BOUNDEDNESS AND COMPACTNESS OF GENERAL KERNEL INTEGRAL OPERATORS FROM A WEIGHTED BANACH FUNCTION SPACE INTO L_∞

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Abstract. We give necessary and sufficient conditions for boundedness and compactness of a general kernel integral operator $Lf(x) = \int_I \ell(x,t)f(t)\mathrm{d}t$, where the kernel ℓ is assumed only to be measurable, from an arbitrary weighted Banach function space into L_∞. We give lower and upper bounds for the distance of L from compact operators. The proofs are carried out by means of a new method based on real-variable techniques.

1. Introduction

The problem of boundedness and compactness of kernel integral operators $Lf(x) = \int_I \ell(x,t)f(t)\mathrm{d}t$, where $\ell(x,t)$ is a general measurable function on I^2 and I is an interval, and their distance from compact operators, have been studied by many authors (cf. e.g. [7], [3], [9], [8], or the monograph [2]). Usually, for L_p–L_q type estimates with $q < \infty$, the authors use rather restrictive assumptions on the kernels. Typically (see for example [[8], (1.3)]), the kernel is supposed to be positive, monotone in each variable, locally uniformly continuous, and satisfying certain triangle inequality.

The situation turns out to be different when the target space is L_∞. For example, in [5], boundedness and compactness of the Hardy operator $Hf(x) = \int_0^x f(t)dt$ from a weighted Banach function space (X,v) into L_∞ was characterized by relatively simple conditions. The methods from [5] can be immediately generalized to kernel operators $Tf(x) = \int_0^\infty k(x,t)f(t)\mathrm{d}t$, but only when k is positive and monotone in the first variable.

In this paper we develop a different method based on real-variable methods and measure-theoretic considerations, which enables us to characterize completely boundedness and compactness of the kernel operator, assuming only that the kernel is measurable. A remarkable fact is that ℓ is allowed to take negative values. We further give sharp lower and upper bounds for the distance of L from the set of compact linear operators. Notably, it turns out that every compact operator can be approximated by operators with kernels of the form $k(x,t) = \sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)$, where $M_i \subset I$ and $\frac{\psi_i}{v} \in (X',v)$ (here (X',v) denotes the associate space to (X,v)).

In the particular case when (X,v) is separable, some of the results were obtained in [4].

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The paper is structured as follows: preliminary material and some basic facts on Banach function spaces are collected in Section 2 (the standard general reference is [6] or [1]). In Section 3 we characterize boundedness of L by means of the norm of ℓ in $L_\infty(X', v)$. This section also contains the key real-variable considerations. In Section 4 we characterize compactness of L, and in Section 5 we present lower and upper bounds for the distance of the Hardy operator from compact operators, recovering thereby, in particular, a result from [5].

2. Preliminaries

Let $-\infty \le a < b \le \infty$ and let $I = (a, b)$. Let $\mathfrak{M}(I)$ and $\mathfrak{M}(I^2)$ denote the sets of all measurable functions on I and I^2. Let v be a *weight* (that is, a measurable and a.e. positive and measurable function) on I.

Definition 2.1. *We say that a normed linear subspace (X, v) of $\mathfrak{M}(I)$ is a weighted Banach function space if the following five axioms are satisfied:*

(2.1) *the norm $\|f\|_{X,v}$ is defined for all $f \in \mathfrak{M}(I)$, and $f \in (X, v)$ if and only if $\|f\|_{X,v} < \infty$;*

(2.2) $\|f\|_{X,v} = \| \, |f| \, \|_{X,v}$ *for all* $f \in \mathfrak{M}(I)$;

(2.3) $0 \le f_n \nearrow f$ *a.e. in* I, *then* $\|f_n\|_{X,v} \nearrow \|f\|_{X,v}$;

(2.4) *if* $v(E) = \int_E v(t)dt < \infty$, *then* $\chi_E \in (X, v)$, *where* χ_E *denotes the characteristic function of* E;

(2.5) *for every E with $v(E) < \infty$ there exists a constant C_E such that* $\int_E f(t)v(t)dt \le C_E \|f\|_{X,v}$ *for all* $f \in (X, v)(I)$.

In what follows, (X, v) will be a fixed weighted Banach function space.

Definition 2.2. *The set*

$$(X', v) = \Big\{ f;\ \int_I |f(t)g(t)|v(t)dt < \infty \quad \textit{for all} \quad g \in (X, v) \Big\}$$

is called the associate space of (X, v). The space (X', v), equipped with the norm

$$\|f\|_{X',v} := \sup_{\|g\|_{X,v} \le 1} \Big| \int_I f(t)g(t)v(t)dt \Big|,$$

is also a weighted Banach function space. The Hölder inequality

$$\int_I |fg|v \le \|f\|_{X,v} \|g\|_{X',v} \tag{2.6}$$

holds, and it is saturated in the sense that for every $g \in \mathfrak{M}(I)$ and $\varepsilon > 0$ there exists a function, f, such that $\|f\|_{X,v} = 1$ and

$$(1 - \varepsilon)\|g\|_{X',v} \le \int_I fgv. \tag{2.7}$$

Throughout the paper we shall work with a kernel operator L, defined for $f \in (X, v)$ by

$$Lf(x) = \int_I \ell(x,t)f(t)dt,$$

where ℓ is a *kernel*, that is, $\ell \in \mathfrak{M}(I^2)$.

Of course, $\int_I \ell(x,t)f(t)dt$ need not have a sense for some functions from (X, v). We say that the kernel ℓ is *admissible*, $\ell \in \mathcal{A}$, if there is a set $J \subset I$, $|I \setminus J| = 0$ (where $|E|$ denotes the Lebesgue measure of E), such that for every $f \in (X, v)$ the function $x \mapsto \int_I \ell(x,t)f(t)dt$ is defined everywhere in J.

Lemma 2.3. *$\ell \in \mathcal{A}$ if and only if $\frac{\ell(x,.)}{v(.)} \in (X', v)$ for each $x \in J$.*

Proof. Let $x \in J$ and $\frac{\ell(x,.)}{v(.)} \in (X', v)$. By Hölder's inequality we have for $f \in (X', v)$

$$\int_I |\ell(x,t)f(t)|dt \le \left\| \frac{\ell(x,.)}{v(.)} \right\|_{(X',v)} \|f\|_{(X,v)} < \infty.$$

Thus, $\int_I \ell(x,t)f(t)dt$ has a sence and $\ell \in \mathcal{A}$.

Let $\ell \in \mathcal{A}$, $x \in J$. Assume $\frac{\ell(x,.)}{v(.)} \notin (X', v)$.

I. Suppose first $\ell(x,t) \ge 0$ and set $I^+ = \{t \in I; \ell(x,t) > 0\}$. Since $\sup\{\int_I |\ell(x,t)f(t)|dt; \|f\|_{(X,v)} \le 1\} = \infty$ there exists a sequence $0 \le f_n$, $\|f_n\|_{(X,v)} \le 1$ and $\int_I \ell(x,t)f(t)dt \ge n^3$. Setting $f_0(t) = \chi_{I^+}(t) \sum_{n=1}^\infty n^{-2} f_n(t)$ we easily obtain $f_0(t) \ge 0$, $f_0(t) = 0$ for $t \in I \setminus I^+$, $f_0(t) \in (X, v)$ and

$$\int_I \ell(x,t)f_0(t)dt = \infty. \tag{2.8}$$

Set $I_k = [2^k, 2^{k+1})$ and $A_k = f_0^{-1}(I_k)$ for each $k \in \mathbb{Z}$. Thus, $I^+ = \bigcup_{k\in\mathbb{Z}} A_k$ and

$$\sum_{k=-\infty}^{\infty} 2^k \int_{A_k} \ell(x,t)dt \le \int_{I^+} \ell(x,t)f_0(t)dt \le 2 \sum_{k=-\infty}^{\infty} 2^k \int_{A_k} \ell(x,t)dt$$

which yields with (2.8)

$$\sum_{k=-\infty}^{\infty} 2^k \int_{A_k} \ell(x,t)dt = \infty.$$

Let $\mathbb{Z}_1$, $\mathbb{Z}_2$ be disjoint subsets of $\mathbb{Z}$ with $\mathbb{Z}_1 \cup \mathbb{Z}_2 = \mathbb{Z}$ and $\sum_{k\in\mathbb{Z}_1} 2^k \int_{A_k} \ell(x,t)dt = \infty$ and $\sum_{k=\mathbb{Z}_2} 2^k \int_{A_k} \ell(x,t)dt = \infty$. Set $f_1(t) = \sum_{k\in\mathbb{Z}_1} f_0(t)\chi_{A_k}(t)$, $f_1(t) = \sum_{k\in\mathbb{Z}_2} f_0(t)\chi_{A_k}(t)$ and $g = f_1 - f_2$. Then $g \in (X, v)$ and $\int_I \ell(x,t)g(t)dt = \int_I \ell(x,t)f_1(t)dt - \int_I \ell(x,t)f_2(t)dt$ has no sense which is a contradiction with $\ell \in \mathcal{A}$.

II. Let $\ell(x,.) \in \mathfrak{M}(I)$ then we can write $\ell = \ell^+ - \ell^-$ and either $\frac{\ell^+(x,.)}{v(.)} \notin (X', v)$ or $\frac{\ell^-(x,.)}{v(.)} \notin (X', v)$. Without loss of generality assume $\frac{\ell^+(x,.)}{v(.)} \notin (X', v)$. Take $g \in (X, v)$ as bellow such that $\{t \in I; g(t) > 0\} \subset \{t \in I; \ell^+(x,t) > 0\}$ and $\int_I \ell^+(x,t)g(t)$ has no sense. Then also $\int_I \ell(x,t)g(t) = \int_I \ell^+(x,t)g(t)$ has no sense which contradicts to $\ell \in \mathcal{A}$.

□

Let us recall that $f \in L_\infty$ if $f \in \mathfrak{M}(I)$ and

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x\in I} |f(x)| = \inf_{|M|=0} \sup_{x\in(I\setminus M)} |f(x)| < \infty.$$

By $\|L\|$ we denote the operator norm of L from (X,v) into L_∞, i.e.,

$$\|L\| = \sup_{\|f\|_{X,v}\le 1} \operatorname{ess\,sup}_{x\in I} \Big| \int_I \ell(x,t) f(t)dt \Big|. \tag{2.9}$$

3. Boundedness of a general kernel operator

Our goal in this section is to establish

Theorem 3.1. *Let I be an arbitrary interval, (X,v) a Banach function space, and $\ell\in\mathcal{A}$. Then the function $x\mapsto \left\|\frac{\ell(x,\cdot)}{v(\cdot)}\right\|_{X',v}$ is measurable. Moreover, setting*

$$\|\ell\|_{L_\infty(X',v)} = \operatorname{ess\,sup}_{x\in I} \left\|\frac{\ell(x,\cdot)}{v(\cdot)}\right\|_{X',v} \tag{3.1}$$

we have $\|L\| = \|\ell\|_{L_\infty(X',v)}$.

Let us start with the proof of the inequality $\|L\|\ge\|\ell\|_{L_\infty(X',v)}$. Basically, we have to interchange the supremum and the essential supremum in the definition of $\|L\|$ (cf. (2.9)). We start with two measure-theoretic lemmas.

For $A\subset I^2$, we denote by A_x the intersection of A with $\{x\}\times I$, i.e., $A_x = \{y;\,(x,y)\in A\}$. As usual, $A\div B$ denotes the symmetric difference of A and B.

Convention. In the rest of this section we assume that (X,v) is a fixed weighted Banach function space. We also, without any loss of generality, will assume that $I=[0,1]$.

