ANALYTICITY AND NONEXISTENCE OF CLASSICAL STEADY HELE-SHAW FINGERS

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Abstract. This paper concerns analyticity of a classical steadily translating finger in a Hele-Shaw cell and nonexistence of solutions when relative finger width λ is smaller than $\frac{1}{2}$. It is proved that any classical solution to the finger problem, if it exists for sufficiently small but nonzero surface tension and is close to some Saffman-Taylor zero-surface-tension solution and satisfies some algebraic decay conditions at ∞, must belong to the analytic function space chosen in a previous study [1] of existence of finger solutions. Further, it is proved that for any fixed $\lambda \in (0, \frac{1}{2})$, there can be no classical steady finger solution when surface tension is sufficiently small, in disagreement with a previous conclusion based on numerical simulation.

1. Introduction

The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell has been the subject of numerous investigations since 1950's. Reviews of the subject from a number of perspectives can be found in [2]-[7]. In a seminal paper, Saffman & Taylor [8] found experimentally that an unstable planar interface evolves through finger competition to a steady translating finger, with relative finger width λ close to a half at large displacement rates. Theoretical calculations [9], [8] ignoring surface tension revealed a one-parameter family of exact steady solutions, parametrized by width λ. When the experimentally determined λ is used, the theoretical shape (usually referred to in the literature as the Saffman-Taylor finger) agreed well with experiment for relatively large displacement rates, or equivalently for small surface tension. However, in the zero-surface-tension steady-state theory, λ remained undetermined in (0, 1) interval. The selection of λ remained unresolved until the mid 1980's. Numerical calculations [10], [11], [12], supported by formal asymptotic calculations in the the steady finger [13], [14], [15], [16], [17], [18], [19] and closely-related steady Hele-Shaw bubble problem [16], [21], suggests that a discrete family of solutions exist for which the the limiting shape, as surface tension tends to zero, approached the Saffman-Taylor with $\lambda = \frac{1}{2}$. Subsequent numerical [22] and formal asymptotic calculations [23] suggest that only branch is stable. However, the conclusion about existence of steady states is not universally accepted. Based on numerical simulation of a time-evolving interface for small but nonzero surface tension, and with the same model equations used in [10]-[18], it was suggested [24] that the limiting steady shape was a Saffman-Taylor solution with $\lambda < \frac{1}{2}$. In this paper, we conclude otherwise through rigorous mathematical analysis. It is to be noted that selection of Saffman-Taylor finger with $\lambda < \frac{1}{2}$ is possible for a more mathematically complicated model, which incorporates thin-film effects [25]-[26], as shown in [27]-[28]. The same is true when anisotropy [17] in surface tension or other perturbations near the tip are introduced.

There has been a rigorous study [29] for a problem mathematically similar, though not identical, to the steady viscous fingering problem considered here. In that case, it has been proved that at least one finger solution exists for fixed surface tension, though the relative finger width and shape remains unknown. On the other hand, our primary focus is the selection of finger width as surface tension tends to zero. A mathematically rigorous study of selection is difficult in this limit since exponentially small terms in surface tension play a critical role. While a rigorous theory of exponential asymptotics for nonlinear ordinary differential equations is by now well developed ([30], for instance), this is not the case for integro-differential equations, even though such problems have arisen in a number of other physical contexts like dendritic crystal growth and water waves (See, for instance, [31]). Formal calculations rely on the assumption that integral terms do not contribute to exponentially small terms, at least to the leading order. With this assumption, integro-differential equations are simplified to essentially nonlinear ordinary differential equation, where variants of the procedure due to Kruskal & Segur [32]-[33] have been used. Recently ([34], [1]), we have shown how integral terms can be controlled and a rigorous theory was developed for the integro-differential equation presented here.

Following [7], a steady symmetric finger is equivalent to finding function F analytic in the upper-half ξ plane ($\mathbf{Z}^+$) and twice differentiable in its closure, *i.e.* in $\mathcal{C}^2(\bar{\mathbf{Z}}^+)$, such that the following conditions are satisfied:

Condition (i): On the real ξ axis, F satisfies

$$ReF = \frac{\epsilon^2}{|F' + H|} Im \left[\frac{F'' + H'}{F' + H} \right]; \tag{1.1}$$

where

$$H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1}, \text{with } \gamma = \frac{\lambda}{1-\lambda}, \;\; \epsilon^2 = \frac{\pi^2 \lambda \mathcal{B}}{4(1-\lambda)^2}, \tag{1.2}$$

where λ is the relative finger width and $\mathcal{B}$ is a non-dimensional surface tension parameter.

