$$

$$|m_{ij}| \leq 1 \quad for \quad 1 \leq i, j \leq 2L, i \neq j. \tag{53}$$

**Proof:** Since $L = 2^n$ we have by Lemma 7.1 a matrix $A$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1L} \\ a_{21} & a_{22} & \dots & a_{2L} \\ \vdots & \vdots & \ddots & \\ a_{L1} & a_{L2} & \dots & a_{LL} \end{pmatrix},$$

which satisfies (49) and (50). For $1 \leq i \leq L$, set

$$m_{ij} := \{ 0 \quad \text{for} \quad 1 \leq j \leq L, \ i \neq j, K \quad \text{for} \quad j = i, a_{i,j-L} \quad \text{for} \quad L+1 \leq j \leq 2L (54)$$

and let $r_1, r_2, \dots, r_L$ be $2L$-dimensional vectors, $r_i = (m_{i1}, m_{i2}, \dots, m_{i,2L})$. Set for $1 \leq i \leq L$

$$r_{i+L} = \frac{1}{K} \sum_{j=1}^{L} a_{ji} r_j \tag{55}$$

Let $M$ be the matrix consisting of the rows $r_1, r_2, \ldots, r_{2L}$, i.e. $v_i(M) = r_i$. Denote the elements of $M$ by $m_{ij}$, so that

$$M = \begin{pmatrix} m_{11} & m_{12} & \ldots\, m_{1,2L} \\ m_{21} & m_{22} & \ldots\, m_{2,2L} \\ \vdots & \vdots & \ddots \\ m_{2L,1} & m_{2L,2} & \ldots\, m_{2L,2L} \end{pmatrix}.$$

We claim that $M$ satisfies (51), (52) and (53).

Let $L+1 \leq i \leq 2L$. Then $r_i$ is by (55) a linear combination of $u_1, u_2, \ldots, u_L$ and then $h(M) \leq L$.

Next, we calculate $m_{ii}$. If $1 \leq i \leq L$, $m_{ii} = K$ by (54). Let $L+1 \leq i \leq 2L$ and write $s = i - L$. Then by (49) and (55) we have

$$\begin{aligned} m_{ii} = m_{s+L,s+L} &= \frac{1}{K}\sum_{j=1}^{L} m_{j,s+L}\ m_{j,s+L} = \frac{1}{K}\sum_{j=1}^{L} a_{js}\ a_{js} \\ &= \frac{1}{K}\|u_s(A)\|^2 = \frac{1}{K}L = K. \end{aligned}$$

We now (53). Calculate $m_{ij}$, $i \neq j$. We have four posibilities:

(i) If $1 \leq i, j \leq L$ then by (54) we have $m_{ij} = 0$ and thus, $m_{ij} = 0$ satisfies (53).

(ii) If $1 \leq i \leq L$, $L+1 \leq j \leq 2L$ then $m_{ij} = a_{i,j-L}$ and due to (49) it is $|m_{ij}| \leq 1$.

(iii) If $L+1 \leq i \leq 2L$, $1 \leq j \leq L$ then setting $s = i - L$ we have by (54) and (55)

$$m_{ij} = m_{s+L,j} = \frac{1}{K}\sum_{k=1}^{L} a_{ks}\ m_{kj} = \frac{1}{K} a_{js}\ m_{jj} = a_{js}$$

which gives by (49) $|m_{ij}| \leq 1$.

(iv) If $L+1 \leq i \leq 2L$, $L+1 \leq j \leq 2L$ denote $s = i - L$, $t = j - L$. By (54) and (55) we obtain

$$m_{ij} = m_{s+L,j} = \frac{1}{K}\sum_{k=1}^{L} a_{ks}\ m_{kj} = \frac{1}{K}\sum_{k=1}^{L} a_{ks}\ a_{kt} = \frac{1}{K} u_s(A)\ u_t(A)$$

which gives with (50) that $m_{ij} = 0$ and proves (53). □

Let $e_i|$ denote the sequence which has 1 on $i$-th coordinate and 0 on other.

