Behaviour of the approximation numbers of a Sobolev embedding in the one-dimensional case.

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Abstract

We consider the Sobolev embeddings

$$E_1 : W_0^{1,p}(a,b) \to L^p(a,b) \quad \text{or} \quad E_2 : L^{1,p}(a,b)/\{1\} \to L^p(a,b)/\{1\},$$

with $-\infty < a < b < \infty$ and $1 < p < \infty$. We show that the approximation numbers $a_n(E_i)$ of E_i have the property that

$$\lim_{n\to\infty} na_n(E_i) = c_p(b-a) \qquad (i = 1,2)$$

where c_p is a constant dependent only on p. Moreover we show the precies value of $a_n(E_1)$ and we study the unbounded Sobolev embedding $E_3 : L^{1,p}(a,b) \to L^p(a,b)$ and determine precisely how closely it may be approximated by n-dimensional linear maps.

Key words: Approximation numbers, Sobolev Embedding, Hardy-type operators, Integral operators
1991 MSC: 47G10, 47B10

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1 Introduction.

Let Ω be a bounded subset of $\mathbf{R}^n$ with smooth boundary, let $1 < p < \infty$ and consider the embedding

$$E_1 : W_0^{1,p}(\Omega) \to L^p(\Omega)$$

where $W_0^{1,p}(\Omega)$ is the usual first-order Sobolev space of functions with zero trace. This space is a closed subspace of the Sobolev space $W^{1,p}(\Omega)$. It is well-known that E_1 is compact. More precise information about E_1 is available via its approximation numbers, for there are positive constants c_1 and c_2, depending only on p and Ω, such that the m-th approximation number $a_m(E_1)$ of E_1 satisfies

$$\frac{c_1}{m} \leq a_m(E_1) \leq \frac{c_2}{m}, \qquad m \in \mathbf{N} \tag{1}$$

Of course, this is a very special case of quite general results concerning the approximation numbers of embeddings between function spaces, for which we refer to (T) and (ET).

When $p = 2$ it is possible to sharpen (1) by using the familiar relation

$$a_m(E_1) = \frac{1}{\lambda_m^{1/2}}$$

between the approximation numbers of E_1 and the eigenvalues λ_m of the Dirichlet Laplacian. Since the behaviour of the eigenvalues is well-known, it follows that $\lim m a_m(E_1)$ exists; and even sharper statements about the asymptotic behaviour of $a_m(E_1)$ can be made. It is natural to ask whether or not $\lim m a_m(E_1)$ exists when p is not equal to 2.

In (EHL) a new technique was given for the study of the approximation numbers of the Hardy-type operator T on a tree Γ:

$$(Tf)(x) = v(x) \int_0^x f(t)u(t)dt, \qquad x \in \Gamma.$$

Using this it was shown that $T : L^p(\Gamma) \to L^p(\Gamma)$ has approximation numbers $a_m(T)$ for which $\lim m a_m(T)$ exists, when $1 \leq p \leq \infty$. This technique was improved and extended in (EKL), where in the case in which Γ is an interval and $p = 2$, remainder estimates were obtained. These results were extended in (L) to cover the cases $1 < p < \infty$.

In the present paper we obtain sharper information about $a_m(E_1)$ than was previously known. We deal only with the case in which $n = 1$ and Ω is a bounded interval in the line. The techniques of this paper are based on methods derived from (EHL), (EKL), (L), (Li2) and (DM). In more detail, for the Sobolev embeddings

$$E_1 : W_0^{1,p}(I) \to L^p(I)$$
$$E_2 : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\},$$

where $I = (a, b)$, $-\infty < a < b < \infty$ and $L^{1,p}(I)$ is the space of all $u \in L^p_{loc}(I)$ with derivative $u' \in L^p(I)$, we show that there is a positive constant α_p such that

$$\lim_{m\to\infty} m a_m(E_i) = \alpha_p |I| \qquad \text{for } i = 1 \text{ or } 2.$$

Moreover, it turns out that for every $m \in \mathbf{N}$, there is a linear map P_m with $\operatorname{rank} P_m = m$ such that

$$\|E_2 - P_m\| = \alpha_p |I|/m \geq a_{m+1}(E_2) \geq \alpha_p |I|/(m+1).$$

For embedding E_1 we have that for every $m \in \mathbf{N}$, there is a linear map B_m with $\operatorname{rank} B_m = m$ such that

$$\|E_1 - B_m\| = \alpha_p |I|/(m+1) = a_{m+1}(E_1).$$

We also study the best approximation of the unbounded Sobolev embedding

$$E_3 : L^{1,p}(I) \to L^p(I)$$

by linear maps of finite rank. We show that for every $m \in \mathbf{N}$, there is a linear map R_m with $\operatorname{rank} R_m = m$ such that

$$\|E_3 - R_m\| = \alpha_p |I|/(m) = \inf\{\|E_3 - P\|; P \text{ linear map}, \operatorname{rank} P < m+1\}.$$

We also show that $\alpha_p = (\frac{1}{\lambda_{n,I}})^{1/p}$ where $\lambda_{n,I}$ is the first eigenvalue of a $p-$Laplacian eigenvalue problem.