Condition (ii):

$$F(\xi) \;,\; \xi F'(\xi) \to 0 \text{ as } \xi \to \pm\infty; \tag{1.3}$$

Condition (iii)(symmetry condition):

$$\text{Re } F(-\xi) = \text{Re } F(\xi) \;,\; \text{Im } F(-\xi) = -\text{Im } F(\xi) \text{ for real } \xi. \tag{1.4}$$

Definition 1.1. Let $\mathcal{R}$ be the open connected set between Im $\xi = 0$ and $\ell^+ \cup \ell^-$ where

$$\ell^+ = \{\xi : \xi = -ib + re^{-i\varphi_0}, 0 < r < \infty, b > 0, \frac{\pi}{2} > \varphi_0 > 0 \text{ fixed}\}$$
$$\ell^- = \{\xi : \xi = -ib - re^{i\varphi_0}, 0 < r < \infty, \}$$

Also, we define $\mathcal{R}^- = \mathcal{R} \cap \{\xi : \text{Re } \xi \ < \ 0\}$ and $\mathcal{R}^+ = \mathcal{R} \cap \{\xi : \text{Re } \xi \ > \ 0\}$

Definition 1.2. For fixed $\tau \in (0,1)$,

$$\mathbf{A}_j = \{F : \ F(\xi) \text{ is analytic in } \{Im\ \xi \geq 0\} \cup \mathcal{R}$$
$$\text{with } \|F\|_j \equiv \sup_{\xi \in \overline{\mathcal{R}}} |(\xi - 2i)^{j+\tau} F(\xi)| < \infty\}, j = 0, 1, 2,$$
$$\mathbf{A}_{0,\hat{\delta}} = \{F : F \in \mathbf{A}_0, \|F\|_0 \leq \hat{\delta}\}, \mathbf{A}_{1,\hat{\delta}_1} = \{F : F \in \mathbf{A}_1, \|F\|_1 \leq \hat{\delta}_1\},$$

Our previous result on the existence of solution [1] satisfying conditions (i)-(iii) involved $F \in \mathbf{A}_{0,\hat{\delta}}, F' \in \mathbf{A}_{1,\hat{\delta}_1}$, where $\hat{\delta}, \hat{\delta}_1$ are assumed *a priori* to be small but independent of ϵ. In this function space, for $\lambda \in [\frac{1}{2}, \lambda_m)$, with $\lambda_m - \frac{1}{2}$ small enough (though independent of ϵ), it was shown that solution existed if and only if $\frac{2\lambda-1}{1-\lambda} = \epsilon^{4/3}\ \beta_n(\epsilon^{2/3})$, where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of functions, analytic at the origin.

However, there are two limitations of this result. The first is the choice of the function space. Non-existence in this function space need not mean non-existence of a classical solution F, analytic in $\mathbf{Z}^+$ and $\mathcal{C}^2$ in its closure $\bar{\mathbf{Z}}^+$. The second limitation is the restriction on λ. In this paper, we prove two theorems (Theorems 1.3 & 1.8) to relax these restrictions to a great degree.

Theorem 1.3. *For small enough ϵ, any analytic function F in the upper half ξ-plane $\mathbf{Z}^+$, which is C^2 on its closure and satisfies conditions (i)-(iii) belongs to function space $\mathbf{A}_{0,\hat{\delta}}$, with $F' \in \mathbf{A}_{1,\hat{\delta}_1}$, where $\hat{\delta} = O(\epsilon^2)$ and $\hat{\delta}_1 = O(\epsilon)$, provided Assumptions (i) and (ii) as stated below are also satisfied:*

Assumption (i): There exists τ independent of ϵ, $0 < \tau < 1$ so that

$$\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau} |F(\xi)| \equiv \delta < \infty \tag{1.5}$$

Assumption (ii): We assume that each of δ_1 and $\epsilon \ln \frac{1}{\epsilon}\, \delta_2$ are sufficiently small, where

$$\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{1+\tau} |F'(\xi)| = \delta_1, \quad \sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{2+\tau} |F''(\xi)| = \delta_2, \tag{1.6}$$