**Lemma 7.3** *Let $2 < p \le \infty$ and $n \in \mathbb{N}$. Set $K = 2^n$ and $L = K^2$. Then there exists a subspace $X$ of $l^p$, $\dim X \le L$ such that for each $i$, $1 \le i \le 2L$.*

$$\mathrm{dist}_p(e_i, X) \le \frac{2^{1/p}}{K^{1-2/p}}.$$

**Proof:** Let $M$ be the matrix of rank $2L$ from Lemma 7.2. Set for $1 \le i \le 2L$

$$x_i = (m_{i1}, m_{i2}, \ldots, m_{i,2L}, 0, 0, \ldots).$$

and

$$X = \mathrm{lin}\{x_1, x_2, \ldots, x_{2L}\}.$$

By (51), $\dim X \le L$.

Next, we estimate $\mathrm{dist}_p(e_k, X)$ for $1 \le k \le 2L$.

Assume first $p < \infty$. Then

$$\begin{aligned}\mathrm{dist}_p^p(e_k, X) &\le \|e_k - \frac{1}{K}x_k\|_p^p \\ &= \|(\frac{1}{K}m_{k1}, \ldots, \frac{1}{K}m_{k,k-1}, 0, \frac{1}{K}m_{k,k+1}, \ldots, \frac{1}{K}m_{k,2L}, 0, 0, \ldots)\|_p^p \\ &\le \sum_{i=1}^{2L-1} \frac{1}{K^p} \le \sum_{i=1}^{2L} \frac{1}{K^p} = \frac{2L}{K^p} = \frac{2}{K^{p-2}}.\end{aligned}$$

This gives $\mathrm{dist}_p(e_k, X) \le \frac{2^{1/p}}{K^{1-2/p}}$.

Next, assume $p = \infty$, so that

$$\begin{aligned}\mathrm{dist}_\infty(e_k, X) &\le \|e_k - \frac{1}{K}x_k\|_\infty \\ &= \|(\frac{1}{K}m_{k1}, \ldots, \frac{1}{K}m_{k,k-1}, 0, \frac{1}{K}m_{k,k+1}, \ldots, \frac{1}{K}m_{k,2L}, 0, 0, \ldots)\|_\infty \le \frac{1}{K}\end{aligned}$$

This concludes the proof. □

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## REFERENCES

(1) D.E. Edmunds, W.D. Evans and D.J. Harris. Approximation numbers of certain Volterra integral operators. *J. London Math. Soc.* (2) 37 (1988), 471–489.

(2) D.E. Edmunds, W.D. Evans and D.J. Harris. Two–sided estimates of the approximation numbers of certain Volterra integral operators. *Studia Math.* 124 (1) (1997), 59–80.

(3) D.E. Edmunds and W.D. Evans, Spectral Theory and Differential Operators, *Oxford Univ. Press, Oxford,* 1987.

(4) D.E. Edmunds, P. Gurka and L. Pick. Compactness of Hardy–type integral operators in weighted Banach function spaces. *Studia Math.* 109 (1) (1994), 73–90.

(5) W.D. Evans, D.J. Harris and J. Lang. Two-sided estimates for the approximation numbers of Hardy-type operators in $L^\infty$ and $L^1$. *Studia Math.* 130 (2) (1998), 171–192.

(6) J. Lang, A. Nekvinda and L. Pick. Boundedness and compactness of integral operators with general kernels from a weighted Banach function space into $L^\infty$, *preprint.*

(7) E. Lomakina and V. Stepanov. On asymptotic behavior of the Approximation numbers and estimates of Shatten - Von Neumann norms of the Hardy-type integral operators, *preprint*

(8) J. Newman and M. Solomyak. Two–sided estimates of singular values for a class of integral operators on the semi–axis, *Integral Equations Operator Theory* 20 (1994), 335–349.

(9) B. Opic and A. Kufner, Hardy–type Inequalities, *Pitman Res. Notes Math. Ser. 219, Longman Sci. & Tech., Harlow,* 1990.

(10) I. Singer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspace,*Springer-Verlag, New York* 1970.

(11) P. Wojtaszyk, Banach spaces for analysts, *Cambridge studies in advanced mathematics 25, Cambridge University Press, Cambridge* 1991.