Our conclusion appears to be the first result of this kind in the literature, apart from the special case $p = 2$. It remains to be seen whether or not this can be extended to higher dimensions.

2 Preliminaries and technical results.

Throughout the paper we shall assume that $-\infty < a < b < \infty$ and that $I = (a, b)$. We also assume that $1 < p < \infty$ and denote by $\|.\|_p$ or $\|.\|_{p,I}$ the usual norm on the Lebesgue space $L_p(I)$.

By the Sobolev space $W_0^{1,p}(I)$ we understand, as usual, the space of all functions $u \in L^p(I)$ with finite norm $\|u'\|_{p,I}$ and zero trace. We consider the embedding

$$E_1 : W_0^{1,p}(I) \to L^p(I) \tag{2}$$

and define the norm of E_1 by

$$\|E_1\| = \sup_{\|u'\|_{p,I}>0} \frac{\|u\|_{p,I}}{\|u'\|_{p,I}}. \tag{3}$$

Plainly $\|E_1\| < \infty$; moreover, it is well known (see, for example, (EE), Theorem V.4.18) that E_1 is compact.

We will consider in this paper also the approximation numbers for the embedding

$$E_2 : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\},$$

where $L^{1,p}(I)$ is the space of all functions $u \in L^p_{loc}(I)$ with finite pseudonorm $\|u'\|_{1,p}$ which vanishes on the subspace of all constant functions. By $L^{1,p}/\{1\}$ we mean the factorization of the space $L^{1,p}(I)$ with respect to constant functions, equipped with the norm $\|u'\|_{p,I}$. Then we have $f \in L^{1,p}/\{1\}$ if and only if $\|f\|_{p,I} = \inf_{c\in\Re} \|f - c\|_{p,I}$. In a similar way $L^p(I)/\{1\}$ is defined. The norm of E_2 is defined by

$$\|E_2\| = \sup_{\|u'\|_p>0} \frac{\|u\|_p}{\|u'\|_p}.$$

It is obvious that $\|E_2\| = a_1(E_2) < \infty$ and also $\lim_{n\to\infty} a_n(E_2) = 0$.

We will also consider the unbounded embedding

$$E_3 : L^{1,p}(I) \to L^p(I).$$

Since $L^{1,p}(I)$ is defined by the pseudonorm $\|u'\|_{1,p}$ and E_3 is unbounded, we will study the best approximation of E_3 by linear maps of finite rank ($a_n(E_3)$ are not well defined).

Definition 2.1 *Let $J = (c, d) \subset I$. We define*

$$A_0(J) = \sup_{\|u'\|_{p,J}>0} \inf_{\alpha\in\Re} \frac{\|u-\alpha\|_{p,J}}{\|u'\|_{p,J}}.$$

Since every function in $W^{1,p}(J)$ is absolutely continuous, we can rewrite $A_0(J)$ as

$$A_0(J) = \sup_{\|u'\|_{p,J}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt + u(c) - \alpha\|_{p,J}}{\|u'\|_{p,J}}.$$

From this we can see the connection between A_0 and the Hardy operator.

Lemma 2.2 *Let I_n be a decreasing sequence of subintervals of I with $|I_n| \to 0$ as $n \to \infty$. Then $\{A_0(I_n)\}$ is a decreasing sequence bounded above by $A_0(I)$ and with limit 0.*

Proof. In this proof we extend $u \in W^{1,p}(I_{n+1})$ outside I_{n+1} by a constant, i.e. $u' = 0$ outside I_{n+1}. From the definition of A_0 we have for $I_{i+1} \subset I_i$,

$$\begin{aligned} A_0^p(I_{i+1}) &= \sup_{\|u'\|_{p,I_{i+1}}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|_{p,I_{i+1}}^p}{\|u'\|_{p,I_{i+1}}^p} \\ &\leq \sup_{\|u'\|_{p,I_{i+1}}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|_{p,I_i}^p}{\|u'\|_{p,I_i}^p} \\ &\leq \sup_{\|u'\|_{p,I_i}>0} \inf_{\alpha\in\Re} \frac{\| \int_c^x u'(t)dt - \alpha\|_{p,I_i}^p}{\|u'\|_{p,I_i}^p} = A_0^p(I_i) \end{aligned}$$

and so $A_0(I_i) \geq A_0(I_{i+1})$. For $A_0(J)$ we have

$$\begin{aligned} A_0(J) &\leq \sup_{\|u'\|_{p,J}=1} \frac{\|\int_c^x u'(t)dt\|_{p,J}}{\|u'\|_{p,J}} \\ &= \sup_{\|u'\|_{p,J}=1} \left\| \int_c^x |u'(t)|dt \right\|_{p,J} \\ &\leq \sup_{\|u'\|_{p,J}=1} \left\| \left(\int_J |u'|^p \right)^{1/p} |J|^{1/p'} \right\|_{p,J} = |J|^{1/p'}. \end{aligned}$$