Remark 1.4. Assumption (i), though stronger than condition (ii), is consistent with the results from Mclean-Saffman's formal procedure [10] near the tail of a finger that in our formulation implies $F \sim a_0\, e^{i\pi\hat{\tau}/2}\, \xi^{-\hat{\tau}}$ as $\xi \to +\infty$, where $\hat{\tau}$ is a positive root of the transcendental equation $\cot(\frac{\pi}{2}\hat{\tau}) = \epsilon^2 \hat{\tau}^2$ and a_0 is real. The symmetry condition (iii) implies similar behavior as $\xi \to -\infty$. This asymptotic relation was also found to be consistent with numerical calculations [10]. While $\hat{\tau}$ depends on ϵ, we can clearly choose $\tau < \hat{\tau}$ independent of ϵ for small ϵ ($\tau = \frac{1}{2}$ would suffice for instance). The following lemma shows that we need not assume *a priori* that δ_1 and δ_2 in *assumption (ii)* exists and are finite; only that δ_1 and $\epsilon \ln \frac{1}{\epsilon}\, \delta_2$ are small. These assumptions are mild since the slope deviation from some Saffman-Taylor solution (which scales as δ_1 in the above theory) is observed to be small in experiment for large displacement rates and in all numerical calculations for small ϵ; we are not making any *a priori* assumption on how this deviation scales with ϵ. Also, the curvature deviation (which scales as F'', and hence δ_2) *a priori* is allowed to be large, though not as large as $\frac{1}{\epsilon \ln \frac{1}{\epsilon}}$.

Lemma 1.5. *If F satisfies assumption (i) in addition to being analytic in $\mathbf{Z}^+$, C^2 in $\bar{\mathbf{Z}}^+$ and satisfying condition (i)-(iii), then*

$$(a) \sup_{\xi \in \mathbf{Z}^+} |\xi + 2i|^{\tau} |F(\xi)| = \delta < \infty, \tag{1.7}$$

$$(b) \sup_{\xi \in \mathbf{Z}^+} |\xi + 2i|^{1+\tau} |F'(\xi)| = \delta_1 < \infty, \tag{1.8}$$

$$(c) \sup_{\xi \in \mathbf{Z}^+} |\xi + 2i|^{2+\tau} |F''| \; = \; \delta_2 \; < \; \infty, \tag{1.9}$$

Proof of Lemma 1.5 relies on some straight forward properties of Hilbert transform and use of Phragmen-Lindelof methods and is relegated to appendix A1.

Remark 1.6. From examining (1.1), $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^\tau |Re\ F(\xi)| = O(\epsilon^2 \delta_2)$. From Hilbert transform of $Re\ F$, (which gives $Im\ F$ on the real axis), and using Lemma A.1 with $g = Re\ F$ and $k = \epsilon$, it follows from assumption (ii) and lemma 1.5 that $\delta = o(\epsilon)$.

Definition 1.7. F will be called a classical solution if F is analytic in the upper-half ξ-plane ($\mathbf{Z}^+$), C^2 in its closure $\bar{\mathbf{Z}}^+$, satisfies Conditions (i)-(iii) and assumptions (i) and (ii).

In section 4, we are going to prove

Theorem 1.8. *For any fixed* $\lambda \in (0, \frac{1}{2})$*, there exists* $\epsilon_0 > 0$ *small so that there can be no classical solution* F *for any* ϵ *in the interval* $(0, \epsilon_0]$.

2. Analytic continuation to the lower half plane

Definition 2.1. Let F be analytic in the upper half ξ-plane, $\bar{F}$ is an analytic function in the lower half ξ-plane defined by

$$\bar{F}(\xi) = [F(\xi^*)]^* \tag{2.1}$$

$$\bar{H} = \frac{\xi - i\gamma}{\xi^2 + 1} \tag{2.2}$$

We define operator $\mathcal{G}$ so that

$$\mathcal{G}(f,g)[t] := \frac{1}{(f'(t)+H(t))^{1/2}(g'(t)+\bar{H}(t))^{1/2}} \times \left[\frac{f''(t)+H'(t)}{f'(t)+H(t)} - \frac{g''(t)+\bar{H}'(t)}{g'(t)+\bar{H}(t)}\right] \tag{2.3}$$

Lemma 2.2. *If F is a classical solution as in definition 1.7, then*

$$\mathcal{G}(F,\bar{F})[\xi] = O(\xi^{-\tau}), \ \text{as } \xi \to \pm\infty; \tag{2.4}$$

