SPECTRA OF $1D$ PERIODIC DIRAC OPERATORS AND SMOOTHNESS OF POTENTIALS

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Abstract. Consider on $[0,1]$ the operator

$$Ly = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + \begin{pmatrix}0 & p(x)\\ q(x) & 0\end{pmatrix} y, \qquad y = \begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

where $p(x), q(x)$ are 1-periodic functions, with Dirichlet ($y_1(0) = y_2(0)$, $y_1(1) = y_2(1)$), periodic ($y(0) = y(1)$), or antiperiodic ($y(0) = -y(1)$) boundary conditions. For large $|n|$ the operator L has close to $n\pi$ a triple of a Dirichlet eigenvalue μ_n and periodic (if n is even), or antiperiodic (if n is odd) eigenvalues λ_n^+, λ_n^-. Let Δ_n be the diameter of the spectral triangle with vertices $\mu_n, \lambda_n^+, \lambda_n^-$.

This note gives a series of results about the relationship between the decay rate of the sequence (Δ_n) and the smoothness of the potential functions p and q (measured by appropriate weighted Hilbert norms). Moreover, finite-zone potentials are dense in the case where the Hilbertian norm is defined by subexponential weights; the potentials, which lead to divergent spectral decompositions, are dense as well.

0. We consider Dirac operators

$$L = iJ\frac{d}{dx} + V, \quad J = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad V = \begin{pmatrix}0 & p\\ q & 0\end{pmatrix}$$

on $I = [0,1]$ with boundary conditions (bc) of three types: for $F = \begin{pmatrix}f_1\\ f_2\end{pmatrix} \in H^1 \times H^1$

$$Per^+: \quad F(0) = F(1); \qquad Per^-: \quad F(0) = -F(1)$$

and

$$Dir: \quad f_1(0) = f_2(0), \quad f_1(1) = f_2(1).$$

A potential V is assumed to be in $L^2(I)$, i.e. *complex valued* functions p, q are in $L^2(I)$. We consider them as periodic functions on $\mathbb{R}$, $V(x+1) = V(x)$, and their smoothness is measured by a weight sequence

$$\Omega(k) \geq 1, \quad \Omega(0) = 1, \quad \Omega(k) = \Omega(-k), \quad \Omega(k) \nearrow \infty, \tag{1}$$

i.e. $V \in H(\Omega)$ if $p = \sum p_k \exp(2\pi i k x) \quad q = \sum q_k \exp(2\pi i k x)$ and

$$\|V|H(\Omega)\|^2 = \sum \left(|p_k|^2 + |q_k|^2\right)(\Omega(2k))^2 < \infty.$$

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If $\Omega(k) = (1+k^2)^{\alpha/2}$, $\alpha > 0$, then $H(\Omega)$ is a Sobolev space H^α.

Operators L with symmetric (or skew-symmetric) matrix functions V help to solve - after Zaharov-Shabat [15] - non-linear Schroedinger equations. Although our results are new even for special V, let us point again that we consider *any L^2-potentials with complex valued p, q.*

1. To localize the spectrum $\sigma_{bc}(L)$ we need the following

Lemma 1. *([11]) If $V \in H(\Omega)$ and $\|V|L^2\| \le m$, $\|V|H(\Omega)\| \le M$ then $\sigma_{bc}(L)$ lies in the domain*

$$\Pi(X,Y) \cup \bigcup_{|k|>N} D_k,$$

where

$$\Pi(X,Y) = \{z \in \mathbb{C} : \quad |Rez| \le \pi N, \ |Imz| \le Y\},$$

(2)

$$D_k = D(\pi k, \delta_k) = \{z \in \mathbb{C} : \ |z - \pi k| < \delta_k\},$$

with

(3)

$$Y = K(1+m^2), \quad N = \min\{n : \ \delta_n^2 \le 1/(20M)\}, \quad \delta_n = (1/\Omega(n) + 1/\sqrt{n})^{1/2}.$$

Moreover, if

$$P_k = \frac{1}{2\pi i}\int_{\partial D_k} (z - L_{bc})^{-1} dz, \quad |k| > N,$$

then $\dim P_k = 2, 0, 1$ *if $bc = Per^+$, Per^-, Dir and k is even, and* $\dim P_k = 0, 2, 1$ *correspondingly if k is odd.*

This lemma (see [11], Thm 1 and 2) improves (and completes) weaker versions of "Counting Lemma" given in [10], [6], [7]. It guarantees that each disk D_k, $|k| > N$, contains three and only three spectral points $\lambda_k^+, \lambda_k^-, \mu_k$, where $\{\lambda_k^+, \lambda_k^-\} = \sigma(P_k L_{bc} P_k)$, $bc = Per^+$ if k is even, $bc = Per^-$ if k is odd, and $\{\mu_k\} = \sigma(P_k L_{Dir} P_k)$, all k, $|k| > N$. (We put $\lambda_k^+ = \lambda_k^-$ if this is a double root of $det(z - P_k L_{bc} P_k)$, $bc = Per^\pm$, even if its geometric multiplicity is *one*, i.e. this block $P_k L_{bc} P_k$ is Jordan.)