From this observation it follows that $A_0(I_n) \to 0$ as $I_n \to 0$. $\square$

Lemma 2.3 *Let $J = (x, y) \subset I$. Then $A_0((x, y))$ is a continuous function of x and y.*

Proof. Let us suppose that there are $x, y \in I$ and $\varepsilon > 0$ such that $A_0(x, y + h_n) - A_0(x, y) > \varepsilon$ for some sequence $\{h_n\}$ with $0 < h_n \searrow 0$. Then we have that there is $\varepsilon_1 > 0$ such that $A_0^p(x, y + h_n) - A_0^p(x, y) > \varepsilon_1$ for any $n \in \mathbf{N}$. But for all $h > 0$,

$$
\begin{aligned}
A_0^p(x, y+h) - A_0^p(x, y) = &\sup_{\|u'\|_{p,(x,y+h)}>0} \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y+h)}}{\|u'\|^p_{p,(x,y+h)}} \\
&- \sup_{\|u'\|_{p,(x,y)}>0} \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y)}} \\
\le &\sup_{\|u'\|_{p,(x,y+h)}>0} \Big(\inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y+h)}}{\|u'\|^p_{p,(x,y+h)}} \\
&- \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}} \Big) \\
\le &\sup_{\|u'\|_{p,(x,y+h)}>0} \Big(\inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}} \\
&+ \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(y,h)}}{\|u'\|^p_{p,(x,y+h)}} - \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{\|u'\|^p_{p,(x,y+h)}} \Big) \\
\le &\sup_{\|u'\|_{p,(x,y+h)}>0} \inf_{\alpha\in\mathbf{R}} \frac{\| \int_x^t u' - \alpha\|^p_{p,(x,y)}}{|u'|^p_{p,(x,y+h)}} \\
\le &\sup_{\|u'\|_{p,(y,y+h)}>0} \frac{\| \int_y^t u'\|^p_{p,(y,y+h)}}{\|u'\|^p_{p,(y,y+h)}} \\
\le &|(y, y+h)|^{p/p'} \le h^{p/p'},
\end{aligned}
$$

and we have a contradiction. Hence $A_0(x, y+h) \to A_0(x, y)$ as $h \to 0$. Similarly we find that $A_0(x + h, y) \to A_0(x, y)$ as $h \to 0$ and the result follows. $\square$

Lemma 2.4 *Let $J = (c, d) \subset I$. Then there is a function $f \in W^{1,p}(J)$ such that*

$$
A_0(J) = \frac{\|f\|_{p,J}}{\|f'\|_{p,J}} = \inf_{\alpha\in\mathbf{R}} \frac{\|f - \alpha\|_{p,J}}{\|f'\|_{p,J}}.
$$

Proof: It is possible to find a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $W^{1,p}(J)$ such that for each n in $\mathbf{N}$,

$$\frac{\|f_n\|_{p,J}}{\|f_n'\|_{p,J}} + 1/n = \inf_{\alpha \in \mathbf{R}} \frac{\|f_n - \alpha\|_{p,J}}{\|f_n'\|_{p,J}} + 1/n > A_0(J)$$

and $\|f_n\|_{W^{1,p}(J)} = 1$. Since E is compact, it follows that there exists a subsequence of $\{f_n\}$, again denoted by $\{f_n\}$ for convenience, which converges weakly in $W^{1,p}(J)$, to f, say, and this subsequence converges strongly to f in $L^p(J)$. By a standard compactness argument we get that f_n converges strongly to f in $W^{1,p}(J)$ and then

$$A_0(J) = \frac{\|f\|_{p,J}}{\|f'\|_{p,J}} = \inf_{\alpha \in \mathbf{R}} \frac{\|f - \alpha\|_{p,J}}{\|f'\|_{p,J}}.$$

□

Lemma 2.5 *Let $J = (c, d) \subset I$ and let f be as in the previous lemma. Then $f(x) = 0$ only for $x = (c + d)/2$, f is monotone and $f'(c_+) = f'(d_-) = 0$.*

Proof: Let f be from the previous lemma. Let $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$; then $\|f_+\|^p_{p,J} = \|f_-\|^p_{p,J}$, $f = f_+ - f_-$ and $|\{x : f(x) = 0\}| = 0$. Since we know that for any $g \in W^{1,p}(J)$, $g \geq 0$ we have $\|g'\|_{p,J} \geq \|(g^*)'\|_{p,(0,|J|)}$ (where g^* is the non-increasing rearrangament of the function g). Then we have that

$$\frac{\|f_+^*\|^p_{p,(0,|J|)} + \|f_-^*\|^p_{p,(0,|J|)}}{\|(f_+^*)'\|^p_{p,(0,|J|)} + \|(f_-^*)'\|^p_{p,(0,|J|)}} = A_0^p(J).$$

Now define $r = |\{x : f(x) > 0\} \cap J|$ and $g(x) = f_+^*(c + r - x)$ for $c \leq x \leq c + r$ and $g(x) = -f_-^*(c + r + x)$ for $c + r \leq x \leq d$. Then

$$\frac{\|g\|_{p,J}}{\|g'\|_{p,J}} = A_0(J),$$

and $\|g_+\|^p_{p,J} = \|g_-\|^p_{p,J}$.