Lemma 3.2. *Let $\Omega\subset I^2$ be an open set. Let $M\subset I$ be a measurable set such that $|(x-\delta,x+\delta)\cap M|>0$ for all $x\in M$ and $\delta>0$. Then for every $\varepsilon>0$ there exist a $Z\subset I$ and an $N\subset M$ such that $|N|>0$ and $|\Omega_x\div Z|<\varepsilon$ for every $x\in N$.*

Proof. Assume the contrary. Let $\varepsilon>0$ be such that for every $Z\subset I$ and $N\subset M$, $|N|>0$, there is an $x\in N$ such that $|\Omega_x\div Z|>\varepsilon$. Let $x_0\in M$. Then Ω_{x_0} is an open subset of I, whence either $\Omega_{x_0}=\emptyset$ or $\Omega_{x_0}=\bigcup_{i=1}^\infty(a_i,b_i)$ for some $0\le a_i<b_i\le 1$. By the regularity of measure, there is a $K_0=\bigcup_{i=1}^{n_0}[c_i,d_i]$ such that $K_0\subset\Omega_{x_0}$ and

$$|\Omega_{x_0}\setminus K_0| < \frac{\varepsilon}{4}. \tag{3.2}$$

Now, K_0 is compact. Therefore, the distance of $\{x_0\}\times K_0$ from $I^2\setminus\Omega$ is positive. Thus, for a $\delta_0>0$ small enough we have

$$(x_0-\delta_0,x_0+\delta_0)\times K_0\subset\Omega. \tag{3.3}$$

Set $Z=\Omega_{x_0}$ and $N=(x_0-\delta_0,x_0+\delta_0)\cap M$. By our assumption, there is an $x_1\in(x_0-\delta_0,x_0+\delta_0)\cap M$ such that

$$|\Omega_{x_0}\div\Omega_{x_1}|>\varepsilon. \tag{3.4}$$

Now, by (3.3), $K_0\subset\Omega_{x_1}$, (3.2), and (3.4),

$$|\Omega_{x_1}\setminus K_0| > \frac{3\varepsilon}{4}. \tag{3.5}$$

Since $\Omega_{x_1}\setminus K_0$ is open, there exists a set $R_1=\bigcup_{i=n_0+1}^{n_1}[c_i,d_i]\subset(\Omega_{x_1}\setminus K_0)$ such that

$$|\Omega_{x_1}\setminus(R_1\cup K_0)| < \frac{\varepsilon}{4}, \tag{3.6}$$

and as a consequence of (3.5) and (3.6) we have

$$|R_1| > \frac{\varepsilon}{2}.$$

Denote $K_1 = R_1 \cup K_0 = \bigcup_{i=1}^{n_1}[c_i, d_i]$. Now, as above, K_1 is compact, whence, for $\delta_1 > 0$ small enough, we have

$$(x_1 - \delta_1, x_1 + \delta_1) \times K_1 \subset \Omega.$$

Let $Z = \Omega_{x_1}$ and $N = (x_1 - \delta_1, x_1 + \delta_1) \cap M$. By our assumption, there is an $x_2 \in (x_1 - \delta_1, x_1 + \delta_1) \cap M$ such that $|\Omega_{x_1} \div \Omega_{x_2}| > \varepsilon$. As above, $K_1 \subset \Omega$ and $|\Omega_{x_2} \setminus K_1| > \frac{3\varepsilon}{4}$. Since $\Omega_{x_2} \setminus K_1$ is an open set, there is a set $R_2 = \bigcup_{i=n_1+1}^{n_2}[c_i, d_i] \subset (\Omega_{x_2} \setminus K_1)$ such that $|\Omega_{x_2} \setminus (R_2 \cup K_1)| < \frac{\varepsilon}{4}$, and, consequently, $|R_2| > \frac{\varepsilon}{2}$. Let $K_2 = K_1 \cup R_2 = \bigcup_{i=1}^{n_2}[c_i, d_i]$. Then $|K_2| > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Since K_2 is a compact set, we have for δ_2 small enough $(x_2 - \delta_2, x_2 + \delta_2) \times K_2 \subset \Omega$. Let $Z = \Omega_{x_2}$ and $N = (x_2 - \delta_2, x_2 + \delta_2) \cap M$. Continuing this process we obtain after m steps for large $m \in \mathbb{N}$ $|K_m| > |I|$, which is a contradiction. □

Lemma 3.3. *Let $A \subset I^2$ be a measurable set and let $M \subset I$, $|M| > 0$. Then for every $\varepsilon > 0$ there is a set $N \subset M$, $|N| > 0$, such that*

$$|A_x \div A_y| < \varepsilon \quad \textit{for all} \quad x, y \in N.$$

Proof. Let $\varepsilon > 0$ be fixed. In the case $|A| = 0$ it suffices to put $N = M$. Let $|A| > 0$. Define $P = \{x; |A_x| > 0\}$. Clearly, $|P| > 0$, whence either $|M \setminus P| > 0$ or $|M \cap P| > 0$. In the case $|M \setminus P| > 0$, it suffices to put $N = M \setminus P$. Assume that $|M \setminus P| = 0$. Denote $M_1 = M \cap P$ and $B = \cup_{x \in M_1}\{x\} \times A_x$. Clearly, $|B| > 0$. By the regularity of the Lebesgue measure there is an open set Ω, $B \subset \Omega$, such that $|\Omega \setminus B| < \frac{\varepsilon}{4}|M_1|$. Set

$$Q = \{x \in M_1; |\Omega_x \setminus B_x| \geq \frac{\varepsilon}{4}\}. \tag{3.7}$$

If $|Q| = |M_1|$, then the Fubini theorem implies $|\Omega \setminus B| \geq \frac{\varepsilon}{4}|Q| = \frac{\varepsilon}{4}|M_1|$, which is a contradiction. Therefore, $|Q| < |M_1|$.

Let M_2 be a set of all density points of $M_1 \setminus Q$. By the Lebesgue density theorem we have $|M_2| = |M_1 \setminus Q| > 0$, and by (3.7) we obtain

$$|\Omega_x \setminus B_x| = |\Omega_x \setminus A_x| < \frac{\varepsilon}{4} \qquad \text{for all} \qquad x \in M_2.$$

By Lemma 3.2, there are sets $Z \subset I$ and $N \subset M_2$, $|N| > 0$, such that

$$|\Omega \div Z| < \frac{\varepsilon}{8} \qquad \text{for all} \qquad x \in N. \tag{3.8}$$

Now, we fix $x, y \in N$. We shall estimate $|A_x \div A_y|$. Since

$$A_x \setminus A_y \subset \Omega_x \setminus A_y \subset (\Omega_x \setminus \Omega_y) \cup (\Omega_y \setminus A_y),$$

it is easy to verify that

$$|A_x \setminus A_y| < |\Omega_x \setminus \Omega_y| + \frac{\varepsilon}{4}. \tag{3.9}$$

Moreover, $\Omega_x \setminus \Omega_y \subset (\Omega_x \setminus Z) \cup (Z \setminus \Omega_y)$ and $\Omega_y \setminus \Omega_x \subset (\Omega_y \setminus Z) \cup (Z \setminus \Omega_x)$, which together with (3.8) yields

$$|\Omega_x \div \Omega_y| \leq |\Omega_x \div Z| + |\Omega_y \div Z| < \frac{\varepsilon}{4}.$$

Using (3.9), we obtain $|A_x \setminus A_y| < \frac{\varepsilon}{2}$, and, consequently, $|A_x \div A_y| < \varepsilon$. The proof is complete. □

In the sequel we denote by $f|_K$ the restriction of a function f to a set K.

We shall use the following notation. Let $\mathcal{F}$ be some subset of the unit ball of (X, v). Then we define

$$F_{\mathcal{F}}(x) = F(x) = \sup_{f\in\mathcal{F}} \int_I \ell(x,t)f(t)dt, \qquad x \in I,$$

and

$$P = \operatorname*{ess\,sup}_{x\in I} F(x).$$

We have to prove that F is measurable on I and that $\|L\| \geq P$. We first adopt certain restrictions on ℓ and $\mathcal{F}$, which will be gradually chipped away later on.

Lemma 3.4. *Let $K \subset I^2$ be a compact set. Assume that $\ell \in \mathcal{A}$, $\ell \geq 0$, $\ell|_K$ is continuous, and $\ell = 0$ on $I^2 \setminus K$. Let $C > 0$ and $\mathcal{F} = \{f;\ \|f\|_{X,v} \leq 1,\ 0 \leq f \leq C\}$. Then F is a measurable function on I and $\|L\| \geq P$.*

Proof. By our assumptions on K and ℓ, there exists a constant $D > 0$ such that $0 \leq \ell \leq D$ on I^2. Let $\varepsilon > 0$. We divide the proof into three steps. In the first two steps we prove that F is measurable and in the third step we show $\|L\| \geq P$.

Step 1. We claim that for a set $M \subset I$, $|M| > 0$, there is a set $N \subset M$, $|N| > 0$, and $f \in \mathcal{F}$, such that

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq \int_I \ell(x,t)f(t)dt \leq F(x) \quad \text{for all } x \in N.$$

Let $|M| > 0$. By Lemma 3.3, there is $M_0 \subset M$, such that $|M_0| > 0$ and

$$|K_x \div K_y| < \varepsilon \qquad \text{for all} \quad x, y \in M_0. \tag{3.10}$$

Now, choose $x_0 \in M_0$ such that for every $\delta > 0$ we have $|(x_0 - \delta, x_0 + \delta) \cap M_0| > 0$. By the uniform continuity of $\ell|_K$, there is a $\delta_0 > 0$ small enough and such that, for $x \in M_1 = (x_0 - \delta_0, x_0 + \delta_0) \cap M_0$ and $t \in K_x \cap K_{x_0}$,

$$|\ell(x,t) - \ell(x_0,t)| < \varepsilon. \tag{3.11}$$

From the definition of $F(x)$ we obtain the existence of $f_0 \in \mathcal{F}$ satisfying

$$F(x_0) - \varepsilon \leq \int_I \ell(x_0,t)f_0(t)dt \leq F(x_0). \tag{3.12}$$

Set

$$G(x) = \int_I \ell(x,t)f_0(t)dt \quad \text{for } x \in I.$$

Clearly,

$$G(x) \leq F(x) \quad \text{for a.e. } x \in I. \tag{3.13}$$

Let $x \in M_1$ and $f \in \mathcal{F}$ be fixed. We denote

$$R(x,f) = \int_I (\ell(x,t) - \ell(x_0,t))f(t)dt.$$

Using (3.10), (3.11), and $0 \le \ell \le D$, we get

$$\begin{aligned} |R(x,f)| &\le \int_I |\ell(x,t)-\ell(x_0,t)|f(t)dt \\ &= \int_{K_x\cap K_{x_0}} |\ell(x,t)-\ell(x_0,t)|f(t)dt \\ &+ \int_{K_x\setminus K_{x_0}} \ell(x,t)f(t)dt + \int_{K_{x_0}\setminus K_x} \ell(x_0,t)f(t)dt \\ &\le \varepsilon C + 2\varepsilon CD = \varepsilon(C+2CD). \end{aligned}$$

Now, setting $f \equiv f_0$ we have

$$|G(x)-G(x_0)| \le \varepsilon(C+2CD) \quad \text{for } x \in M_1. \tag{3.14}$$

Let $R(x) = \sup_{f\in\mathcal{F}} |R(x,f)|$. We immediately obtain

$$0 \le R(x) \le \varepsilon(C+2CD) \qquad \text{for all} \quad x \in M_1. \tag{3.15}$$

Since

$$\int_I \ell(x,t)f(t)dt = \int_I \ell(x_0,t)f(t)dt + R(x,f),$$

we have

$$F(x) = \sup_{f\in\mathcal{F}} \Big(\int_I \ell(x_0,t)f(t)dt + R(x,f)\Big),$$

and, consequently,

$$F(x) \le \sup_{f\in\mathcal{F}} \int_I \ell(x_0,t)f(t)dt + R(x).$$

We can rewrite the last inequality as $F(x) \le F(x_0) + R(x)$, $x \in M_1$. Thus, by the above inequalities, (3.14), (3.12), and (3.15), we obtain for $x \in M_1$

$$\begin{aligned} G(x) &\ge G(x_0) - \varepsilon(C+2CD) \ge F(x_0) - \varepsilon(1+C+2CD) \\ &\ge F(x) - R(x) - \varepsilon(1+C+2CD) \ge F(x) - \varepsilon(1+2C+4CD). \end{aligned}$$

Now, it suffices to use (3.13) and the last inequality, and to set $f = f_0$ and $N = M_1$ to prove our claim.

Step 2. Let $M = I$. By Step 1, there exist $N_1 \subset I$, $|N_1| > 0$, and $f_1 \in \mathcal{F}$ such that

$$F(x) - \varepsilon(1+2C+4CD) \le \int_I \ell(x,t)f_1(t)dt \le F(x) \qquad \text{for} \quad x \in N_1.$$

Assume that we have constructed sets N_β and functions f_β for all ordinal numbers $\beta < \alpha$, where α is a fixed countable ordinal number. Set $M = I \setminus \bigcup_{\beta<\alpha} N_\beta$. If $|M| = 0$, we stop the construction. If $|M| > 0$, then by Step 1 we have $N_\alpha \subset I \setminus \bigcup_{\beta<\alpha} N_\beta$, $|N_\alpha| > 0$, and there is an $f_\alpha \in \mathcal{F}$ such that

$$F(x) - \varepsilon(1+2C+4CD) \le \int_I \ell(x,t)f_\alpha(t)dt \le F(x) \qquad \text{for} \quad x \in N_\alpha.$$

This process will stop after countably many steps. Hence, there exists a countable ordinal number γ such that for $x \in N_\beta$, $\beta < \gamma$,

$$F(x) - \varepsilon(1+2C+4CD) \le \int_I \ell(x,t)f_\beta(t)dt \le F(x), \tag{3.16}$$

and moreover

$$N_\alpha \cap N_\beta = \emptyset \quad \text{for} \quad \alpha \neq \beta, \qquad |N_\beta| > 0 \quad \text{for} \quad \beta < \gamma; \tag{3.17}$$

$$|I \setminus \bigcup_{\beta<\gamma} N_\beta| = 0. \tag{3.18}$$

Define a function H_ε by $H_\varepsilon(x) = \sum_{\beta<\gamma} \chi_{N_\beta}(x) \int_I \ell(x,t) f_\beta(t)dt$. Since for every $\beta < \gamma$ the functions $x \mapsto \int_I \ell(x,t) f_\beta(t)dt$ and $\chi_{N_\beta}(x)$ are measurable, H_ε is measurable as well. Moreover, according to (3.16), (3.17)and (3.18) we have for a.e. $x \in I$

$$F(x) - \varepsilon(1 + 2C + 4CD) \leq H_\varepsilon(x) \leq F(x),$$

which implies

$$F(x) = \lim_{n\to\infty} H_{1/n}(x) \qquad \text{for a.e.} \quad x \in I,$$

and, consequently, F is measurable.
Step 3. We claim $\|L\| \geq P$. Observe that

$$\|L\| = \sup_{\|f\|_{X,v}\leq 1} \operatorname{ess\,sup}_{x\in I} \Big| \int_I \ell(x,t) f(t)dt \Big| = \sup_{\|f\|_{X,v}\leq 1} \operatorname{ess\,sup}_{x\in I} \int_I \ell(x,t) f(t)dt.$$