Proof. Since the right hand side of (1.1) can be written as $\frac{\epsilon^2}{2i}\mathcal{G}(F,\bar{F})(\xi)$, the lemma follows from (1.5). □

Definition 2.3. we define operator $\mathcal{I}$ so that

$$I(\xi) \equiv \mathcal{I}(F)[\xi] = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathcal{G}(F,\bar{F})[t]dt}{t-\xi} \quad \text{for } Im\ \xi\ <\ 0. \tag{2.5}$$

Lemma 2.4. *For $I(\xi)$ in the lower half plane $\mathbf{Z}^- = \{\xi : Im\ \xi < 0\}$, we have*

$$\sup_{\xi\in\mathbf{Z}^-} |\xi - 2i|^\tau \epsilon^2 |I(\xi)| \ = \ \sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^\tau |F(\xi)| = \delta \tag{2.6}$$

Proof. From (1.1), (2.3) and (2.5) $\lim_{Im\ \xi\ \to\ 0^-} \epsilon^2 I(\xi) = -\bar{F}(\xi)$ for ξ real. Since $I(\xi)$ is analytic in the lower half plane, the above lemma follows from Lemma A.5 in Appendix 1. □

Lemma 2.5. *Let F be a classical solution to the finger problem. If $F(\xi)$ can be analytically continued at least to a part of $\mathbf{Z}^-$, then F satisfies :*

$$\epsilon^2 F''(\xi) + L(\xi)F(\xi) = \mathcal{N}(F,I,\bar{F})[\xi], \ \text{for } \{\xi \in \mathbf{Z}^-\}; \tag{2.7}$$

where

$$L(\xi) = -iH^{3/2}(\xi)\bar{H}^{1/2}(\xi) = -\frac{i\sqrt{\gamma^2+\xi^2}(\xi+i\gamma)}{(\xi^2+1)^2}, \ \bar{F}(\xi) \equiv [F(\xi^*)]^*; \tag{2.8}$$

and the operator $\mathcal{N}$ *is defined as*

$$\mathcal{N}(F, I, \bar{F}) = \epsilon^2 \left(\frac{\bar{H}' H}{\bar{H}} - H' \right) - i\epsilon^2 (F' + H)^{3/2} (\bar{F}' + \bar{H})^{1/2} I$$
$$+ iF \ \left[(F' + H)^{3/2} (\bar{F}' + \bar{H})^{1/2} - H^{3/2} \bar{H}^{1/2} \right] + \epsilon^2 \left[(\bar{F}'' + \bar{H}') \frac{F' + H}{\bar{F}' + \bar{H}} - \frac{\bar{H}' H}{\bar{H}} \right]; \tag{2.9}$$

Proof. Since F is analytic in upper half ξ-plane and satisfies equation (1.1), using Poission formula, we have in the upper half plane:

$$\begin{aligned} F(\xi) &= \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{(t-\xi)} \frac{1}{|F'(t) + H(t)|} \text{Im} \ \left[\frac{F''(t) + H'(t)}{F'(t) + H(t)} \right] \\ &= -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{G}(F, \bar{F})[t] dt}{t - \xi}, \ \text{Im } \xi > 0; \end{aligned} \tag{2.10}$$

Using Plemelj Formula [35], analytic continuation to the lower half ξ-plane leads to

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i} \mathcal{G}(F, \bar{F})(\xi), \ \text{for Im } \xi < 0; \tag{2.11}$$

Multiplying the above by $i(F' + H)^{3/2} (\bar{F}' + \bar{H})^{1/2}$ results in (2.7) ☐

Definition 2.6.

$$g_1(\xi) = L^{-1/4}(\xi) \exp\{-\frac{P(\xi)}{\epsilon}\}, \tag{2.12}$$

$$g_2(\xi) = L^{-1/4}(\xi) \exp\{\frac{P(\xi)}{\epsilon}\}, \tag{2.13}$$

where

$$P(\xi) \ = \ i \int_{-i\gamma}^{\xi} L^{1/2}(t) dt \ = \ i \int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4} \ (\gamma + it)^{1/4}}{(1 + t^2)} dt; \tag{2.14}$$

We will use in this paper the following properties of $P(\xi)$ that are shown in Appendix 2. Some of these properties were shown in the appendix of Xie & Tanveer [1] for the restricted case $\lambda \in [\frac{1}{2}, \lambda_m)$.