From all this we can see that we have found a function g such that: g is monotone, $g(c + r) = 0$ where $c < c + r < d$ and $(\|g\|_{p,J}/\|g'\|_{p,J}) = A_0(J)$.

Now we show that $g((c + d)/2) = 0$ (i.e. $r = (c + d)/2$). Put $J_1 = (c, c + r)$ and $J_2 = (c + r, d)$; then we have

$$\frac{\|g\|^p_{p,J_1} + \|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_1} + \|g'\|^p_{p,J_2}} = A_0^p(J). \tag{4}$$

Since $A_0(J) = |J|A_0((0,1))$, we see that

$$\frac{\|g\|^p_{p,J_1}}{\|g'\|^p_{p,J_1}} \leq A_0^p((0,1))|J_1|^p 2^p.$$

For if not then we can define $h(x) = g(x)$ on $(c, c+r)$ and $h(x) = -g(-x + 2(r+c))$ on $(c+r, c+2r)$ and we have that $\inf_{\alpha\in\Re} \|h-\alpha\|_{p,(c,c+2r)} = \|h\|_{p,(c,c+2r)}$ and

$$\frac{\|h\|^p_{p,(c,c+2r)}}{\|h'\|^p_{p,(c,c+2r)}} > A_0^p((c, c+2r)),$$

which is a contradiction with the definition of A_0. Similarly we have

$$\frac{\|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_2}} \leq A_0^p((0,1))|J_2|^p 2^p.$$

Observe that (4) holds if and only if

$$\frac{\|g\|^p_{p,J_1}}{\|g'\|^p_{p,J_1}} = \frac{\|g\|^p_{p,J_2}}{\|g'\|^p_{p,J_2}} = A_0^p(J)$$

(do not forget that $\|g\|^p_{p,J_1} = \|g\|^p_{p,J_2}$). This means that $c + r = (c+d)/2$ and moreover we can suppose that $g(x) = -g(-x + (c+d))$ (i.e. $g(x)$ is odd with respect to $(c+d)/2$).

Next we show that $g'(c) = g'(d) = 0$. Note that $g(c) = -g(d) \geq 0$. Suppose that $g'(c) = -g'(d) < 0$; then there are a number $z > 0$ and a sequence of numbers $\{x_n\}_{n=1}^\infty$ such that $x_n > c$, $x_n \to c$ and

$$\frac{g(c) - g(x_n)}{c - x_n} < z < 0$$

(i.e. $\int_c^{x_n} g'(t)dt < (x_n - c)z$). A similar procedure can be carried out in the neighbourhood of d.

Then we have $|z|(x_n - c) < \int_c^{x_n} |g'(t)|dt \leq (\int_c^{x_n} |g'(t)|^p dt)^{1/p}(x_n - c)^{1/p'}$. And also we have

$$A_0^p(J) = \frac{\int_{x_n}^d |g|^p + \int_c^{x_n} |g|^p}{\int_{x_n}^d |g'|^p + \int_c^{x_n} |g'|^p} \leq \frac{\int_{x_n}^d |g|^p + (x_n - c)|g(c)|^p}{\int_{x_n}^d |g'|^p + (x_n - c)|z|^p}$$

Since $A_0(J) > 0$ and $|z| > 0$, plainly

$$|g(c)|^p < |z|^p A_0^p(J) + |g(c)|^p$$

and there exists $n_1 \in \mathbf{N}$ such that for any $n > n_1$ we have

$$(x_n - c)|g(c)|^p < (x_n - c)|z|^p \frac{\int_{x_n}^d |g|^p}{\int_{x_n}^d |g'|^p} + (x_n - c)|g(c) - z(x_n - c)|^p$$

and then

$$\begin{aligned}(\int_{x_n}^{d} |g|^p)(\int_{x_n}^{d} |g'|^p) + (x_n - c)|g(c)|^p(\int_{x_n}^{d} |g'|^p) <& \\ & (\int_{x_n}^{d} |g|^p)(\int_{x_n}^{d} |g'|^p) + (x_n - c)|z|^p(\int_{x_n}^{d} |g|^p) \\ & + (x_n - c)|g(c) - z(x_n - c)|^p(\int_{x_n}^{d} |g'|^p) \\ & + |z|^p|g(c) - z(c - x_n)|^p(c - x_n)^2.\end{aligned}$$

From this it follows that for any $n > n_1$,

$$\frac{\int_{x_n}^d |g|^p + (x_n - c)|g(c)|^p}{\int_{x_n}^d |g'|^p + (x_n - c)|z|^p} < \frac{\int_{x_n}^d |g|^p + (x_n - c)|g(x_n)|^p}{\int_{x_n}^d |g'|^p}.$$

But this means that for $l_n = \chi_{(x_n,d)}g + \chi_{(c,x_n)}g(x_n)$ we have:

$$A_0^p(J) < \frac{\int_c^d |l_n|^p}{\int_c^d |l'_n|^p} \qquad \text{for any } n > n_1.$$