By the definition of P, there is a set M, $|M| > 0$, such that

$$P - \varepsilon \leq F(x) \leq P \qquad \text{for all} \quad x \in M.$$

Now, by Step 1, there is a set $N \subset M$, $|N| > 0$, and a function $f_0 \in \mathcal{F}$ such that

$$\int_I \ell(x,t) f_0(t)dt \geq F(x) - \varepsilon(1 + 2C + 4CD) \qquad \text{for all} \quad x \in N.$$

Thus,

$$\begin{aligned}
\|L\| &= \sup_{\|f\|_{X,v}\leq 1} \operatorname{ess\,sup}_{x\in I} \int_I \ell(x,t) f(t)dt \\
&\geq \sup_{f\in\mathcal{F}} \operatorname{ess\,sup}_{x\in N} \int_I \ell(x,t) f(t)dt \\
&\geq \operatorname{ess\,sup}_{x\in N} \int_I \ell(x,t) f_0(t)dt \geq F(x) - \varepsilon(1 + 2C + 4CD) \\
&\geq P - \varepsilon(2 + 2C + 4CD).
\end{aligned}$$

Letting ε tend to zero we complete the proof. □

Lemma 3.5. *Assume that $\ell \in \mathcal{A}$ and $0 \leq \ell \leq D$ for some $D > 0$. Let $C > 0$. Define $\mathcal{F} = \{f;\ \|f\|_{X,v} \leq 1$ and $0 \leq f \leq C\}$. Then F is measurable and $\|L\| \geq P$.*

Proof. Let K_n be a sequence of compact sets, $K_n \nearrow I^2$, such that $\ell|_{K_n}$ are continuous functions. Set $\ell_n(x,t) = \ell(x,t)\chi_{K_n}(x,t)$. Since $0 \leq \ell_n \leq \ell$ we have by Lemma 2.3 $\frac{\ell(x,.)}{v(.)} \in (X', v)$ almost everywhere in I and so, $\frac{\ell_n(x,.)}{v(.)} \in (X', v)$ which implies again by Lemma 2.3 $\ell_n \in \mathcal{A}$. Set

$$\begin{aligned}
L_n f(x) &= \int_I \ell_n(x,t) f(t)dt \quad \text{for } f \in (X, v), \\
F_n(x) &= \sup_{f\in\mathcal{F}} L_n f(x) \text{ and,} \\
P_n &= \operatorname{ess\,sup}_{x\in I} F_n(x).
\end{aligned}$$

Clearly, $0 \le \ell_n \nearrow \ell$ a.e. in I^2. Then there is a set $I_1 \subset I, |I \setminus I_1| = 0$, such that for every $x \in I_1$ we have $0 \le \ell_n(x,\cdot) \nearrow \ell(x,\cdot)$ a.e. in I. Thus, $0 \le \ell_n(x,t)f(t) \nearrow \ell(x,t)f(t)$ for $x \in I_1$ and $f \ge 0$, whence

$$0 \le \int_I \ell_n(x,t)f(t)dt \nearrow \int_I \ell(x,t)f(t)dt \quad \text{for } x \in I_1 \text{ and } f \ge 0.$$

Now, it is not difficult to verify that

$$0 \le F_n(x) \nearrow F(x) \text{ for } x \in I_1.$$

By Lemma 3.4, F_n are measurable. Thus F is measurable as a pointwise limit of F_n. Moreover, it is readily seen that

$$0 \le P_n \nearrow P. \tag{3.19}$$

Moreover, it is clear that

$$\|L\| = \sup_{\|f\|_{X,v} \le 1, f \ge 0} \operatorname{ess\,sup}_{x \in I} \int_I \ell(x,t)f(t)dt,$$

and

$$\|L_n\| = \sup_{\|f\|_{X,v} \le 1} \operatorname{ess\,sup}_{x \in I} \int_I \ell_n(x,t)f(t)dt.$$

For $f \ge 0$ we have $\int_I \ell_n(x,t)f(t)dt \le \int_I (x,t)f(t)dt$. We thus obtain

$$\|L\| \ge \|L_n\| \qquad \text{for any} \quad n \in \mathbb{N}. \tag{3.20}$$

Observe that the kernels ℓ_n satisfy the assumptions of Lemma 3.5 and therefore $\|L_n\| \ge P_n$, which via (3.19) and (3.20) implies $\|L\| \ge P$. The proof is complete. □

For $M \subset \mathbb{R}$ measurable we denote by $\mathfrak{D}(M)$ the set of all points of M which are the Lebesgue density points of M. Recall that $|M \setminus \mathfrak{D}(M)| = 0$.

Lemma 3.6. *Let $A \subset I^2$ and $M \in I$ be measurable sets, $|M| > 0$. Then there exists $N \subset M$, $|M \setminus N| = 0$, $N = \mathfrak{D}(N)$ with the following property: for every $x \in N$ and $\varepsilon > 0$ there is a set $B \subset N$, such that*

$$B = \mathfrak{D}(B), \tag{3.21}$$

$$x \in B, \tag{3.22}$$

$$|A_y \div A_z| < \varepsilon \qquad \textit{for} \quad y,z \in B. \tag{3.23}$$

Proof. Fix $\varepsilon > 0$. By Lemma 3.3, there is a set $\widetilde{M}_{\varepsilon,1} \subset M$, $|\widetilde{M}_{\varepsilon,1}| > 0$, such that $|A_y \div A_z| < \varepsilon$ for $y,z \in \widetilde{M}_{\varepsilon,1}$. Set $M_{\varepsilon,1} = \mathfrak{D}(\widetilde{M}_{\varepsilon,1})$. Clearly, $|\widetilde{M}_{\varepsilon,1} \setminus M_{\varepsilon,1}| = 0$ and, consequently, $|M_{\varepsilon,1}| > 0$. Assume that we have constructed for an ordinal number α the sets $M_{\varepsilon,\beta}, \beta < \alpha$, such that for any β we have

$$|A_y \div A_z| < \varepsilon \qquad \text{for} \quad y,z \in M_{\varepsilon,\beta} \qquad \textit{and} \qquad M_{\varepsilon,\beta} = \mathfrak{D}(M_{\varepsilon,\beta}).$$

If $|M \setminus \bigcup_{\beta<\alpha} M_{\varepsilon,\beta}| = 0$, we set $M_\varepsilon = \bigcup_{\beta<\alpha} M_{\varepsilon,\beta}$ and we stop the construction. If $|M \setminus \bigcup_{\beta<\alpha} M_{\varepsilon,\beta}| > 0$, then, by Lemma 3.3, there is an $\widetilde{M}_{\varepsilon,\alpha} \subset M \setminus \bigcup_{\beta<\alpha} M_{\varepsilon,\beta}$, $|\widetilde{M}_{\varepsilon,\alpha}| > 0$ and $|A_y \div A_z| < \varepsilon$ for $y,z \in \widetilde{M}_{\varepsilon,\alpha}$. Set $M_{\varepsilon,\alpha} = \mathfrak{D}(\widetilde{M}_{\varepsilon,\alpha})$.

This process will stop after a countable number of steps. Hence there is a countable ordinal γ_ε such that

$$|M \setminus \bigcup_{\beta<\gamma_\varepsilon} M_{\varepsilon,\beta}| = 0,$$

and for $\beta < \gamma_\varepsilon$ we have

$$M_{\varepsilon,\beta} = \mathfrak{D}(M_{\varepsilon,\beta}), \qquad \text{and} \qquad |A_y \div A_z| < \varepsilon \qquad \text{for} \quad y, z \in M_{\varepsilon,\beta}.$$

Let us define

$$M_\varepsilon = \bigcup_{\beta<\gamma_\varepsilon} M_{\varepsilon,\beta}.$$

Set

$$\widetilde{N} = \bigcap_{n=1}^{\infty} M_{\frac{1}{n}} \text{ and } N = D(\widetilde{N}).$$

Evidently, $N \subset M$. Moreover, $|M \setminus M_{\frac{1}{n}}| = 0$ for $n \in \mathbb{N}$, hence $|M \setminus \bigcap_{n=1}^{\infty} M_{\frac{1}{n}}| = |M \setminus \widetilde{N}| = 0$, and, as $|\widetilde{N} \setminus N| = 0$, we have $|M \setminus N| = 0$. Clearly, $N = \mathfrak{D}(N)$.

Let $\varepsilon > 0$ and $x \in N$. Fix n such that $\frac{1}{n} < \varepsilon$. Then $x \in M_{\frac{1}{n}} = \bigcup_{\beta<\gamma_{\frac{1}{n}}} M_{\frac{1}{n},\beta}$. Let $\alpha < \gamma_{\frac{1}{n}}$ be an ordinal number such that $x \in M_{\frac{1}{n},\alpha}$. By the construction we have $M_{\frac{1}{n},\alpha} = \mathfrak{D}(M_{\frac{1}{n},\alpha})$. Moreover,

$$|A_y \div A_z| < \frac{1}{n} < \varepsilon \qquad \text{for} \quad y, z \in M_{\frac{1}{n},\alpha}.$$

Now, to prove (3.21), (3.22) and (3.23), if suffices to take $B = M_{\frac{1}{n},\alpha} \cap N$. □

Lemma 3.7. *Let $\ell \in \mathcal{A}$ and let $|\ell| \le D$ a.e. in I^2 for some $D > 0$. Let $C > 0$. Set $\mathcal{F} = \{f;\ \|f\|_{X,v} \le 1$ and $|f| \le C\}$. Then F is measurable and $\|L\| \ge P$.*

Proof. Since $\ell \in \mathcal{A}$, there is a $J \subset I$, $|I \setminus J| = 0$, and the function $x \mapsto \int_I \ell(x,t) f(x) dt$ is well-defined for all $x \in J$ and $f \in (X, v)$. Then for $x \in J$

$$F(x) = \sup_{f\in\mathcal{F}} \int_I |\ell(x,t)| f(t) dt \tag{3.24}$$

By Lemma 3.5, the last expression is a measurable function. Thus, F is measurable.

Let $\varepsilon > 0$. Then there is a set $M \subset J, |M| > 0$, such that

$$P - \varepsilon \le F(x) \le P \qquad \text{for all} \quad x \in M. \tag{3.25}$$

Set

$$K^+ = \{(x,y) \in I^2;\ \ell(x,t) > 0\}, \quad K^- = \{(x,y) \in I^2;\ \ell(x,t) < 0\}, \quad K = K^+ \cup K^-,$$

and

$$PK^+ = \{x \in J;\ |K_x^+| > 0\}, \qquad PK^- = \{x \in J;\ |K_x^-| > 0\}.$$

Let further

$$M_1 = PK^+ \cap PK^- \cap M, \qquad M_2 = (PK^+ \setminus PK^-) \cap M,$$
$$M_3 = (PK^- \setminus PK^+) \cap M, \qquad M_4 = (I \setminus (PK^+ \cup PK^-)) \cap M.$$

Clearly, $M = \bigcup_{i=1}^{4} M_i$, and at least one of these sets has a positive measure. Set

$$\ell_i(x,t) = \ell(x,t)\chi_{M_i}(x),$$
$$(L_i f)(x) = \int_I \ell_i(x,t) f(t) dt.$$

Clearly, by Lemma 2.3 we have $\ell_i \in \mathcal{A}$ for $1 \le i \le 4$. Fix $i \in \{2,3,4\}$ and assume $|M_i| > 0$. It is easy to see that $0 \le \ell_2 \le D$, $-D \le \ell_3 \le 0$, $\ell_4 = 0$ and

$$\|L\| \ge \|L_i\|. \tag{3.26}$$

Define

$$P_i = \operatorname*{ess\,sup}_{x\in I} \sup_{\|f\|_{X,v}\le 1, 0\le f\le C} \int_I |\ell_i(x,t)| f(t)dt.$$

Then, by the definition of ℓ_i and (3.24), we have

$$\begin{aligned} P_i &\ge \operatorname*{ess\,sup}_{x\in M_i} \sup_{\|f\|_{X,v}\le 1, 0\le f\le C} \int_I |\ell_i(x,t)| f(t)dt \\ &= \operatorname*{ess\,sup}_{x\in M_i} \sup_{\|f\|_{X,v}\le 1, 0\le f\le C} \int_I |\ell(x,t)| f(t)dt = \operatorname*{ess\,sup}_{x\in M_i} F(x). \end{aligned}$$

By Lemma 3.5 we have $\|L_i\| \ge P_i$. Therefore, using also (3.25) and (3.26), we have

$$\|L\| \ge P - \varepsilon. \tag{3.27}$$

Now let us assume that $|M_2| = |M_3| = |M_4| = 0$. Then $|M_1| > 0$. There exist sequences of compact sets K_n^+, K_n^- such that $K_n^+ \nearrow K^+$, $K_n^- \nearrow K^-$ and moreover $\ell|_{K_n^+}, \ell|_{K_n^-}$ are continuous functions. Set $K_n = K_n^+ \cup K_n^-$. Fix $n \in \mathbb{N}$. Now, Lemma 3.6 guarantees the existence of sets $N_n^+ \subset M_1, |M_1 \setminus N_n^+| = 0$ and $N_n^+ = \mathfrak{D}(N_n^+)$ such that for any $x \in N_n^+$ there is a set $N_{x,n}^+ \subset N_n^+$ which satisfies

$$N_{x,n}^+ = \mathfrak{D}(N_{x,n}^+), \tag{3.28}$$

$$x \in N_{x,n}^+, \tag{3.29}$$

$$|K_{n,y}^+ \div K_{n,z}^+| < \varepsilon \qquad \text{for} \quad y, z \in N_{x,n}^+. \tag{3.30}$$