Property 1: Re $P(\xi)$ decreases along negative Re ξ axis $(-\infty, 0)$ with Re $P(-\infty) = \infty$. Re $P(\xi)$ decreases monotonically on imaginary ξ axis from $-ib$ to 0 where $0 < b < min\{1, \gamma\}$.

Property 2: There exists a constant R independent of ϵ so that for $|\xi| \geq R$, Re $P(t)$ increases with increasing s along any ray $r = \{t : t = \xi - se^{i\varphi}, 0 < s < \infty, 0 \leq \varphi \leq \varphi_0 \ < \frac{\pi}{2}\}$ in $\mathcal{R}$ from ξ to $\xi + \infty e^{i\varphi}$ and $C_1 |t - 2i|^{-1} \leq |\frac{d}{ds}$ Re $P(t(s)| \leq C_2 |t - 2i|^{-1}$, where C_1 and C_2 are constants, independent of ϵ, with $C_1 \ > \ 0$.

Property 3: There exists sufficiently small $\nu > 0$ independent of ϵ so that $\frac{d}{ds}[ReP(t(s))] \geq C > 0$ on the parametrized straight line $\{t(s) = -\nu + se^{-i\frac{\pi}{4}}, 0 \leq s \leq \sqrt{2}\nu\}$, C is some constant independent of ϵ and ν.

Property 4: There exists b, φ_0, with $\nu < b < min\{1, \gamma\}, 0 < \varphi_0 < \frac{\pi}{2}$, each independent of ϵ, so that $\frac{d}{ds} Re\ P(t(s)) \geq \frac{C}{|t(s) - 2i|}$ on $t(s) = -bi + se^{i(\pi + \varphi_0)}$, where $C \ > \ 0$ is independent of ϵ.

$g_1(\xi), g_2(\xi)$ are the two WKB solutions of the homogenueous equation corresponding to (2.7); they satisfy the following equation exactly:

$$\epsilon^2 g''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi)) g(\xi) = 0; \tag{2.15}$$

where

$$L_1(\xi) \ = \ \frac{L''(\xi)}{4L(\xi)} \ - \ \frac{5L'^2(\xi)}{16L^2(\xi)} \tag{2.16}$$

Remark 2.7. By (2.8) and (2.16),$L_1(\xi) \sim O(\xi^{-2})$, as $|\xi| \to \infty$.

The Wronskian of g_1 and g_2 is

$$W(\xi) = g_1(\xi) g_2'(\xi) - g_2(\xi) g_1'(\xi) = \frac{2i}{\epsilon}, \tag{2.17}$$

Definition 2.8. We define operator $\mathcal{V}$ so that

$$\mathcal{V}F(\xi) \equiv \epsilon^2 F''(\xi) + (L(\xi) + \epsilon^2 L_1(\xi))F(\xi) \tag{2.18}$$

Remark 2.9. Equation (2.7) implies

$$\mathcal{V}F(\xi) = N_1(\xi) \equiv \mathcal{N}(F, I, \bar{F})[\xi] + \epsilon^2 L_1(\xi) F(\xi), \tag{2.19}$$

Definition 2.10. Let $\mathcal{D}$ be an open connected (see Figure 1) domain in the lower left complex ξ plane bounded by lines

$$R_1 = \{\xi : Im\ \xi = 0, -\infty < Re\ \xi < -\nu\},$$

$$R_2 = \{\xi : \xi = -\nu + se^{-\pi i/4}\}, 0 \leq s \leq \sqrt{2}\nu\}$$

$$R_3 = \{\xi : Re\ \xi = 0, -b < Im\ \xi < -\sqrt{2}\nu\}$$

$$R_4 = \{\xi : \xi = -bi + se^{i(\pi+\varphi_0)}, 0 \leq s < \infty\}$$

where ν, ϕ_0 and b are chosen so that Properties 3 and 4 are satisfied.

In addition to Properties 1-4 above, we show in Appendix 2 two other properties.:

Property 5: For any $\xi \in \mathcal{D}$, there is a path $\mathcal{P}(-\nu, \xi) = \{t : t = t(s)\}$, parametrized by arclength s, going from $-\nu$ to ξ, entirely contained in $\mathcal{D}$, so that $\frac{d}{ds} ReP(t(s)) \geq C > 0$ for a constant C independent of ϵ.