In view of the antisymmetry of g we define a function
$r_n(x) = \chi_{(c,d+c-x_n)}g(x) + \chi_{(d+c-x_n,d)}g(d + c - x_n)$, and have

$$A_0^p(J) < \frac{\int_c^d |r_n|^p}{\int_c^d |r'_n|^p} \qquad \text{for any } n > n_1.$$

Finally we define $k_n(x) = \chi_{(x_n,d+c-x_n)}g(x)+\chi_{(d+c-x_n,d)}g(d+c-x_n)+\chi_{(c,x_n)}g(x_n)$. Then for n large enough we have

$$A_0(J) < \inf_{c\in\mathbf{R}} \frac{\|k_n - c\|_{p,J}}{\|k'_n\|_{p,J}}.$$

But this contradicts the definition of $A_0(J)$: hence $g'(c) = g'(d) = 0$. $\square$

Now we recall the p-Laplacian eigenvalue problem, which is defined, for $1 < p < \infty$, $\lambda > 0$ and $T > 0$ by

$$(|u'|^{p-2}u')' + \lambda|u|^{p-2}u = 0, \text{ on } (0,T),$$
$$u'(0) = 0, \qquad u'(T) = 0.$$

The set of eigenvalues of this problem is given by

$$\lambda_n := \left(\frac{2n\pi_p}{T}\right)^p \frac{1}{p'p^{p-1}} \text{ for each } n \in \mathbf{N}.$$

The corresponding eigenfunctions are $u_0(t) = c$, $c \in \mathbf{R} \setminus \{0\}$ and

$$u_n(t) = \frac{T}{n\pi_p} \sin_p\left(\frac{n\pi_p}{T}(t - \frac{T}{2n})\right).$$

Here for $p > 1$ we put $p' = \frac{p}{p-1}$ and $\pi_p = 2B(\frac{1}{p}, \frac{1}{p'}) = \pi/\sin(\pi/p)$, where B denotes the beta function. Moreover $\sin_p(.)$ can be defined as the unique (global) solution to the initial–value problem

$$(|u'|^{p-2}u')' + \frac{2^p}{p'p^{p-1}}|u|^{p-2}u = 0$$
$$u(0) = 0, \qquad u'(0) = 1.$$

Also $\sin_p$ can be expressed in terms of hypergeometric functions, see ((AS), p.263),

$$\operatorname{arcsin}_p(s) = ps^{1/p}F(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}; s),$$

or

$$\operatorname{arcsin}_p(s) = B(\frac{1}{p}, \frac{1}{p'}, (\frac{2s}{p})^p)$$

where $F(a, b, c; s)$ denotes the hypergeometric function and B is the incomplete beta function

$$B(1/q, 1/p', x) = \int_0^x z^{1/q-1}(1-z)^{-1/p}dz,$$

see (AS).

Moreover, for $s \in [0, p/2]$ we have

$$\arcsin_p(s) = \frac{p}{2} \int_0^{\frac{2s}{p}} \frac{dt}{(1-t^p)^{1/p}},$$

(note that this integral converges for all $s \in [0, p/2]$).

We note that in this paper we are using the definition of π_p and $\sin_p$ functions from the paper (DM) which is slightly different from the definition of π_p and the $\sin_p$ function used in (Li1) and (Li2).

Note that as $\arcsin_p : [0, p/2] \to [0, \pi_p/2]$ is strictly increasing then its inverse function $\sin_p : [0, \pi_p/2] \to [0, p/2]$ is also strictly increasing.

We extended $\sin_p$ from $[0, \pi_p/2]$ to all $\mathbf{R}$ as a $2\pi_p$ periodic function by the usual way as in the $p = 2$ case (i.e. from sin).

For later use let us define $\cos_p(t) := \sin_p'(t)$. We have that

$$\left(\frac{p}{2}\right)^p |\cos_p(t)|^p + |\sin_p(t)|^p = 1 \text{ for all } t \in \mathbf{R},$$

and

$$\pi_p = \pi_{p'}.$$

From (DM) we have

$$\int_0^T |\sin_p(\frac{n\pi_p}{T}t)|^p dt = \frac{Tp'p^p}{2^p(p'+p)}$$

and

$$\int_0^T |\frac{d}{dt}\sin_p(\frac{n\pi_p}{T}t)|^p dt = \frac{n^p\pi_p^p p}{T^{p-1}(p'+p)}.$$

See (Li2) for more information about $\sin_p(.)$ and $\cos_p(.)$ functions.