Analogously, there is a set N_n^- with $N_n^- \subset M_1, |M_1 \setminus N_n^-| = 0, N_n^- = \mathfrak{D}(N_n^-)$ and for any $x \in N_n^-$ we have a set $N_{x,n}^- \subset N_n^-$ such that

$$N_{x,n}^- = \mathfrak{D}(N_{x,n}^-), \tag{3.31}$$

$$x \in N_{x,n}^-, \tag{3.32}$$

$$|K_{n,y}^- \div K_{n,z}^-| < \varepsilon \qquad \text{for} \quad y, z \in N_{x,n}^-. \tag{3.33}$$

Set $\widetilde{N}_1 = \bigcap_{n=1}^{\infty}(N_n^+ \cap N_n^-)$ and $N_1 = \mathfrak{D}(\widetilde{N}_1)$. Obviously, $|M_1 \setminus N_1| = 0$ and, consequently, $|N_1| > 0$. Denote for $n \in \mathbb{N}$

$$PK_n^+ = \{x \in J;\, |K_{n,x}^+| > 0\}, \qquad PK_n^- = \{x \in J;\, |K_{n,x}^-| > 0\}.$$

By the Fubini theorem, $|M_1| > 0$, and $K_n \nearrow K$, we can choose an $n_0 \in \mathbb{N}$ large enough in order that

$$|PK_n^+ \cap PK_n^- \cap M_1| > 0 \qquad \text{for all} \quad n \ge n_0.$$

Set

$$A_n = \{x \in J;\, |K_x^+ \setminus K_{n,x}^+| < \varepsilon \text{ and } |K_x^- \setminus K_{n,x}^-| < \varepsilon\}.$$

Since for any $n \in \mathbb{N}$ the functions $x \mapsto |K_x^+ \setminus K_{n,x}^+|$ and $x \mapsto |K_x^- \setminus K_n|$ are measurable, A_n are measurable as well. Moreover, A_n is a non-decreasing sequence of sets.

Since $K_n^+ \nearrow K^+$ and $K_n^- \nearrow K^-$, we obtain by the Fubini theorem $K_{n,x}^+ \nearrow K_x^+$ and $K_{n,x}^- \nearrow K_x^-$ for a.e. $x \in J$. So, there is a set $J_1 \subset J$, $|J \setminus J_1| = 0$ such that

$$K_{n,x}^+ \nearrow K_x^+,\ K_{n,x}^- \nearrow K_x^- \quad \text{for all } x \in J_1. \tag{3.34}$$

Using the Fubini theorem again, we can find an $n_1 \in \mathbb{N}$ such that $|A_n \cap N_1| > 0$ for any $n \ge n_1$. Let $n_2 = \max\{n_0, n_1\}$. Then we see that

$$|K_x^+ \setminus K_{n,x}^+| < \varepsilon \quad \text{and} \quad |K_x^- \setminus K_{n,x}^-| < \varepsilon \quad \text{for all } x \in A_n \cap N_1, \quad n \ge n_2. \tag{3.35}$$

Define $\ell_n^+ = \ell\chi_{K_n^+}$ and $\ell_n^- = \ell\chi_{K_n^-}$. Set $\ell_n = \ell_n^+ - \ell_n^-$. Let $N_2 = A_{n_1} \cap N_1 \cap J_1$. Clearly, $|N_2| > 0$.

Let $x_0 \in \mathfrak{D}(N_2)$. Since $x_0 \in N_2 \subset N_1 \subset M_1 \subset M$, we have from (3.25) and the definition of $F(x_0)$ a function $f_0(t)$ such that

$$P - 2\varepsilon \le \Big| \int_I \ell(x_0,t) f_0(t) dt \Big| \le P. \tag{3.36}$$

Clearly, (3.34) and the fact that $x_0 \in J_1$ imply $\ell_n^+(x_0,t) \nearrow \ell^+(x_0,t), \ell_n^-(x_0,t) \nearrow \ell^-(x_0,t)$. Since the constant function CD can serve as an integrable majorant, we can write

$$\int_I \ell_n^+(x_0,t) f_0(t) dt \to \int_I \ell^+(x_0,t) f_0(t) dt,$$

and

$$\int_I \ell_n^-(x_0,t) f_0(t) dt \to \int_I \ell^-(x_0,t) f_0(t) dt.$$

Hence, there exists an $n_3 \ge n_2$ such that

$$\int_I \ell^+(x_0,t) f_0(t) dt - \varepsilon \le \int_I \ell_{n_3}^+(x_0,t) f_0(t) dt \le \int_I \ell^+(x_0,t) f_0(t) dt + \varepsilon,$$

and

$$\int_I \ell^-(x_0,t) f_0(t) dt - \varepsilon \le \int_I \ell_{n_3}^-(x_0,t) f_0(t) dt \le \int_I \ell^-(x_0,t) f_0(t) dt + \varepsilon.$$

Using these inequalities, $\ell_n(x_0,t) = \ell_n^+(x_0,t) - \ell_n^-(x_0,t)$, and $\ell(x_0,t) = \ell^+(x_0,t) - \ell^-(x_0,t)$, we obtain

$$\Big| \int_I \ell(x_0,t) f_0(t) dt \Big| - 2\varepsilon \le \Big| \int_I \ell_{n_3}(x_0,t) f_0(t) dt \Big| \le \Big| \int_I \ell(x_0,t) f_0(t) dt \Big| + 2\varepsilon.$$

Together with (3.36), this yields

$$P - 4\varepsilon \le \Big| \int_I \ell_{n_3}(x_0,t) f_0(t) dt \Big| \le P + 2\varepsilon. \tag{3.37}$$

Since $\ell|_{K_{n_3}}$ is continuous, there is an $\alpha_0 > 0$ such that, for $x \in (x_0 - \alpha_0, x_0 + \alpha_0)$ and $t \in K_{n_3,x_0} \cap K_{n_3,x}$,

$$|\ell_{n_3}(x,t) - \ell_{n_3}(x_0,t)| < \varepsilon \tag{3.38}$$

Set $N_3 = (x_0 - \alpha_0, x_0 + \alpha_0) \cap N_2 \cap N_{x_0,n_3}^+ \cap N_{x_0,n_3}^-$. We know that $x_0 \in \mathfrak{D}(N_2)$. By (3.28), (3.29), (3.31) and (3.32) we have $x_0 \in N_3$, $x_0 \in \mathfrak{D}(N_3)$ and, consequently, $|N_3| > 0$.

Recall that N_3 satisfies the following inclusions:

$$N_3 \subset N_2 \subset A_{n_1}, \tag{3.39}$$

$$N_3 \subset (x_0 - \alpha_0, x_0 + \alpha_0), \tag{3.40}$$

$$N_3 \subset N_{x_0,n_3}^+ \cap N_{x_0,n_3}^-. \tag{3.41}$$

We shall estimate

$$G(x) = \int_I \ell(x,t) f_0(t) dt, \qquad x \in N_3.$$

Clearly, for a fixed $x \in N_3$, we have

$$
\begin{aligned}
G(x) &= \int_I (\ell(x,t) - \ell_{n_3}(x,t)) f_0(t)dt \\
&+ \int_{K_{n_3,x_0}\cap K_{n_3,x}} (\ell_{n_3}(x,t) - \ell_{n_3}(x_0,t)) f_0(t)dt \\
&+ \int_{K_{n_3,x_0}\setminus K_{n_3,x}} \ell_{n_3}(x,t) f_0(t)dt \\
&+ \int_{K_{n_3,x}\setminus K_{n_3,x_0}} \ell_{n_3}(x_0,t) f_0(t)dt \\
&+ \int_I \ell_{n_3}(x_0,t) f_0(t)dt = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Now, evidently,

$$
\begin{aligned}
|I_1| &\le \int_I |\ell(x,t) - \ell_{n_3}(x,t)||f_0(t)|dt \\
&\le \int_{K_x^+\setminus K_{n_3,x}^+} \ell^+(x,t)|f_0(t)|dt + \int_{K_x^-\setminus K_{n_3,x}^-} |\ell^-(x,t)||f_0(t)|dt.
\end{aligned}
$$

By (3.39), $x \in N_3 \subset A_{n_1}$. Since $n_3 \ge n_2 \ge n_1$, we have by (3.35) $|I_1| \le 2\varepsilon CD$. By (3.38) and (3.40), $|I_2| \le \varepsilon C$. Using (3.30), (3.33) and (3.41), we get $|K_{n,x}^+ \div K_{n,x_0}^+| < \varepsilon$ and $|K_{n,x}^- \div K_{n,x_0}^-| < \varepsilon$, and therefore

$$|I_3| \le \varepsilon CD, \qquad |I_4| < \varepsilon CD.$$

Now, (3.37) and the estimates of I_1, I_2, I_3, I_4 give

$$
\begin{aligned}
\|L\| &\ge \sup_{f\in\mathcal{F}} \operatorname*{ess\,sup}_{x\in I} \Big| \int_I \ell(x,t)f(t)dt \Big| \ge \operatorname*{ess\,sup}_{x\in N_3} \Big| \int_I \ell(x,t) f_0(t)dt \Big| \\
&= \operatorname*{ess\,sup}_{x\in N_3} |G(x)| \ge |I_5| - |I_1| - |I_2| - |I_3| - |I_4| \ge P - \varepsilon(4 + C + 4CD).
\end{aligned}
$$

We have proved that if $|M_2| = |M_3| = |M_4| = 0$, then either $\|L\| \ge P - \varepsilon$ or $\|L\| \ge P - \varepsilon(4 + C + CD)$. Together with (3.27) this yields

$$\|L\| \ge P - \varepsilon(4 + C + 4CD).$$

Letting ε tend to zero, we obtain $\|L\| \ge P$, and the proof is complete. □

Lemma 3.8. *Let $\ell \in \mathcal{A}$, $|\ell| \le D$ in I^2 for some $D > 0$ and let $\mathcal{F}$ be the unit ball of (X, v). Then F is measurable. Moreover, $\|L\| \ge P$.*

Proof. Let $\mathcal{F}_C = \{f;\ \|f\|_{X,v} \le 1,\ |f| \le C\}$ for any $C > 0$. Let $a > 0$ and $h \in \mathfrak{M}(I)$. We denote

$$
h_a(t) = \begin{cases} a & \text{if} \quad h(t) > a \\ h(t) & \text{if} \quad |h(t)| \le a \\ -a & \text{if} \quad h(t) < -a. \end{cases}
$$

We define $F_C(x) = \sup_{f\in\mathcal{F}_C} |\int_I \ell(x,t)f(t)dt|$.

Let $J \subset I$, $|I \setminus J| = 0$ and assume that $\int_I \ell(x,t)f(t)dt$ exists for any $x \in J$ and $f \in (X, v)$. Fix $x \in J$. Clearly, $\int_I h_C \to \int_I h$ for $C \to \infty$ if $\int_I h$ exists in the Lebesgue sense. Then

$$\Big| \int_I \ell(x,t) f_C(t)dt \Big| \to \Big| \int_I \ell(x,t)f(t)dt \Big| \quad \text{for } C \to \infty.$$

It is not difficult to prove from the above convergence that

$$F_C(x) \nearrow F(x).$$

By Lemma 3.7, the functions $F_C(x)$ are measurable and therefore $F(x)$ is measurable as a monotone pointwise limit of $F_C(x)$.

Moreover, as $F_C(x) \nearrow F(x)$ a.e. in I, we have also $P_C \nearrow P$. Now, Lemma 3.7 implies $\|L\| \geq P_C$ for any $C > 0$, and thus $\|L\| \geq P$. The proof is complete. $\square$

Lemma 3.9. *Let $I = [0,1]$. Let $\delta > 0$ and denote*

$$\mathcal{M} = \{A;\ A \in \mathfrak{M}(I), |A| < \delta\}.$$

Then there exists a countable system $\mathcal{C} = \{M_i\}_{i\in\mathbb{N}}$ of sets, $|M_i| < \delta$, such that for every $A \in \mathcal{M}$ and $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $|A \div M_k| < \varepsilon$.

Proof. Set

$$\mathcal{C} = \{M \subset I;\ M = \bigcup_{i=1}^{n}(a_i, b_i),\ n \in \mathbb{N},\ a_i, b_i \text{ rational},\ (a_i,b_i)\cap(a_j,b_j) = \emptyset \text{ for } i \neq j$$
$$\text{and } \sum_{i=1}^{n}(b_i - a_i) < \delta\}.$$

Clearly, $\mathcal{C}$ is a countable system, i.e. $\mathcal{C} = \{M_i\}_{i\in\mathbb{N}}$.