Property 6: For any $\xi \in \mathcal{D}$, there is a path $\mathcal{P}(\xi, -\infty) = \{t : t = t(s)\}$ parametrized by arclength s going from ξ to $-\infty$ contained entirely in $\mathcal{D}$ so that $\frac{d}{ds}[ReP(t(s))] \geq \frac{C}{|t-2i|} > 0$, where $C > 0$ is independent of ϵ.

We introduce spaces of functions:

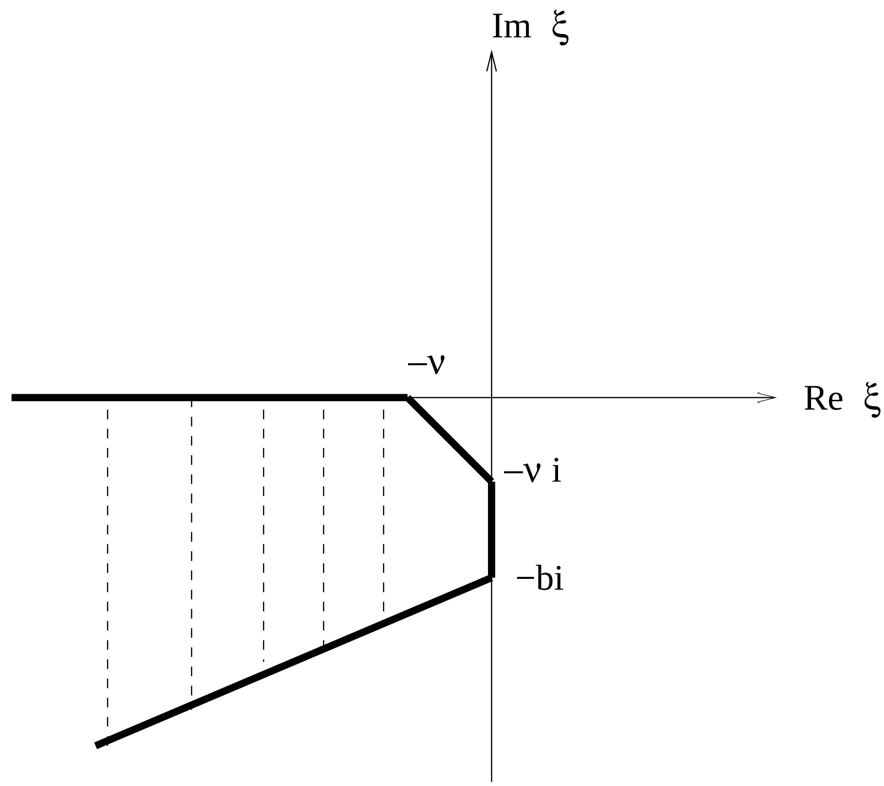

FIGURE 1. Region $\mathcal{D}$ in complex ξ plane.

Definition 2.11.

$$\mathbf{B}_j = \{F(\xi) : F(\xi) \text{ is analytic in } \mathcal{D} \text{ and continuous in } \overline{\mathcal{D}},$$

$$\text{with } \sup_{\xi\in\overline{\mathcal{D}}} |(\xi - 2i)^{j+\tau} F(\xi)| < \infty\}, j = 0, 1, 2,$$

$$\|F\|_j := \sup_{\xi\in\overline{\mathcal{D}}} |(\xi - 2i)^{j+\tau} F(\xi)|$$

.

Remark 2.12. $\mathbf{B}_j$ are Banach spaces and $\mathbf{B}_0 \supset \mathbf{B}_1 \supset \mathbf{B}_2$.

Definition 2.13. Let $\mathcal{Q}$ be any connected set in complex ξ-plane,we introduce norms: $\|F(\xi)\|_{j,\mathcal{Q}} := \sup_{\xi\in\mathcal{Q}} |(\xi - 2i)^{j+\tau} F(\xi)|, j = 0, 1, 2.$

Definition 2.14. Let $\tilde{\delta} > 0, \tilde{\delta}_1 > 0$ be two constants, define spaces:

$$\mathbf{B}_{0,\tilde{\delta}} = \{f : f \in \mathbf{B}_0, \|f\|_0 \le \tilde{\delta}\};\ \mathbf{B}_{1,\tilde{\delta}_1} = \{g : g \in \mathbf{B}_1, \|g\|_1 \le \tilde{\delta}_1\}$$

Remark 2.15. A remark is in order about use of symbol C for constants that occur through out the paper. In order to avoid proliferation of constants, we have used C (and sometimes C_1 and C_2) as generic constant, whose value is allowed to differ from Lemma to Lemma, and sometimes even from step to step within a Lemma. However, C does not depend on ϵ. For more specific constants, we have reserved constant K, K_1, K_2, etc.