Definition 2.6 *Given $J = [c, d] \subset \mathbf{R}$ we denote by $u_{n,J}(t)$ the n-th eigenfunction of the p-Laplacian eigenvalue problem on J and by $\lambda_{n,J}$ the corresponding n-th eigenvalue.*

Note that

$$u_{0,J} = C,$$

$$u_{n,J}(t) = \frac{|J|}{n\pi_p} \sin_p \left(\frac{n\pi_p}{|J|}(t - \frac{|J|}{2n} - c) \right), \qquad \text{for } n \geq 1$$

and

$$\lambda_{n,J} = \left(\frac{2n\pi_p}{|J|} \right)^p \frac{1}{p' p^{p-1}}, \qquad \text{for each } n \in \mathbf{N},$$

where $\pi_p = \pi / \sin(\pi/p)$. It is simple to see that for any $n \in \mathbf{N}$, $\{u_{i,J}\}_{i=1}^n$ is a linearly independent set.

Lemma 2.7 *Let $J = (c, d) \subset I$. Then*

$$A_0(J) = \frac{\|u_{1,J}\|_{p,J}}{\|u'_{1,J}\|_{p,J}} = \inf_{\alpha \in \mathbf{R}} \frac{\|u_{1,J} - \alpha\|_{p,J}}{\|u'_{1,J}\|_{p,J}} = \left(\frac{1}{\lambda_{1,J}} \right)^{1/p}.$$

Proof: We can see that

$$A_0(J) = \sup_{u \in K(J)} \frac{\|u_{1,J}\|_{p,J}}{\|u'_{1,J}\|_{p,J}}$$

where $K(J) = \{f; 0 < \|f'\|_{p,J} < \infty, \inf_\alpha \|f - \alpha\|_{p,J} = \|f\|_{p,J}\}$. After taking the Fréchet derivative of $A_0^p(J)$ we can see that this lemma follows from the previous observation about eigenfunction and eigenvalues for the p-Laplacian problem with Neumann boundary value conditions together with Lemma 4 (more can be found in (DKN)) $\square$

We recall that, given any $m \in \mathbf{N}$, the m–th approximation number $a_m(T)$ of a bounded linear operator $T : X \to Y$, where X and Y are Banach spaces, is defined by

$$a_m(T) := \inf \|T - F|X \to Y\|,$$

where the infimum is taken over all bounded linear maps $F : X \to Y$ with rank less than m.

A measure of non-compactness of T is given by

$$\beta(T) := \inf \|T - P|X \to Y\|,$$

where the infimum is taken over all compact linear maps $P : X \to Y$. In our case we have $X = W^{1,p}(I)$ and $Y = L^p(I)$. Then since $L^p(I)$ has the approximation property for $1 \le p \le \infty$, T is compact if and only if $a_m(T) \to 0$ as $m \to \infty$, and $\beta(T) = \lim_{n\to\infty} a_n(T)$.

3 The Main Theorem.

Definition 3.1 *Let $\varepsilon > 0$ and $I = (a,b) \subset \mathbf{R}$. We define*

$$N(\varepsilon, I) = \inf\{n; I = \cup_{i=1}^{n} I_i, A(I_i) \le \varepsilon, |I_i \cap I_j| = 0 \text{ for } i \ne j\}.$$

From our previous observation that $A_0(J) = \left(\frac{1}{\lambda_{1,J}}\right)^{1/p} = (p'p^{p-1})^{1/p}\frac{|J|}{2\pi_p}$ we have:

Observation 3.2 *i) Given any $\varepsilon > 0$ we have $N(\varepsilon, I) < \infty$.*

ii) Let $\varepsilon > 0$. Then there is a covering set of intervals (that is, a set of non-overlapping intervals)

$\{I_i\}_{i=1}^{N(\varepsilon)}$ such that $A_0(I_i) = \varepsilon$ for $i = 1, ..., N(\varepsilon)$ and $A_0(I_{N(\varepsilon,I)}) \le \varepsilon$.

iii) For any $n \in \mathbf{N}$ there exist $\varepsilon > 0$, such that $n = N(\varepsilon, I)$ and corresponding covering sets $\{I_i\}_{i=1}^{N(\varepsilon,I)}$ for which $A_0(I_i) = \varepsilon$ for $i = 1, ...N(\varepsilon, I)$.

Moreover we can see:

Observation 3.3 *Let $n \in \mathbf{N}$ and $\varepsilon \in \left[\frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}, \frac{|I|}{2(n-1)\pi_p}(p'p^{p-1})^{1/p}\right)$. Then $N(\varepsilon, I) = n$.*

From this observation we obtain the following two lemmas as in (EEH2).

Lemma 3.4 *Let $n \in \mathbf{N}$. Then*

$$a_n(E_1) \le \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

and

$$a_{n+1}(E_2) \le \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

and

$$\inf \|E_3 - P_{n+1}\| \leq \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ *with* rank *less than* $n+1$.