In the case of symmetric V, and $bc = Per^+$ or Per^-, λ_k's are real, $\gamma_k = \lambda_k^+ - \lambda_k^-$ are spectral gaps of L on $\mathbb{R}$. In general case, we analyze relationship between the rate of decay of sequences

$$\gamma_k = |\lambda_k^+ - \lambda_k^-|, \quad \delta_k = |\mu_k - \frac{1}{2}(\lambda_k^+ - \lambda_k^-)| \tag{4}$$

and the smoothness of potential V.

We will often require that Ω is a submultiplicative sequence, i.e.

$$\Omega(k+n) \le \Omega(k)\Omega(n), \quad k, n \in \mathbb{Z}. \tag{5}$$

This inequality holds if $\Omega(-n) = \Omega(n)$, and

$$\Omega(n) = \exp\ h(n), \ n \ge 0, \quad h(0) = 0, \ h(n) \uparrow \infty, \ h - \text{ concave}.$$

A weight $\Omega \in (1)$ is said to be *slowly increasing* if

$$\sup_n \Omega(2n)/\Omega(n) < \infty. \tag{6}$$

We consider also a class of *rapidly increasing* weights of the form

$$\Omega(n) = \exp(\varphi(\log|n|)), \quad |n| \geq 1, \tag{7}$$

where $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, is twice differentialble function such that

(7.a) $\varphi'(t) \nearrow \infty$ as $t \nearrow \infty$;

(7.b) $e^t/\varphi'(t) \nearrow \infty$ as $t \nearrow \infty$;

(7.c) $[\varphi'(t) - \varphi''(t)]/\log\varphi'(t) \to \infty$ as $t \to \infty$.

1. A first set of our results is about the rate of decay of the sequences (4).

Theorem 2. *If $V \in H(\Omega)$, and $\Omega \in (5) + (1)$ then*

$$\sum |\lambda_n^+ - \lambda_n^-|^2 (\Omega(2n))^2 < \infty. \tag{8}$$

Theorem 3.

$$\sum |\mu_n - \frac{1}{2}(\lambda_n^+ - \lambda_n^-)|^2 (\Omega(2n))^2 < \infty. \tag{9}$$

In the case of Schroedinger operators this type statements have been proven in [8], [9], and for Dirac operators - under special restrictions on growth of Ω (at least, faster than $n^\delta, \exists \delta > 0$) - in [6], [7].

We also follow the general schemes of [8], [9] but in technical details there are new essential difficulties. Eigenvalues $\{\lambda_n^\pm\}$, $|n| \geq N_*$, $bc = Per^\pm$, are roots of quasi-quadratic equation

$$\det \begin{pmatrix} \alpha_n^{11}(z) - z & \beta_n^{12}(z) \\ \beta_n^{21}(z) & \alpha_n^{22}(z) - z \end{pmatrix} = 0. \tag{10}$$

(compare (2.14), p. 624 in [8], or (14), p. 93 in [4]).

This matrix comes from 2D operator $P_k L_{bc} P_k$ where (with D_k from Lemma 1)

$$P_k = \frac{1}{2\pi i}\int_{\partial D_k} (z - L_{bc})^{-1} dz.$$

With $\alpha^{11} = \alpha^{22}$ (compare Lemma 2.2 in [8]) the distance between the two roots of (10) depends on β^{12} and β^{21}. We do not write the explicit formulas for β's but some adjustment and simplification of their structure (with elimination of unessential dependence on z, $|z| \leq \pi/2$) can be seen in formulas below.

Lemma 4. *If $\Omega \in (5) + (1)$ and $V \in H(\Omega)$ then for $|z| < \pi/2$, and $|n| \geq N_* = N_*(m, M)$,*

$$\sum_{|n|\geq N} |\beta^{12}(n,z)|^2 (\Omega(2n)|^2 \leq \left(\sum_{\nu=1}^{\infty} S_\nu\right)^2,$$

where

$$S_\nu^2 := \sum_{|n|\geq N} |\beta_\nu^{12}(n,z)|^2 (\Omega(2n))^2 \leq \sum_{|n|\geq N} \sigma_\nu^2,$$

$$\sigma_n := \sum_{j_1,\dots,j_{2\nu}\neq n} \frac{r(n+j_1)r(-j_1-j_2)r(j_2+j_3)\dots r(j_{2\nu}+n)}{|n-j_1||n-j_2|\dots|n-j_{2\nu}|},$$

with $r(m) = (|p(m)| + |q(m)|)\Omega(m)$, *and for* $N \geq N_*$,

$$\sum_{\nu=1}^{\infty} \left(\sum_{|n|\geq N} \sigma_\nu^2(n) \right)^{1/2} \leq \frac{C\|r|\ell^2\|^2}{N^2} + \varepsilon(N) R^2(N),$$

where $R^2(N) = \sum_{|n|\geq N} r^2(m)$, *and* $\varepsilon(N) \to 0$ *as* $N \to 0$.