Lemma 2.16. *Let $N \in \mathbf{B}_2$, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_2(\xi) \int_\xi^{-\infty} \frac{N(t)}{W(t)} g_1(t) dt \in \mathbf{B}_0, \textit{ and } \|f_1\|_0 \le K_1 \ \|N\|_2;$$

where K_1 is a constant independent of ϵ.

Proof. **Case 1**: $|\xi| \ge R$, where R is large enough for Property 2 to hold, but independent of ϵ. On path $\mathcal{P}(\xi, -\infty) = \{t : t = \xi - s, 0 < s < \infty\}$, Re $(P(t) - P(\xi))$ increases monotonically from 0 to ∞ as s increases.

$$(2.20)\quad |f_1(\xi)| = \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(\xi,-\infty)} N(t) L^{-1/4}(t) \exp\{\frac{1}{\epsilon}(P(\xi) - P(t))\} dt \right|$$
$$\le \|N\|_2 |L^{-1/4}(\xi)|$$
$$\times \int_0^1 \frac{|(t(s)-2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\frac{d}{ds} ReP(t(s))} d\left[\exp\{\frac{1}{\epsilon}(ReP(\xi) - ReP(t(s)))\}\right]$$

Since $|\xi - 2i| \le |t(s) - 2i|$ for any s, we have $|L^{-1/4}(\xi)| \le C|\xi - 2i|^{1/2}$ and

$$\frac{d}{ds}\ Re\ P(t(s)) = Re\,(P'(t)t'(s)) \ge C|L^{1/2}(t)| \ge C|t(s) - 2i|^{-1}$$

$$|L^{-1/4}(\xi)| \frac{|(t(s)-2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\frac{d}{ds} ReP(t(s))} \le C|\xi - 2i|^{-\tau};$$

So $\|f_1\|_0 \le K_1 \|N\|_2$ and the lemma follows.

Case 2: For $\xi \in \mathcal{D} \cap \{|\xi| \le R\}$, by Property 6, there exists a path $\mathcal{P}(\xi, -\infty)$, so that Re $(P'(t(s)t'(s)) \ge \frac{C}{|t(s)-2i|}$, then in (2.20)

$$|L^{-1/4}(\xi)| \frac{|(t(s)-2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\left|\frac{d}{ds} Re P(t(s))\right|} \le C \tag{2.21}$$

and therefore lemma follows since ξ is bounded in this region. □

Lemma 2.17. *Let $N \in \mathbf{B}_2$, then for sufficiently small $\epsilon_0 > 0$, we have for all $\epsilon \in (0, \epsilon_0]$,*

$$f_2(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt \in \mathbf{A_0}, \textit{and } \|f_2\|_0 \le \; K_2 \; \|N\|_2.$$

where K_2 is independent of ϵ.

Proof. **Case 1**:For the case $|\xi| \le 4R^2$, by Property 5, there is path $\mathcal{P}(-\nu, \xi)$ entirely in $\mathcal{D}$ so that $\frac{d}{ds} Re P(t(s)) \ge C > 0$ for $t(s)$ going from $-\nu$ to ξ. So

$$|f_2(\xi)| \le C \|N\|_2 |L^{-1/4}(\xi)| \\ \times \int_0^1 |(t-2i)^{-2-\tau}||L^{-1/4}(t)| \frac{\left|d[\exp\{-\frac{1}{\epsilon}(Re P(\xi) - Re P(t))\}]\right|}{\frac{d}{ds} Re\; P(t(s))}$$

Since (2.21) holds here too, the result follows since $|\xi - 2i|^\tau$ is bounded in this case as well.