Proof: Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Set $Pf = \sum_{i=1}^n P_i f$ where

$$P_i f(x) := \chi_{I_i}(x)(f((a_i+b_i)/2)), \text{ where } I_i = (a_i, b_i).$$

We can see that $P_i f$ is a linear map from $L^{1,p}(I_i)$ into $L^p(I_i)$ (not necessarily bounded) and it is a bounded linear map from $L^{1,p}(I_i)/\{1\}$ into $L^p(I_i)$ with rank less or equal to 1. Then rank $P \leq n$ and P is a linear map from $L^{1,p}(I)$ into $L^p(I)$ and it is a linear map from $L^{1,p}(I)/\{1\}$ into $L^p(I)$. From (Li1) and Lemma 5 we have that $A_0(I_i) = \sup_{\|u'\|_{p,I_i}>0} \frac{\|u-P_i u\|_{p,I_i}}{\|u'\|_{p,I_i}}$. Then we have:

$$\begin{aligned}
\|(E_3 - P)f\|_{p,I}^p &= \sum_{i=1}^n \|(E_3 - P)f\|_{p,I_i}^p \\
&= \sum_{i=1}^n \|(f(.) - f((a_i+b_i)/2)\|_{p,I_i}^p \\
&\leq \sum_{i=1}^n A_0^p(I_i)\|f'\|_{p,I_i}^p \\
&\leq \sum_{i=1}^n \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I_i}^p \\
&\leq \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I}^p.
\end{aligned}$$

(Note that $\|f - f((a+b)/2)\|_{p,I}/\|f'\|_{p,I} < \infty$ for any $f \in L^{1,p}(I)$.) From this follows the third inequality for E_3.

The proof of the inequality for E_2 is the same.

For the first inequality for $a_n(E_1)$ we have to define a new partition of I. Let $\{I_i\}_1^n$ by the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Put $J_i = (a_i + |I_i|/2, b_i + |I_i|/2)$ for $i = 1, \ldots, n-1$ and $J_0 = (a, a + |I_1|/2)$, $J_n = (a_n + |I_n|/2, b)$ where $I_i = (a_i, b_i)$. Define $\{c_i\}_0^n$ and $\{d_i\}_0^n$ by $J_i = (c_i, d_i)$.

Set $Gf = \sum_{i=0}^n G_i f$ where $G_i f(x) := \chi_{J_i}(x)(f((c_i+d_i)/2))$, for $i = 1, \ldots, n-1$, $G_0 f(x) := f(a) = 0$ and $G_n f(x) := f(b) = 0$ where $I_i = (a_i, b_i)$. Then

$\operatorname{rank} G \le n_1$ and G is a bounded linear map from $W_0^{1,p}(I)$ into $L^p(I)$. Since $A_0^p(I_i) = A_0^p(J_j)$ then as before we have for $f \in W_0^{1,p}(I)$:

$$\begin{aligned}
\|(E_1 - G)f\|_{p,I}^p &= \sum_{i=0}^{n} \|(E-P)f\|_{p,J_i}^p \\
&= \sum_{i=1}^{n-1} \|(f(.) - f((a_i+b_i)/2)\|_{p,J_i}^p + \|f\|_{p,J_0}^p + \|f\|_{p,J_n}^p \\
&\le \sum_{i=1}^{n-1} A_0^p(J_i)\|f'\|_{p,J_i}^p + A_0^p(I_1)\|f'\|_{p,I_0}^p + A_0^p(I_n)\|f'\|_{p,J_n}^p \\
&\le \sum_{i=0}^{n} \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I_i}^p \\
&\le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_{p,I}^p.
\end{aligned}$$

From this follows the first inequality for $a_n(E_1)$.

□

From the proof of Lemma 6 we can see that for any n there exists K_n, an n-dimensional linear subspace of L^p, such that for any $f \in L^{1,p}(I)/\{1\}$ (or from any $f \in L^{1,p}(I)$) we have

$$\inf_{g \in K_n} \|f - g\|_p^p \le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_p^p.$$

Moreover, for any n there exists R_{n-1}, an $n-1$ dimensional linear subspace of L^p, such that for any $f \in W_0^{1,p}(I)$ we have

$$\inf_{g \in R_{n-1}} \|f - g\|_p^p \le \frac{|I|}{2n\pi_p}(p'p^{p-1})\|f'\|_p^p.$$

Lemma 3.5 *Let $n \in \mathbf{N}$. Then*

$$a_n(E_1) \ge \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

and

$$a_n(E_2) \ge \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}.$$

and

$$\inf \|E_3 - P_{n+1}\| \geq \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ *with* rank *less than* $n+1$.

Proof: First we prove the second inequality for E_2. Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$.

From the definition of $A_0(I_i)$ we know that for $i = 1, ..., n$ there exists $\varphi_i \in W^{1,p}(I_i)$, $\|\varphi_i'\|_{p,I_i} = 1$ such that

$$\inf_{\alpha \in \mathbf{R}} \|\varphi_i - \alpha\|_{p,I_i} = A_0(I_i) = \varepsilon.$$

We extend each φ_i to I by taking $\varphi_i' = 0$ outside I_i and define $\phi_i = \varphi_i + c_i$ where $c_i \in \mathbf{R}$ is such that $\phi_i \in L^{1,p}/\{1\}$.