This statement lists a series of inequalities which lead to Thm 2 and 3.

2. Now we go to the opposite direction.

Theorem 5. *If V is symmetric, and $V \in L^2(\Omega)$, with Ω satisfying (1), (5) and either (6) or (7.a-c) then (8) implies $V \in H(\Omega)$.*

Theorem 6. *Under conditions of Thm 5 but without the assumption that V is symmetric, (8) together with (9) imply $V \in H(\Omega)$.*

Again, the general plan of the proof follows our constructions in [4], [5]. But instead of non-linear equations (39) in [4], Thm 9, or (37) in [5], Prop 3, now we have to deal with an equation

$$\xi_n = w(2n) + \sum_{\nu=1}^{\infty} \Psi_\nu(n,w), \tag{11}$$

where

$$\Psi_\nu(n,w) = \sum_{j_1,\dots,j_{2\nu}\neq n} \frac{w(n-j_1)w(j_1+j_2)(-j_2-j_3)\dots w(-j_{2\nu}+n)}{(n-j_1)\dots(n-j_{2\nu})} \tag{12}$$

Lemma 7. *(compare* [4]*, Thm 9). If*

$$w \in \ell^2(\mathbb{Z}) \quad \textit{and} \quad \xi \in \ell^2(\Omega) \tag{13}$$

with Ω as in Thm 5 then $w \in \ell^2(\Omega)$.

Remark. V. Tkachenko, [13], proved that in the case of geometric decay

$$|\lambda_n^+ - \lambda_n^-| + |\mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-)| \leq A\exp(-a|n|), \quad \exists A, a > 0, \forall n \in \mathbb{Z}, \tag{14}$$

the potential V is analytic as well, i.e.

$$|p_n| + |q_n| \leq B\exp(-b|n|), \quad B, b > 0.$$

However, like in Schroedinger case (compare the discussion in [4], Sect 5.4) the choice of b, the type of decay, or the width of the strip of analyticity of an extension of $V(x)$, cannot be determined by a in (14). It depends on

$\|V|L^2\|$ as well. This phenomenon leads us with necessity to restriction on the weight-sequences Ω we are dealing with in Thm 5 and 6. Certainly, Gevrey weights

$$\Omega(n) = (1+n^2)^\alpha \exp(-a|n|^\gamma), \quad a > 0, \gamma \in (0,1),$$

satisfy these restrictions because (7.a-c) hold.

3. This section deals with *density* of good (Thm 8) or bad (Thm 9) potentials V. Theorems 8 and 9 are analogues of B.Mityagin [12] results in the case of Schroedinger-Hill operators.

We say that V is *a finite-zone potential* if for some $N^* \geq N \in (3)$ we have $\lambda_k^+ = \lambda_k^- = \mu_k,\ |k| > N^*$ and $P_k L_{bc} P_k$, $bc = Per^\pm$, are not Jordan.

Theorem 8. *If*

$$\sup \log \Omega(k)/k < \infty, \tag{15}$$

then finite-zone potentials are dense in $H(\Omega)$. If V is symmetric (or skew-symmetric), i.e. $q(x) = \overline{p(x)}$ (or $q(x) = -\overline{p(x)}$), then approximating finite-zone potentials can be chosen in the same class.

Quite different are potentials V's such that

$$\lambda_k^+ \neq \lambda_k^- \quad \text{for } |k| \geq N_{**} \geq N, \quad k \text{ even if } bc = Per^+, \text{ odd if } bc = Per^- \tag{16}$$

In this case 2D projectors

$$P_k^{bc} = \frac{1}{2\pi i} \int_{\partial D_k} (z - L_{bc})^{-1} dz \tag{17}$$

could be refined, i.e. presented in the form

$$P_k F = \langle F, \Phi_{2k-1} \rangle E_{2k-1} + \langle F, \Phi_{2k} \rangle E_{2k}, \tag{18}$$

with

$$\|E_j\| = 1, \langle E_j, \Phi_k \rangle = \delta_{jk}, \quad |j|, |k| > N_{**}, \quad \Phi_i \in L^2.$$

The spectral decompositions

$$F = P_*^{bc} F + \sum P_k^{bc} F \tag{19}$$

where

$$P_*^{bc} = \frac{1}{2\pi i} \int_{\partial \Pi} (z - L_{bc})^{-1} dz, \quad \Pi \in (2),$$

converge in L^2 for any $F \in L^2$, (or even in L^p for any $F \in L^p,\ 1 < p < \infty$) according to [11], Thm 3.

But if we want to write the eigenfunction decomposition

$$F \equiv P_*^{bc} F + \sum \langle F, \Phi_j \rangle E_j \tag{20}$$

its convergence depends on boundedness of the sequence $(\Phi_j) \in (18)$.

Theorem 9. *If (15) holds then in $H(\Omega)$ there is a dense set of potentials V with the following properties:*

(a) (16) holds, but

(b) decompositions (20) do not necessarily converge in L^2 if $F \in L^2$; more specifically,

$$\|\Phi_j\|_2 \to \infty \quad if \quad j \to \pm\infty.$$

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