Now, let $A \in \mathcal{M}$ and $\varepsilon > 0$. Fix a γ such that $0 < \gamma < \min\{\frac{\varepsilon}{3}, \delta - |A|\}$. By the regularity of the Lebesgue measure, there is an open set $G = \bigcup_{i=1}^{\infty}(c_i, d_i)$ such that $A \subset G$ and

$$|G \setminus A| < \gamma. \tag{3.42}$$

Since $G \subset I$, we have $\sum_{i=1}^{\infty}(c_i - d_i) \leq 1$, and there exists an $n \in \mathbb{N}$ such that the set G_n, defined by $G_n = \bigcup_{i=1}^{n}(c_i, d_i)$, satisfies

$$|G \setminus G_n| < \gamma. \tag{3.43}$$

Let $a_i, b_i \in \mathbb{Q}$, $i = 1, 2, \dots, n$, are such that

$$(a_i, b_i) \subset (c_i, d_i) \qquad \text{and} \qquad |(c_i,d_i) \setminus (a_i,b_i)| < \frac{\gamma}{n}.$$

Set $M = \bigcup_{i=1}^{n}(a_i, b_i)$. Clearly, $M \subset G_n$ and

$$|G_n \setminus M| \leq \sum_{i=1}^{n}|(c_i,d_i) \setminus (a_i,b_i)| < \gamma. \tag{3.44}$$

Evidently, we have from (3.42)

$$|M| \leq |M \setminus A| + |A| \leq |G \setminus A| + |A| \leq \gamma + |A| < \delta,$$

which implies $M \in \mathcal{C}$. Moreover, due to (3.42), (3.43), and (3.44) we can write

$$|A \div M| = |A \setminus M| + |M \setminus A| \leq |G \setminus G_n| + |G_n \setminus M| + |G \setminus A| \leq 3\gamma < \varepsilon,$$

which completes the proof. $\square$

Lemma 3.10. *Let $\ell \in \mathcal{A}$ and let $\mathcal{F}$ be the unit ball of (X, v). Then F is measurable and $\|L\| \geq P$.*

Proof. Since $\ell \in \mathcal{A}$, there is a set $J \subset I$, $|I \setminus J| = 0$, and such that $\int_I \ell(x,t)f(t)dt$ has a sense for any $x \in J$ and $f \in (X,v)$. Set $B_n = \{(x,y) \in I^2;\ |\ell(x,t)| \le n\}$. Define

$$\ell_n(x,t) = \ell(x,t)\chi_{B_n}(x,t), \qquad (L_n f)(x) = \int_I \ell_n(x,t)f(t)dt.$$

Remark that by Lemma 2.3 we immediately obtain $\ell_n \in \mathcal{A}$.

Step 1. Now we claim that F is measurable. Note that $F(x)$ is defined for every $x \in J$. Fix $x \in J$. Clearly, $|\ell_n(x,t)| \nearrow |\ell(x,t)|$ in I^2, and, consequently,

$$|\ell_n(x,t)|f(t) \nearrow |\ell(x,t)|f(t) \qquad \text{for any} \quad f \ge 0. \tag{3.45}$$

It is easy to see that

$$F(x) = \sup_{\|f\|_{X,v} \le 1, f \ge 0} \int_I |\ell(x,t)|f(t)dt.$$

By (3.45), we have for $f \ge 0$

$$\int_I |\ell_n(x,t)|f(t)dt \nearrow \int_I |\ell(x,t)|f(t)dt,$$

and, consequently,

$$F_n(x) := \sup_{\|f\|_{X,v} \le 1, f \ge 0} \int_I |\ell_n(x,t)|f(t)dt \nearrow F(x).$$

The function F_n can be expressed also by

$$F_n(x) = \sup_{\|f\|_{X,v} \le 1} \Big| \int_I \ell_n(x,t)f(t)dt \Big|. \tag{3.46}$$

Since $\ell_n \in \mathcal{A}$, we have from Lemma 3.8 that F_n are measurable. Then the fact that $F_n(x) \nearrow F(x)$ for any $x \in J$ shows that F is measurable.

Step 2. We will prove the inequality

$$\|L\| \ge \|L_n\| \text{ for any } \qquad n \in \mathbb{N}. \tag{3.47}$$

Fix $n \in \mathbb{N}$. The norms $\|L\|$ and $\|L_n\|$ are well defined because $\ell_n, \ell \in \mathcal{A}$.

If $\|L\| = \infty$, then (3.47) is trivial. Assume that $\|L\| < \infty$.

If $\|L_n\| = 0$, then (3.47) is evident. Suppose $\|L_n\| > 0$. Choose $0 < \varepsilon_0 < \frac{1}{\sqrt{2}}$ such that $\varepsilon_0 \le \|L_n\|$ is $\|L_n\| < \infty$ and $\|L\| < \frac{1}{\varepsilon}_0 - 2\varepsilon_0$ if $\|L_n\| = \infty$. For $\varepsilon \in (0, \varepsilon_0)$ we define

$$D_\varepsilon = \begin{cases} \|L_n\| - \varepsilon & \text{if} \quad \|L_n\| < \infty \quad \text{and} \quad \varepsilon_0 < \|L_n\|; \\ \frac{1}{\varepsilon} & \text{if} \quad \|L_n\| = \infty \quad \text{and} \quad \|L\| < \frac{1}{\varepsilon_0} - 2\varepsilon_0. \end{cases} \tag{3.48}$$

By the definition of $\|L_n\|$, there exists a function f_0, $\|f_0\|_{X,v} \le 1$, and a set $M \subset J$, $|M| > 0$, such that

$$D_\varepsilon \le \Big| \int_I \ell_n(x,t)f_0(t)dt \Big| \qquad \text{for} \quad x \in M. \tag{3.49}$$

Let $J_1 \subset J$, $J_2 \subset J$ be measurable sets such that

$$\begin{cases} |\int_I \ell(x,t)f_0(t)dt| < \infty & \text{for } x \in J_1, \\ |\int_I \ell(x,t)f_0(t)dt| = \infty & \text{for } x \in J_2. \end{cases} \tag{3.50}$$

If $|J_2| > 0$, then

$$\|L\| \geq \operatorname*{ess\,sup}_{x \in J_2} \Big| \int_I \ell(x,t) f_0(t) dt \Big| = \infty,$$

which is a contradiction. Thus, $|I \setminus J_1| = |J \setminus J_1| = 0$, or

$$\int_I |\ell(x,t)||f_0(t)| dt < \infty \quad \text{for a.e. } x \in J_1. \tag{3.51}$$

Let $\delta > 0$. Set $A_\delta = \{x \in J_1;\ \sup_{|A|<\delta} \int_A |\ell(x,t)||f_0(t)|dt < \varepsilon\}$. Observe that $A_{\delta_2} \subset A_{\delta_1}$ for $0 < \delta_1 < \delta_2$.

We will show now that A_δ is a measurable set. For $x \in J_1$ we define the function

$$H_\delta(x) = \sup_{|A|<\delta} \int_A |\ell(x,t)||f_0(t)| dt.$$

By (3.51), $H_\delta(x) < \infty$ for $x \in J_1$. Let $\mathcal{C}$ be a countable system of sets from Lemma 3.9. Let $\lambda > 0$ and fix $x \in J_1$. By the definition of $H_\delta(x)$, there is a set A_0, such that $|A_0| < \delta$ and

$$H_\delta(x) - \lambda \leq \int_{A_0} |\ell(x,t)||f_0(t)| dt \leq H_\delta(x). \tag{3.52}$$

The absolute continuity of the Lebesgue integral and (3.51) now give the existence of $\eta > 0$ such that

$$\int_B |\ell(x,t)||f_0(t)| dt < \lambda \qquad \text{for} \quad |B| < \eta. \tag{3.53}$$

Obviously, we can choose $\eta < \delta$. Let $N_0 \in \mathcal{C}$ such that $|A_0 \div N_0| < \eta$. Then (3.52) and (3.53) yield

$$\begin{aligned} H_\delta(x) &\geq \sup_{N \in \mathcal{C}} \int_N |\ell(x,t)||f_0(x)| dt \geq \int_{N_0} |\ell(x,t)||f_0(t)| dt \\ &= \int_{A_0} |\ell(x,t)||f_0(t)| dt - \int_{A_0 \setminus N_0} |\ell(x,t)||f_0(t)\ dt \\ &+ \int_{N_0 \setminus A_0} |\ell(x,t)||f_0(t)| dt \geq H_\delta(x) - \lambda. \end{aligned}$$

On letting $\lambda \to 0_+$ we have $H_\delta(x) = \sup_{N \in \mathcal{C}} \int_N |\ell(x,t)||f_0(t)|dt$. Since the function $x \mapsto \int_N |\ell(x,t)||f_0(t)|dt$ is measurable for any fixed $N \in \mathcal{C}$, the function H_δ is measurable as a supremum of countably many measurable functions. Moreover, as $A_\delta = H_\delta^{-1}((0,\varepsilon))$, the set A_δ is measurable for any $\delta > 0$.

Now, the absolute continuity of the Lebesgue integral and (3.51) give $\bigcup_{\delta>0} A_\delta = J_1$. Hence there is a δ_0 such that $|A_{\delta_0} \cap M| > 0$. By Lemma 3.6, there is a set $M_1 \subset (A_{\delta_0} \cap M)$ with the following properties: $|(A_{\delta_0} \cap M) \setminus M_1| = 0$, $M_1 = \mathfrak{D}(M_1)$, and for any $x \in M_1$ there is a set $N_{\delta_0,x} \subset M_1$ such that $N_{\delta_0,x} = \mathfrak{D}(N_{\delta_0,x})$, $x \in N_{\delta_0,x}$ and $|B_{n,y} \div B_{n,z}| < \delta_0$ for $y, z \in N_{\delta_0,x}$. Let $x_0 \in M_1$ be fixed. Then N_{δ_0,x_0} satisfies

$$N_{\delta_0,x_0} = \mathfrak{D}(N_{\delta_0,x_0}), \tag{3.54}$$

$$x_0 \in N_{\delta_0,x_0}, \tag{3.55}$$

$$|B_{n,y} \div B_{n,z}| < \delta_0 \qquad \text{for} \quad y, z \in N_{\delta_0,x_0}. \tag{3.56}$$

The properties (3.54) and (3.55) guarantee that $|N_{\delta_0,x_0}| > 0$. Set $f_1(t) = f_0(t)\chi_{B_{n,x_0}}(t)$. Clearly, $\|f_1\|_{X,v} \le 1$. Fix $x \in N_{\delta_0,x_0}$. Note that (3.55) and (3.56) give

$$|B_{n,x_0} \div B_{n,x}| < \delta_0. \tag{3.57}$$

Obviously,

$$\begin{aligned}\int_I \ell(x,t)f_1(t)dt &= \int_{B_{n,x_0}} \ell(x,t)f_0(t)dt\\ &= \int_{B_{n,x_0}\cap B_{n,x}} \ell(x,t)f_0(t)dt + \int_{B_{n,x_0}\setminus B_{n,x}} \ell(x,t)f_0(t)dt\\ &= \int_I \ell_n(x,t)f_0(t)dt - \int_{B_{n,x}\setminus B_{n,x_0}} \ell(x,t)f_0(t)dt + \int_{B_{n,x_0}\setminus B_{n,x}} \ell(x,t)f_0(t)dt.\end{aligned}$$

By (3.49), (3.57) and $N_{\delta_0,x_0} \subset A_{\delta_0}$, we have

$$D_\varepsilon - 2\varepsilon \le \Big|\int_I \ell(x,t)f_1(t)dt\Big| \quad \text{for any } x \in N_{\delta_0,x_0}.$$

Since $|N_{\delta_0,x_0}| > 0$, we have

$$\|L\| \ge D_\varepsilon - 2\varepsilon.$$

If $\|L_n\| = \infty$, then $\|L\| \ge \frac{1}{\varepsilon} - 2\varepsilon$, which is a contradiction with (3.48). Thus, $\|L_n\| < \infty$ and $\|L\| \ge \|L_n\| - 3\varepsilon$. On letting $\varepsilon \to 0_+$ we obtain (3.47).

Step 3. Denote $P_n = \text{ess sup}_{x\in I} F_n(x)$ for $n \in \mathbb{N}$, where $F_n(x)$ is defined by (3.46). Recall that $P = \text{ess sup}_{x\in\mathbb{N}} F(x)$. By Step 2, F_n and F are measurable, and, consequently, P and P_n are well defined. We shall prove

$$\liminf_{n\to\infty} P_n \ge P. \tag{3.58}$$

Denote for $x \in J$ and $f \in (X,v)$

$$F_n(x,f) = \Big|\int_I \ell_n(x,t)f(t)dt\Big| \qquad \text{and} \qquad F(x,f) = \Big|\int_I \ell(x,t)f(t)dt\Big|.$$

Fix $x \in J$ and $f \in (X,v)$. Let

$$I^+ = \{t;\, \ell(x,t)f(t) > 0\}, \qquad I^- = \{t;\, \ell(x,t)f(t) < 0\}.$$

Obviously, the definition of ℓ_n gives

$$\ell_n(x,t)f(t) \ge 0 \quad \text{on } I^+, \qquad \ell_n(x,t)f(t) \le 0 \quad \text{on } I^-,$$

and

$$\begin{aligned}&\ell_n(x,t)f(t) \nearrow \ell(x,t)f(t) \qquad \text{a.e. in} \quad I^+ \quad \text{for} \quad n\to\infty,\\ &\ell_n(x,t)f(t) \searrow \ell(x,t)f(t) \qquad \text{a.e. in} \quad I^- \quad \text{for} \quad n\to\infty.\end{aligned}$$

Of course, $\ell_n(x,t) = \ell(x,t) = 0$ in $I \setminus (I^+ \cup I^-)$. Then

$$\begin{aligned}&(\ell_n(x,t)f(t))^+ \nearrow (\ell(x,t)f(t))^+ \quad \text{for } n\to\infty,\\ &(\ell_n(x,t)f(t))^- \nearrow (\ell(x,t)f(t))^- \quad \text{for } n\to\infty,\end{aligned}$$

and, consequently,

$$\int_I \ell_n(x,t)f(t)dt \to \int_I \ell(x,t)f(t)dt \quad \text{for } n\to\infty.$$