Let $P : L^{1,p}(I)/\{1\} \to L^p(I)/\{1\}$ be a bounded linear operator with rank$(P) < n$. Then there are constants $\lambda_1, ..., \lambda_n$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^n \lambda_i \phi_i.$$

Note that $\phi \in L^p(I)/\{1\}$. Then, noting that the following summation is over $\lambda_i \neq 0$,

$$\begin{aligned} \|E_2\phi - P\phi\|_{p,I}^p = \|E_2\phi\|_{p,I}^p &= \sum_{i=1}^n \|\phi\|_{p,I_i}^p \\ &\geq \sum_{i=1}^n \inf_\alpha \|\phi - \alpha\|_{p,I_i}^p \geq \sum_{i=1}^n \inf_\alpha \|\phi_i - \alpha\|_{p,I_i}^p |\lambda_i|^p \\ &\geq \varepsilon^p \sum_{i=1}^n \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p. \end{aligned}$$

Then we have that $\|E_2 - P\|_{p,I} \geq \varepsilon$, so that $a_n(E_2) \geq \varepsilon$.

We prove the inequality for E_3 in the same way as for E_2. Let $P : L^{1,p}(I) \to L^p(I)$ be a linear operator with rank$(P) < n+1$. Let we have the system of functions $\{\phi_i\}_{i=1}^n$ considered previously and put $\phi_{n+1} = 1$; then we have $n+1$ linearly independent functions from $L^{1,p}(I)$ (note that $W^{1,p}(I)/\{1\} \subset L^{1,p}(I)$).

Then there are constants $\lambda_1, ..., \lambda_{n+1}$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^{n+1} \lambda_i \phi_i.$$

Then, noting that the following summation is over $\lambda_i \neq 0$ we have

$$\begin{aligned} \|E_3\phi - P\phi\|_{p,I}^p = \|E_3\phi\|_{p,I}^p &= \sum_{i=1}^{n+1} \|\phi\|_{p,I_i}^p \\ &\geq \sum_{i=1}^{n+1} \inf_{\alpha} \|\phi - \alpha\|_{p,I_i}^p \geq \sum_{i=1}^{n+1} \inf_{\alpha} \|\phi_i - \alpha\|_{p,I_i}^p |\lambda_i|^p \\ &\geq \varepsilon^p \sum_{i=1}^{n+1} \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p. \end{aligned}$$

Hence $\|E_3 - P\|_{p,I} \geq \varepsilon$ and then the third inequality for E_3 is satisfied.

Now we prove the inequality for $a_n(E_1)$. Take $u_{n,I}$ the n-th eigenfunction of the $p-$Laplacian eigenvalue problem on I with Neumann boundary condition. Let $\{I_i\}_1^n$ be the partition from Observation 4 with $\varepsilon = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$. Then we define $\phi_i = u_{n,I}\chi_{I_i}$ and $\phi_i \in W_0^{1,p}(I_i)$ and $\|\phi_i\|_{p,I}/\|\phi_i'\|_{p,I} = A_0(I_i)$. Let $P : L^{1,p}(I) \to L^p(I)$ be a linear operator with $\text{rank}(P) < n$. Then there are constants $\lambda_1, ..., \lambda_n$, not all zero, such that

$$P\phi = 0, \qquad \phi = \sum_{i=1}^{n} \lambda_i \phi_i.$$

Noting that the following summation is over $\lambda_i \neq 0$ we have

$$\begin{aligned} \|E_1\phi - P\phi\|_{p,I}^p = \|E_1\phi\|_{p,I}^p &= \sum_{i=1}^{n} \|\phi\|_{p,I_i}^p \\ &\geq \sum_{i=1}^{n} \|\phi\|_{p,I_i}^p \geq \sum_{i=1}^{n} \|\phi_i\|_{p,I_i}^p |\lambda_i|^p \\ &\geq \varepsilon^p \sum_{i=1}^{n+1} \|\phi_i'\|_{p,I_i}^p |\lambda_i|^p \geq \varepsilon^p \|\phi'\|_{p,I}^p. \end{aligned}$$

Thus $\|E_1 - P\|_{p,I} \geq \varepsilon$ and so the third inequality for $a_n(E_1)$ is satisfied. □

The previous two lemmas give us:

Theorem 3.6 *If $|I| < \infty$, then*

$$a_n(E_1) = \frac{|I|}{2(n)\pi_p}(p'p^{p-1})^{1/p},$$

$$\frac{|I|}{2(n-1)\pi_p}(p'p^{p-1})^{1/p} \geq a_n(E_2) \geq \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

and

$$\inf \|E_3 - P_{n+1}\| = \frac{|I|}{2n\pi_p}(p'p^{p-1})^{1/p}$$

where the infimum is taken over all linear maps $P_{n+1} : L^{1,p}(I) \to L^p(I)$ with rank less than $n+1$.

Then

$$\lim_{n\to\infty} a_n(E_1)n = \frac{|I|}{2\pi_p}(p'p^{p-1})^{1/p},$$

and

$$\lim_{n\to\infty} a_n(E_2)n = \frac{|I|}{2\pi_p}(p'p^{p-1})^{1/p},$$

where $\pi_p = \pi/\sin(\pi/p)$.

Acknowledgement: Both authors thank the Leverhulme Foundation for its support; the second author is also indebted to the Ohio State Univeristy for an international travel grant.

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