The last relation implies

$$F_n(x,f) \to F(x,f) \qquad \text{for any} \quad x \in J \quad \text{and} \quad f \in (X,v). \tag{3.59}$$

Let $\varepsilon > 0$. Set $J_1 = \{x;\, F(x) < \infty$, $J_2 = \{x;\, F(x) = \infty$. Fix $x \in J_1$. By the definition of $F(x)$, there exists a function f_0, $\|f_0\|_{X,v} \le 1$, such that

$$F(x) - \varepsilon \le F(x,f_0) \le F(x). \tag{3.60}$$

By (3.59), it is possible to choose n_0 such that for any $n \ge n_0$

$$F(x,f_0) - \varepsilon \le F_n(x,f_0) \le F(x,f_0) + \varepsilon,$$

which together with (3.60) gives

$$F(x) - 2\varepsilon \le F_n(x,f_0) \le F(x) + \varepsilon,$$

and, consequently,

$$F(x) - 2\varepsilon \le F_n(x,f_0) \le \sup_{\|f\|_{X,v}\le 1} F_n(x,f) = F_n(x).$$

Thus, for any $\varepsilon > 0$ there is an n_0 such that

$$F(x) - 2\varepsilon \le F_n(x) \qquad \text{for } n \ge n_0,$$

which in turn yields

$$F(x) \le \liminf_{n\to\infty} F_n(x) \qquad \text{for any} \quad x \in J_1. \tag{3.61}$$

Fix $x \in J_2$. By the definition of $F(x)$ there exists a function f_0, $\|f_0\|_{X,v} \le 1$, such that $\frac{1}{\varepsilon} \le F(x,f_0)$. By (3.59), we have n_0 such that for $n \ge n_0$,

$$\frac{1}{\varepsilon} - \varepsilon \le F_n(x,f_0) \le \sup_{\|f\|_{X,v}} \le 1 = F_n(x,f) = F_n(x).$$

It proves $\lim_{n\to\infty} F_n(x) = \infty$ and, consequently, (3.61) holds for $x \in J_2$.

Let $I_n \subset J$, $|J \setminus I_n| = 0$ such that

$$P_n = \operatorname*{ess\,sup}_{x\in I} F_n(x) = \sup_{x\in I_n} F_n(x). \tag{3.62}$$

Let $J_3 = \bigcap_{n=1}^{\infty} I_n$. Clearly, $|J \setminus J_3| = 0$. By (3.61) and (3.62),

$$\begin{aligned} P &\le \operatorname*{ess\,sup}_{x\in I} \liminf_{n\to\infty} F_n(x) \le \sup_{x\in J_3} \liminf_{n\to\infty} F_n(x) \\ &\le \liminf_{n\to\infty} \sup_{x\in J_3} F_n(x) \le \liminf_{n\to\infty} \sup_{x\in I_n} F_n(x) \\ &= \liminf_{n\to\infty} P_n, \end{aligned}$$

which proves (3.58).

Step 4. We know from Step 1 that $\ell_n \in \mathcal{A}$. Moreover, $|\ell_n(x,t)| \le n$ in I^2. Hence, ℓ_n satisfies the assumptions of Lemma 3.8, and thus $\|L_n\| \ge P_n$ for any $n \in \mathbb{N}$. Using (3.47) and (3.58), we get

$$\|L\| \ge \limsup_{n\to\infty} \|L_n\| \ge \limsup_{n\to\infty} P_n \ge \liminf_{n\to\infty} P_n \ge P$$

which completes the proof. □

Remark 3.11. *Let $\ell \in \mathcal{A}$ and let $\mathcal{F}$ be the unit ball of (X,v). Then $P = \|\ell\|_{L_\infty(X',v)}$ where $\|\ell\|_{L_\infty(X',v)}$ is defined by (3.1).*

Proof. Since $\ell \in \mathcal{A}$ we have that the function

$$F(x) = \sup_{f \in F} \int_I \ell(x,t) f(t) dt$$

is measurable. Clearly, using (2.6) and (2.7) we obtain

$$F(x) = \sup_{f \in F} \left| \int_I \ell(x,t) f(t) dt \right| = \left\| \frac{\ell(x,\cdot)}{v(\cdot)} \right\|_{X',v}$$

and thus

$$P = \operatorname{ess\,sup}_{x \in I} F(x) = \|\ell\|_{L_\infty(X',v)}.$$

□

We have proved one part of Theorem 3.1. To completely verify Theorem 3.1 it remains to prove the following lemma.

Lemma 3.12. *Let $\ell \in \mathcal{A}$. Then $\|L\| \le \|\ell\|_{L_\infty(X',v)}$.*

Proof. By Lemma 3.10 the function $x \mapsto \left\| \frac{\ell(x,\cdot)}{v(\cdot)} \right\|_{X',v}$ is measurable. The definition of $\|\ell\|_{L_\infty(X',v)}$ guarantees that there is a set J, $|I \setminus J| = 0$, such that

$$\|\ell\|_{L_\infty(X',v)} = \sup_{x \in J} \left\| \frac{\ell(x,\cdot)}{v(\cdot)} \right\|_{X',v}.$$

For each $\|f\|_{X,v} \le 1$ we obtain

$$\begin{aligned}
\|Lf\|_\infty &= \inf_{|M|=0} \sup_{x \in (I \setminus M)} \left| \int_I \ell(x,t) f(t) dt \right| \\
&\le \sup_{x \in J} \left| \int_I \ell(x,t) f(t) dt \right| \le \sup_{x \in J} \left\| \frac{\ell(x,\cdot)}{v(\cdot)} \right\|_{X',v} \|f\|_{X,v} \\
&\le \sup_{x \in J} \left\| \frac{\ell(x,\cdot)}{v(\cdot)} \right\|_{X',v} = \|\ell\|_{L_\infty(X',v)},
\end{aligned}$$

which yields $\|L\| \le \|\ell\|_{L_\infty(X',v)}$. The proof is complete. □

We define $L_\infty(X',v)$ as the set of all $\ell \in \mathcal{A}$ such that

$$\|\ell\|_{L_\infty(X',v)} < \infty.$$

Lemma 3.13. *The set $L_\infty(X',v)$, equipped with the norm $\|\cdot\|_{L_\infty(X',v)}$, is a Banach space. Moreover, it satisfies*

(i) $\|\ell\|_{L_\infty(X',v)} = \|\,|\ell|\,\|_{L_\infty(X',v)}$;
(ii) *if $0 \le \ell_n \nearrow \ell$ a.e. in I^2 and $\ell \in \mathcal{A}$, then $\|\ell_n\|_{L_\infty(X',v)} \nearrow \|\ell\|_{L_\infty(X',v)}$.*

Proof. Clearly, $L_\infty(X',v)$ is a linear space, and $\|\cdot\|_{L_\infty(X',v)}$ defines a norm. The properties (i) is obvious. Let us prove (ii). By Lemma 2.3 it is $\ell_n \in \mathcal{A}$. Set $F_n(x) = \|\frac{\ell_n(x,\cdot)}{v(\cdot)}\|_{X',v}$, $F(x) = \|\frac{\ell(x,\cdot)}{v(\cdot)}\|_{X',v}$. Since (X',v) and L_∞ are Banach function spaces we have $F_n(x) \nearrow F(x)$ for a.e. $x \in I$ and, consequently, $\|F_n\|_{L_\infty} \nearrow \|F\|_{L_\infty}$, which proves (ii). Let us prove the completeness of $L_\infty(X',v)$. Let ℓ_n be a Cauchy sequence in $L_\infty(X',v)$. Take $J_{n,m} \subset I$ such that $|I \setminus J_{n,m}| = 0$ and

$$\|\ell_n - \ell_m\|_{L_\infty(X',v)} = \sup_{x \in J_{n,m}} \left\| \frac{\ell_n(x,\cdot) - \ell_m(x,\cdot)}{v(\cdot)} \right\|_{X',v}.$$

Set

$$\widetilde{J} = \bigcap_{n,m=1}^{\infty} J_{n,m}.$$

Clearly, $|I \setminus \widetilde{J}| = 0$.

Let $\varepsilon > 0$. Since ℓ_n is a Cauchy sequence in $L_\infty(X', v)$we have n_0 such that

$$\|\ell_n - \ell_m\|_{L_\infty(X',v)} \le \sup_{x\in\widetilde{J}} \Big\| \frac{\ell_n(x,\cdot) - \ell_m(x,\cdot)}{v(\cdot)} \Big\|_{X',v} < \varepsilon \text{ for all } n, m > n_0. \tag{3.63}$$

Fix $x \in \widetilde{J}$. Then the sequence $\frac{\ell_n(x,\cdot)}{v(\cdot)}$ is a Cauchy sequence in (X', v). Since (X', v) is a Banach space there exists a unique function $\ell(x,\cdot)$ such that $\frac{\ell_n(x,\cdot)}{v(\cdot)} \to \frac{\ell(x,\cdot)}{v(\cdot)}$ in (X', v). Thus, $\|\frac{\ell(x,\cdot)}{v(\cdot)}\|_{X',v} < \infty$ and

$$\int_I |\ell(x,t)f(t)|dt \le \Big\| \frac{\ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} \|f\|_{X,v} < \infty$$

for each $f \in (X, v)$ which proves that $\int_I |\ell(x,t)f(t)|dt$ has a sense for all $f \in (X, v)$ and $x \in \widetilde{J}$. Consequently, $\ell \in \mathcal{A}$.

It remains to prove $\ell_n \to \ell$ in $L_\infty(X', v)$. Let $\varepsilon > 0$ and $x \in \widetilde{J}$. Let n_0 satisfies (3.63). Since $\frac{\ell_n(x,\cdot)}{v(\cdot)} \to \frac{\ell(x,\cdot)}{v(\cdot)}$ in (X', v) we can find $m > n_0$ such that $\|\frac{\ell_m(x,\cdot)-\ell(x,\cdot)}{v(\cdot)}\|_{X',v} < \varepsilon$. Then

$$\Big\| \frac{\ell_n(x,\cdot) - \ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} \le \Big\| \frac{\ell_n(x,\cdot) - \ell_m(x,\cdot)}{v(\cdot)} \Big\|_{X',v} + \Big\| \frac{\ell_m(x,\cdot) - \ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} < 2\varepsilon$$

which gives for $n \to \infty$

$$\operatorname*{ess\,sup}_{x\in I} \Big\| \frac{\ell_n(x,\cdot) - \ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} \le \sup_{x\in\widetilde{J}} \Big\| \frac{\ell_n(x,\cdot) - \ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} \to 0$$

which finishes the proof. □

It is a routine procedure to extend the result of Theorem 3.1 to an arbitrary interval $I = [a, b]$, $-\infty < a < b < \infty$. This finishes the proof of Theorem 3.1.

4. Compactness of a general kernel operator

In this section we investigate the distance of the operator L from the set of all compact operators $K : (X, v) \to L_\infty$. Define

$$D = \inf\{\|L - K\|;\ K \in \mathcal{K}\},$$

where $\mathcal{K}$ is the set of all compact operators. Denote by $\mathfrak{R}$ the set of all kernels $k \in \mathfrak{M}(I^2)$ that can be written as

$$k(x,t) = \sum_{i=1}^{n} \chi_{M_i}(x)\psi_i(t)$$

for some $n \in \mathbb{N}$, $\chi_{M_i} \in \mathfrak{M}(I)$, and $\frac{\psi_i}{v} \in (X', v)$. Clearly, $k \in \mathfrak{R}$ implies $k \in L_\infty(X', v)$. Let $\mathfrak{C}$ be the closure of $\mathfrak{R}$ in $L_\infty(X', v)$. Define further

$$d := \inf\{\|\ell - k\|_{L_\infty(X',v)};\ k \in \mathfrak{C}\} = \inf\{\|\ell - k\|_{L_\infty(X',v)};\ k \in \mathfrak{R}\}.$$

Our main aim in this section is to prove that D is comparable to d and, consequently, that an operator L is compact if and only if its kernel ℓ can be approximated in $L_\infty(X', v)$ by kernels from $\mathfrak{R}$.

Let L be a fixed linear operator given by a kernel $\ell \in L_\infty(X', v)$.

Theorem 4.1. *$\frac{d}{2} \le D \le d$.*

The proof will be given in a series of lemmas.

Lemma 4.2. *Let $k \in \mathfrak{R}$. Then the operator $(Kf)(x) = \int_I k(x,t)f(t)dt$ is a finite-dimensional bounded operator. Consequently, K is compact.*

Proof. Let $k(x,t) = \sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)$, $\chi_{M_i} \in \mathfrak{M}(I)$ and $\frac{\psi_i}{v} \in (X', v)$. Since $k \in L_\infty(X', v)$, the operator K is bounded by Theorem 3.1. Moreover,

$$Kf(x) = \sum_{i=1}^n \int_I \psi_i(t)f(t)dt\chi_{M_i}(x) = \sum_{i=1}^n A_i\chi_{M_i}(x).$$

Now, (2.6) gives $|A_i(f)| \le \|\frac{\psi_i}{v}\|_{X',v}\|f\|_{X,v}$ which implies $A_i \in (X,v)^*$ (the dual space) and, consequently, K is a bounded operator. □

Now we are in a position to prove the second inequality in Theorem 4.1.

Lemma 4.3. *$D \le d$.*

Proof. By Lemmas 3.12 and 4.2, we can write

$$\begin{aligned} D &\le \inf_{k\in\mathfrak{R}} \sup_{\|f\|_{X,v}\le 1} \operatorname{ess\,sup}_{x\in I} \int_I |\ell(x,t) - k(x,t)||f(t)|dt \\ &\le \inf_{k\in\mathfrak{R}} \sup_{\|f\|_{X,v}<1} \operatorname{ess\,sup}_{x\in I} \left\| \frac{\ell(x,\cdot) - k(x,\cdot)}{v(\cdot)} \right\|_{X',v} \|f\|_{X,v} \\ &= \inf_{k\in\mathfrak{R}} \|\ell - k\|_{L_\infty(X',v)} = d \end{aligned}$$

which proves the assertion. □

In the rest of this section we show the first inequality in Theorem 4.1.

Definition 4.4. *We say that a finite system of sets $\mathcal{A} = \{\Omega_j;\ j = 1,2,\dots,n\}$ is a partition of I if $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$, and $\bigcup_{j=1}^n \Omega_j = I$.*

Lemma 4.5. *Let n, N be positive integers. Let $\mathcal{A}_i = \{\Omega_j^i;\ j = 1,2,\dots,N\}$, $i = 1,2,\dots,n$ be partitions of I. Then there is a positive integer m and a partition $\mathcal{A} = \{E_k;\ k = 1,2,\dots,m\}$ of I such that*

(4.1) *for any $i \in \{1,2,\dots,n\}$ and $k \in \{1,2,\dots,m\}$ there exists a $j \in \{1,2,\dots,N\}$ such that $E_k \subset \Omega_j^i$.*

Proof. We use the induction on n. Let $n = 1$. Then the assertion is obvious.

Assume that $\mathcal{A}_i = \{\Omega_j^i;\ j = 1,2,\dots,N\}$, $i = 1,2,\dots,n+1$, are partitions of I. By the induction assumption, there is a partition of I, $\widetilde{\mathcal{A}} = \{\widetilde{E}_k;\ k = 1,2,\dots,\widetilde{m}\}$ such that

for any $i \in \{1,2,\dots,n\}$ and $k \in \{1,2,\dots,\tilde{m}\}$ there exists a $j \in \{1,2,\dots,N\}$ such that $\tilde{E}_k \subset \Omega_j^i$.

Set $F_{kj} = \widetilde{E}_k \cap \Omega_j^{n+1}$, $k \in \{1, 2, \ldots, \widetilde{m}\}$, and $j \in \{1, 2, \ldots, N\}$. Define a system of sets $\mathcal{A}$ by

$$\mathcal{A} = \{F_{kj};\ k \in \{1, 2, \ldots, \widetilde{m}\} \text{ and } j \in \{1, 2, \ldots, N\}\}.$$

It is not difficult to verify that $\mathcal{A}$ is a partition of I with the required properties. □

Let B be the unit ball in (X, v). Let $M \subset L_\infty$ and $\eta > 0$. We say that $N \subset L_\infty$ is a η-net in M if for every $f \in M$ there is a $g \in N$ with $\|f - g\|_{L_\infty} \leq \eta$.

Lemma 4.6. *Let*

$$\sigma = \inf\{\eta;\ \textit{there exists a finite } \eta - \textit{net of } L(B)\}.$$

Then $\sigma \leq D$.

Proof. Let $\varepsilon > 0$. Take $K \in \mathcal{K}$ such that

$$\|L - K\| \leq D + \varepsilon.$$

Since $K \in \mathcal{K}$, there exists a finite ε-net $\{g_1, g_2, \ldots, g_n\}$ of $K(B)$. Let $g \in L(B)$. Then there is a function $f \in B$ such that $Lf = g$. Choose g_i with $\|Kf - g_i\|_{L_\infty} \leq \varepsilon$. Then

$$\|g - g_i\|_{L_\infty} = \|Lf - g_i\|_{L_\infty} \leq \|Lf - Kf\|_{L_\infty} + \|Kf - g_i\|_{L_\infty} \leq D + 2\varepsilon.$$

Thus, $\{g_1, \ldots, g_n\}$ is a finite $(D + 2\varepsilon)$-net of $L(B)$ and, consequently, $\sigma \leq D + 2\varepsilon$. On letting $\varepsilon \to 0_+$ we obtain the assertion. □

It is worth noting that Lemma 4.6 remains true under more general assumptions, namely, for Banach spaces X, Y, a bounded linear operator $T : X \to Y$, and σ, D defined in an analogous way.

Lemma 4.7. *Let λ be a measure on I such that λ-measurable sets coincide with the Lebesgue measurable sets, and $\lambda(E) = 0$ if and only if $|E| = 0$. Let $h(x, t) \in \mathfrak{M}(I^2)$, such that $h(x, t)v(t) \in \mathcal{A}$. Then the function $x \mapsto \|h(x, \cdot)\|_{X',v}$ is λ-measurable. Moreover, for $E \subset I$ measurable, $0 < \lambda(E) < \infty$, we have*

$$\Big\|\frac{1}{\lambda(E)} \int_I h(x, \cdot) d\lambda(x)\Big\|_{X',v} \leq \frac{1}{\lambda(E)} \int_I \|h(x, \cdot)\|_{X',v}\, d\lambda(x).$$

Proof. Define $F(x) = \|h(x, \cdot)\|_{X',v}$. Clearly,

$$F(x) = \sup_{\|f\|_{X,v} \leq 1} \int_I |h(x, t)f(t)|v(t)dt = \sup_{\|f\|_{X,v} \leq 1} \Big|\int_I h(x, t)f(t)v(t)dt\Big|.$$

By Theorem 3.1, the last expression is a Lebesgue measurable function, whence F is Lebesgue measurable. Due to the assumptions on λ, F is λ-measurable, which proves the first part of the lemma.

Now, using the Fubini theorem, we have

$$\Big\|\frac{1}{\lambda(E)} \int_I h(x, \cdot) d\lambda(x)\Big\|_{X',v} = \frac{1}{\lambda(E)} \sup_{\|f\|_{X,v} \leq 1} \Big|\int_I \int_I h(x, t)f(t)v(t)dt d\lambda(x)\Big| = A,$$

say. Assume that $A < \infty$ (the case $A + \infty$ can be handled analogously). Let $\varepsilon > 0$. Then there is an $f_0 \in B$ such that

$$A - \varepsilon \le \frac{1}{\lambda(E)} \Big| \int_I \int_I h(x,t) f_0(t) v(t) dt d\lambda(x) \Big| \le \frac{1}{\lambda(E)} \int_I F(x) d\lambda(x) = \frac{1}{\lambda(E)} \int_I \|h(x,\cdot)\|_{X',v} \, d\lambda(x).$$

On letting $\varepsilon \to 0_+$ we obtain the assertion. □

The main idea of the proof of the following lemma is taken from [10].

Lemma 4.8. *The inequality $\frac{d}{2} \le \sigma$ holds.*

Proof. Let $\varepsilon > 0$. Let $\{g_1, g_2, \dots, g_n\}$ be a finite $(\sigma + \varepsilon)$-net of $L(B)$. Since $L(B)$ is bounded in L_∞, the set $\{g_1, g_2, \dots, g_n\}$ is bounded in L_∞, too. Hence there exists an $A > 0$ such that $\operatorname{ess\,sup}_{x \in I} |g_i(x)| \le A, \quad i = 1, 2, \dots, n$. We can even assume that $\sup_{x \in I} |g_i(x)| \le A$ because in the opposite case we simply change every function g_i on a set of measure zero.

Let $\{I_j;\, j = 1, 2, \dots, N\}$ be a partition of $[-A, A]$ such that I_j are intervals and $|I_j| \le \varepsilon$. Let $\Omega^i_j = g_i^{-1}(I_j)$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, N$. Then the systems $\mathcal{A}_i = \{\Omega^i_j;\, j = 1, 2, \dots, N\}$ are partitions of I. By Lemma 3.5, there is a partition of I, say, $\mathcal{A} = \{E_k;\, k = 1, 2, \dots, m\}$, such that (4.1) holds.

Let $B = \{E_k \in \mathcal{A};\, |E_k| > 0\}$. Then we can write $B = \{E_k;\, k = 1, 2, \dots, m_1\}$ where $m_1 \le m$. Clearly,

(4.2) $$E_{k_1} \cap E_{k_2} = \emptyset, \qquad k_1, k_2 \in \{1, \dots, m_1\}, \quad k_1 \ne k_2,$$

(4.3) $$\Big| I \setminus \bigcup_{k=1}^{m_1} E_k \Big| = 0,$$

and

(4.4) for every $\quad i \in \{1, 2, \dots, n\} \quad$ and $\quad k \in \{1, 2, \dots, m_1\} \quad$ there is

(4.5) a $\quad j \in \{1, 2, \dots, N\} \quad$ such that $\quad E_k \subset \Omega^i_j$.

We define the operator

$$(P_\varepsilon f)(x) = \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{\int_{E_k} f(t) e^{-t^2} dt}{\int_{E_k} e^{-t^2} dt}.$$

Then $(P_\varepsilon f)(x)$ is defined on $\bigcup_{k=1}^{m_1} E_k$ and therefore, by (4.3), it is defined a.e. on I. It is not difficult to see that $P_\varepsilon : L_\infty \to L_\infty$ is a bounded linear finite-dimensional operator. Moreover, using (4.2), we obtain

$$\begin{aligned}(P_\varepsilon^2 f)(x) &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{1}{\int_{E_k} e^{-t^2} dt} \int_{E_k} (P_\varepsilon f)(t) e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} \frac{\chi_{E_k}(x)}{\int_{E_k} e^{-t^2} dt} \int_{E_k} \sum_{\ell=1}^{m_1} \chi_{E_\ell}(t) \frac{\int_{E_\ell} f(s) e^{-s^2} ds}{\int_{E_\ell} e^{-s^2} ds} e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{\int_{E_k} f(s) e^{-s^2} ds}{\int_{E_k} e^{-s^2} ds} = (P_\varepsilon f)(x),\end{aligned}$$

which proves

$$P_\varepsilon^2 = P_\varepsilon.$$

In other words, P_ε is a projection. Further, due to (4.2),

$$\|P_\varepsilon f\|_{L_\infty} \le \|f\|_{L_\infty} \operatorname*{ess\,sup}_{x\in I} \sum_{k=1}^{m_1} \chi_{E_k}(x) = \|f\|_{L_\infty},$$

which gives

$$\|P_\varepsilon\| \le 1. \tag{4.6}$$

Let Z be the finite-dimensional subspace of L_∞ defined by

$$Z = \{f = \sum_{k=1}^{m_1} a_k \chi_{E_k}(x);\ (a_1,\dots,a_{m_1}) \in \mathbb{R}^{m_1}\}.$$

In fact, $P_\varepsilon : L_\infty \to Z$. Moreover, let $f = \sum_{k=1}^{m_1} a_k \chi_{E_k}(x) \in Z$. Then, by (4.2), we can write

$$\begin{aligned}(P_\varepsilon f)(x) &= \sum_{k=1}^{m_1} \chi_{E_k}(x) \frac{1}{\int_{E_k} e^{-t^2} dt} \int_{E_k} a_k e^{-t^2} dt \\ &= \sum_{k=1}^{m_1} a_k \chi_{E_k}(x) = f(x),\end{aligned}$$

which shows that

$$P_\varepsilon f = f \quad \text{for any} \quad f \in Z. \tag{4.7}$$

We claim that dist $(g_i, Z) \le \varepsilon$ for any $i \in \{1,2,\dots,n\}$. Fix $i \in \{1,2,\dots,n\}$. By (4.4), for every $k \in \{1,2,\dots,m_1\}$ there is a set $\Omega^i_{j_k}$ such that $E_k \subset \Omega^i_{j_k}$. Consequently, $g_i(E_k) \subset I_{j_k}$. Choose $\gamma_k \in I_{j_k}$, $k = 1,2,\dots,m_1$ and define the function $\bar g_i$ by

$$\bar g_i(x) = \sum_{k=1}^{m_1} \gamma_k \chi_{E_k}(x).$$

Then $\bar g_i \in Z$ and, moreover, $|I_{j_k}| \le \varepsilon$ implies that

$$\|g_i - \bar g_i\|_{L_\infty} = \sup_{k\in\{1,2,\dots,m_1\}} \operatorname*{ess\,sup}_{x\in E_k} |g_i(x) - \gamma_k| \le \varepsilon. \tag{4.8}$$

Let $f \in B$. We shall estimate $\|Lf - P_\varepsilon Lf\|_{L_\infty}$. Choose g_i such that $\|Lf - g_i\|_{L_\infty} \le \sigma + \varepsilon$. Then

$$\begin{aligned}\|Lf - P_\varepsilon Lf\|_{L_\infty} &\le \|Lf - g_i\|_{L_\infty} + \|P_\varepsilon(Lf - g_i)\|_{L_\infty} + \|g_i - \bar g_i\|_{L_\infty} + \|\bar g_i - P_\varepsilon g_i\|_{L_\infty} \\ &\le \sigma + \varepsilon + \|P_\varepsilon\|(\sigma+\varepsilon) + \|g_i - \bar g_i\|_{L_\infty} + \|\bar g_i - P_\varepsilon g_i\|_{L_\infty}.\end{aligned}$$

Using (4.6)–(4.8) and $\bar g_i = P\bar g_i$, we get

$$\|Lf - P_\varepsilon Lf\|_{L_\infty} \le 2\sigma + 3\varepsilon + \|P_\varepsilon(\bar g_i - g_i)\|_{L_\infty} \le 2\sigma + 4\varepsilon,$$

that is,

$$\|L - P_\varepsilon L\| \le 2\sigma + 4\varepsilon. \tag{4.9}$$

Now let us deal with $(P_\varepsilon Lf)(x)$. Clearly,

$$
\begin{aligned}
(P_\varepsilon Lf)(x) &= \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} (Lf)(t) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} \\
&= \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{1}{\int_{E_j} e^{-t^2} dt} \int_{E_j} \int_I \ell(t,s) f(s) ds e^{-t^2} dt \\
&= \int_I \Big(\sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} \ell(t,s) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} \Big) f(s) ds \\
&= \int\limits_I k_\varepsilon(x,s) f(s) ds,
\end{aligned}
$$

where

$$
k_\varepsilon(x,s) = \sum_{j=1}^{m_1} \chi_{E_j}(x) \frac{\int_{E_j} \ell(t,s) e^{-t^2} dt}{\int_{E_j} e^{-t^2} dt} = \sum_{j=1}^{m_1} \chi_{E_j}(x) \psi_j(t) \qquad \text{say.} \tag{4.10}
$$

Thus, $P_\varepsilon L$ is a kernel operator with the kernel $k_\varepsilon(x,s)$. Now, $\ell \in \mathfrak{M}(I^2)$ implies $\psi_j(s) \in \mathfrak{M}(I)$, and, consequently, $k_\varepsilon \in \mathfrak{M}(I^2)$.

Define the measure λ on I by $\lambda(E) = \int_E e^{-t^2} dt$. It is not difficult to prove that λ satisfies the assumptions of Lemma 4.7. Moreover, we have $0 < \lambda(E_j) \le \lambda(I) = \int_I e^{-t^2} dt \le \int_{-\infty}^{\infty} e^{-t^2} dt < \infty$ for any $j \in \{1, 2, \dots, m_1\}$. Setting $h(x,t) = \frac{\ell(x,t)}{v(t)}$, we have $h(x,t)v(t) \in \mathcal{A}$, and, using also Lemma 4.7, we can write

$$
\begin{aligned}
\Big\| \frac{\psi_j(s)}{v(s)} \Big\|_{X',v} &= \Big\| \int_{E_j} \frac{\ell(t,s) e^{-t^2}}{v(s) \int_{E_j} e^{-t^2} dt} dt \Big\|_{X',v} = \Big\| \frac{1}{\lambda(E_j)} \int_{E_j} h(t,\cdot) d\lambda(t) \Big\|_{X',v} \\
&\le \frac{1}{\lambda(E_j)} \int_{E_j} \|h(t,\cdot)\|_{X',v} \, d\lambda(t) = \frac{1}{\lambda(E_j)} \int_{E_j} \Big\| \frac{\ell(t,\cdot)}{v(\cdot)} \Big\|_{X',v} d\lambda(t) \\
&\le \operatorname*{ess\,sup}_{x \in E_j} \Big\| \frac{\ell(x,\cdot)}{v(\cdot)} \Big\|_{X',v} \le \|\ell\|_{L_\infty(X',v)},
\end{aligned}
$$

which gives

$$
\Big\| \frac{\psi_j}{v} \Big\| \le \Big\| \ell \Big\|_{L_\infty(X',v)}. \tag{4.11}
$$

This implies

$$
\begin{aligned}
\|\ell - k_\varepsilon\|_{L_\infty(X',v)} &\le \|\ell\|_{L_\infty(X',v)} + \Big\| \sum_{j=1}^{m_1} \chi_{E_j}(x) \psi_j(s) \Big\|_{L_\infty(X',v)} \\
&\le \|\ell\|_{L_\infty(X',v)} + \sum_{j=1}^{m_1} \|\chi_{E_j}\|_{L_\infty} \Big\| \frac{\psi_j}{v} \Big\|_{X',v}.
\end{aligned}
$$

From (4.11) we obtain $\|\ell - k_\varepsilon\|_{L_\infty(X',v)} \le (1 + m_1)\|\ell\|_{L_\infty(X',v)}$ and, consequently, using also Lemma 3.12, we have $\ell - k_\varepsilon \in \mathcal{A}$. Now, Theorem 3.1 yields

$$
\|\ell - k_\varepsilon\|_{L_\infty(X',v)} = \|L - P_\varepsilon L\|.
$$

Together with (4.9) this implies

$$
\|\ell - k_\varepsilon\| \le 2\sigma + 4\varepsilon.
$$

By (4.10) and (4.11), $k_\varepsilon \in \mathfrak{R}$ and, consequently,

$$d = \inf_{k\in\mathfrak{R}} \|\ell - k\|_{L_\infty(X',v)} \le \|\ell - k_\varepsilon\|_{(X',v)} \le 2\sigma + 4\varepsilon.$$

On letting $\varepsilon \to 0_+$ we obtain $d \le 2\sigma$, and the proof is complete. □

The first inequality in Theorem 4.1 now follows from Lemmas 4.5 and 4.7.

5. Application to the Hardy operator

Let $I = [a, b]$, $-\infty \le a < b \le +\infty$. We define the Hardy operator by $Hf(x) = \int_a^x f(t)dt$. Further, let

$$U(x,\varepsilon) = \begin{cases} (x-\varepsilon, x+\varepsilon)\cap[a,b] & \text{if} \quad -\infty < x < \infty \\ (-\infty, -\frac{1}{\varepsilon})\cap[a,b] & \text{if} \quad x = -\infty \\ (\frac{1}{\varepsilon}, \infty)\cap[a,b] & \text{if} \quad x = \infty. \end{cases}$$

We also denote

$$B(x) = \lim_{\varepsilon\to 0_+} \|\chi_{U(x,\varepsilon)}\frac{1}{v}\|_{X',v}, \qquad \text{and} \qquad B = \sup_{a\le x\le b} B(x).$$

In [5], a characterization of the boundedness and compactness of the Hardy operator was characterized for $I = [0, \infty]$. It was shown that H is bounded if and only if $\frac{1}{v} \in (X', v)$, and that H is compact if and only if $B = 0$. We will apply the results of Sections 3 and 4 to the Hardy operator and $I = [a, b]$. Observe that the Hardy operator is given by the kernel $h(x, t) = \chi_{(a,x)}(t)$, i.e. $\int_a^x f(t)dt = \int_I \chi_{(a,x)}(t)f(t)dt$.

Theorem 5.1. *The operator H is bounded from (X, v) into L_∞ if and only if $\|\frac{1}{v}\|_{X',v} < \infty$.*

Proof. By Theorem 3.1, H is bounded if and only if $\|h\|_{L_\infty(X',v)} < \infty$. Moreover, $\|H\| = \|h\|_{L_\infty(X',v)}$. Then

$$\|H\| = \operatorname{ess\,sup}_{x\in I} \|\frac{h(x,\cdot)}{v(\cdot)}\|_{X',v} = \operatorname{ess\,sup}_{x\in I} \|\frac{\chi_{(a,x)}(t)}{v(t)}\|_{X',v} = \|\frac{1}{v}\|_{X',v},$$

which completes the proof. □

Lemma 5.2. *The inequality $d \le B$ holds.*

Proof. Let $\varepsilon > 0$. From the definition of B we know that for every $x \in [a, b]$ there is an $\eta(x) > 0$ such that

$$\Big\|\chi_{U(x,\eta(x))}(t)\frac{1}{v(t)}\Big\|_{X',v} \le B + \varepsilon.$$

Since $\bigcup_{x\in I} U(x, \eta(x)) \supset I$ and $I = [a, b]$ is a compact set in the topology induced by $U(x, \varepsilon)$, we can choose $x_1, \ldots, x_n \in I$ such that $\bigcup_{i=1}^n U(x_i, \eta(x_i)) \supset I$. Denote $\widetilde{U}_i = U(x_i, \eta(x_i))$. Take $\alpha_i, \beta_i,\ i = 1, 2, \ldots, n$, such that

$$U_i := (\alpha_i, \beta_i) \subset \widetilde{U}_i, \qquad i = 1, 2, \ldots, n, \tag{5.1}$$

and

$$\sum_{i=1} n\chi_{U_i}(x) = 1 \qquad \text{a.e. in } I. \tag{5.2}$$

Let us define $k(x,t)=\sum_{i=1}^n \chi_{U_i}(x)\chi_{(a,\alpha_i)}(t)$. Clearly, by (5.1) and (5.2), we have

$$\begin{aligned} d &\le \operatorname*{ess\,sup}_{x\in I}\Big\|\frac{1}{v(t)}\sum_{i=1}^n \chi_{U_i}(x)(\chi_{(a,x)}(t)-\chi_{(a,\alpha_i)}(t))\Big\|_{X',v}\\ &= \operatorname*{ess\,sup}_{x\in I}\Big\|\frac{1}{v(t)}\sum_{i=1}^n \chi_{U_i}(x)\chi_{(\alpha_i,x)}(t)\Big\|_{X',v}\\ &\le \operatorname*{ess\,sup}_{x\in I}\Big\|\frac{1}{v(t)}\sum_{i=1}^n \chi_{U_i}(x)\chi_{U_i}(t)\Big\|_{X',v}\\ &\le \operatorname*{ess\,sup}_{x\in I}\sum_{i=1}^n \chi_{U_i}(x)\Big\|\frac{\chi_{U_i}(t)}{v(t)}\Big\|_{X',v}\le B+\varepsilon. \end{aligned}$$

Therefore, $d\le B+\varepsilon$ for any $\varepsilon>0$, and the assertion follows. □

Lemma 5.3. *The inequality $B\le 4d$ holds.*

Proof. Let $\varepsilon>0$. Then, for some M_i and ψ_i, $i=1,\dots,n$,

$$\Big\|\chi_{(a,x)}(t)-\sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)\Big\|_{L_\infty(X',v)}\le d+\varepsilon. \tag{5.3}$$

Let $x_0\in[a,b)$. Then there is a $k\in\{1,2,\dots,n\}$ such that $|(x_0,x_0+\sigma)\cap M_k|>0$ for any $\sigma>0$. Set $x_1=\operatorname{ess\,sup} M_k$, i.e., $x_1=\inf\{y;\ |(y,b)\cap M_k|=0\}$. Let $N_k=M_k\cap(x_0,x_1)$. Then (5.3) gives

$$\begin{aligned} d+\varepsilon &\ge \operatorname*{ess\,sup}_{x\in I}\Big\|\frac{1}{v(t)}\Big(\chi_{(a,x)}(t)-\sum_{i=1}^n \chi_{M_i}(x)\psi_i(t)\Big)\chi_{N_k}(x)\chi_{(x_0,x_1)}(t)\Big\|_{X',v}\\ &= \operatorname*{ess\,sup}_{x\in I}\Big\|\frac{1}{v(t)}\Big(\chi_{(x_0,x)}(t)-\chi_{(x_0,x_1)}(t)\psi_k(t)\Big)\chi_{N_k}(x)\Big\|_{X',v}\\ &= \operatorname*{ess\,sup}_{x\in N_k}\Big\|\frac{1}{v(t)}(\chi_{(x_0,x)}(t)(1-\psi_k(t))-\chi_{(x,x_1)}(t)\psi_k(t)\Big\|_{X',v}. \end{aligned}$$

Since $(x_0,x)\cap(x,x_1)=\emptyset$ for every $x\in N_k$, we have

$$\begin{aligned} d+\varepsilon &\ge \operatorname*{ess\,sup}_{x\in N_k}\Big\|\frac{1}{v(t)}\chi_{(x_0,x)}(t)(1-\psi_k(t))\Big\|_{X',v}\\ &= \Big\|\frac{1}{v(t)}\chi_{(x_0,x_1)}(t)(1-\psi_k(t))\Big\|_{X',v} \end{aligned}$$

and

$$d+\varepsilon\ge \operatorname*{ess\,sup}_{x\in N_k}\Big\|\frac{1}{v(t)}\chi_{(x,x_1)}(t)\psi_k(t)\Big\|_{X',v}=\Big\|\frac{1}{v(t)}\chi_{(x_0,x_1)}(t)\psi_k(t)\Big\|_{X',v}.$$

As a consequence we obtain

$$\Big\|\frac{\chi_{(x_0,x_1)}(t)}{v(t)}\Big\|_{X',v}\le\Big\|\chi_{(x_0,x_1)}(t)\frac{1-\psi_k(t)}{v(t)}\Big\|_{X',v} \tag{5.4}$$

$$+\Big\|\chi_{(x_0,x_1)}(t)\frac{\psi_k(t)}{v(t)}\Big\|_{X',v}\le 2d+2\varepsilon. \tag{5.5}$$

Let $B^+(x) = \lim_{\varepsilon\to 0_+} \|\frac{\chi_{(x,x+\varepsilon)}(t)}{v(t)}\|_{X',v}$ for $x \in [a,b)$ and, analogously, $B^-(x) = \lim_{\varepsilon\to 0_+} \|\frac{\chi_{(x-\varepsilon,x)}(x)}{v(t)}\|_{X',v}$ for $x \in (a,b]$. Then $B(a) = B^+(a)$, $B(b) = B^-(b)$ and $B(x) \le B^+(x) + B^-(x)$ for $x \in (a,b)$ which together with (5.4) yields

$$B(x_0) \le B^+(x_0) + B^-(x_0) \le 4d + 4\varepsilon.$$

Letting $\varepsilon \to 0_+$, we obtain $B(x_0) \le 4d$ and, consequently, $B \le 4d$, which completes the proof. □

Corollary 5.4. *The inequalities* $\frac{B}{8} \le D \le B$ *hold. The Hardy operator is compact if and only if* $B = 0$.

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