Case 2: For the case where $\xi \in \mathcal{D}, |\xi| \ge 4R^2$.

we choose path $\mathcal{P}(-\nu, \xi) = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$, where

$$\mathcal{P}_1 = \{t : t = \rho e^{i \arg \xi}, |\xi| \ge \rho \ge \sqrt{|\xi|}\},$$

$$\mathcal{P}_2 = \{t : t = \rho e^{i \arg \xi}, \sqrt{|\xi|} \ge \rho \ge 2R\},$$

$$\mathcal{P}_3 = \mathcal{P}(-\nu, \xi^0), \text{where } \xi_0 = 2R\; e^{i\; arg\; \xi};$$

We break up integral $\int_{\mathcal{P}} = \int_{\mathcal{P}_3} + \int_{\mathcal{P}_2} + \int_{\mathcal{P}_1}$ and accordingly write $f_2 = f_{2,1} + f_{2,2} + f_{2,3}$.

Now from (2.14), from the asymptotics for large $|\xi|$ and $|t|$, it follows that

$$Re(P(t) - P(\xi)) \leq C_1 \int_{|\xi|}^{|t|} \frac{1}{r} dr \leq C_1 \ln(\frac{|t|}{|\xi|}), \text{where } C_1 \text{ is independent of } \epsilon \tag{2.22}$$

$$\begin{aligned} |f_{2,1}(\xi)| &= \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_1} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt \right| \\ &\leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{\sqrt{|\xi|}}^{|\xi|} \rho^{-\frac{3}{2}-\tau} \exp\{\frac{C_1}{\epsilon} \ln(\frac{\rho}{|\xi|})\} d\rho \\ &\leq C \|N\|_2 |\xi|^{-\tau}. \end{aligned} \tag{2.23}$$

Also, (2.22),(2.23) are still valid on $\mathcal{P}_2$, hence

$$\begin{aligned} |f_{2,2}(\xi) &= \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}_2} N(t) L^{-1/4}(t) \exp\{-\frac{1}{\epsilon}(P(\xi) - P(t))\} dt \right| \\ &\leq C \frac{2}{\epsilon} \|N\|_2 |L^{-1/4}(\xi)| \int_{|\xi^0|}^{\sqrt{|\xi|}} \rho^{-\frac{3}{2}-\tau} \ \exp\{\frac{C_1}{\epsilon} \ln(\frac{\rho}{|\xi|})\} d\rho \\ &\leq C \|N\|_2 |L^{-1/4}(\xi)| |\xi|^{-\frac{C_1}{2\epsilon} - \frac{1}{4} - \frac{\tau}{2}} \\ &\leq C \|N\|_2 |\xi|^{-\tau} \text{ for } \epsilon < C_1 \end{aligned}$$

On $\mathcal{P}_3$:

$$|f_{2,3}(\xi)| \leq \|N\|_2 |L^{-1/4}(\xi)| \tag{2.24}$$

$$\times \int_0^{\exp[-\frac{1}{\epsilon}(Re\ P(\xi) - Re\ P(\xi_0))]} \frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\frac{d}{ds} Re P(t(s))} d \left[\exp\{\frac{1}{\epsilon}(Re P(\xi) - Re P(t(s)))\} \right]$$

$$\leq C \|N\|_2 |L^{-1/4}(\xi)| \exp[-\frac{1}{\epsilon}(Re\ P(\xi) - Re\ P(\xi_0))] \leq C \|N\|_2 |\xi|^{-\frac{C_1}{\epsilon} + \frac{1}{2}} \leq C \|N\|_2 |\xi|^{-\tau}, \text{when } \epsilon \leq \frac{C_1}{2}$$

Combining bounds for $f_{2,1}$, $f_{2,2}$ and $f_{2,3}$, the proof of the Lemma follows. □

Definition 2.18. Define operator $\mathcal{U}$: $\mathbf{B}_2 \to \mathbf{B}_0$; $\mathcal{U}_1$: $\mathbf{B}_2 \to \mathbf{B}_1$ so that

$$\mathcal{U} N(\xi) := -\frac{1}{\epsilon^2} g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt; \tag{2.25}$$

$$(2.26)\quad \mathcal{U}_1 N(\xi) := -\frac{1}{\epsilon^2} h_1(\xi) g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} h_2(\xi) g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t) dt;$$

where

$$(2.27)\quad h_1(\xi) = -\frac{L'(\xi)}{4L(\xi)} - \frac{1}{\epsilon} P'(\xi), h_2(\xi) = -\frac{L'(\xi)}{4L(\xi)} + \frac{1}{\epsilon} P'(\xi);$$

Lemma 2.19.

$$(2.28)\quad \sup_{\mathcal{D}} |(\xi - 2i) h_j(\xi)| \le \frac{K_3}{\epsilon}; j = 1, 2$$

where K_3 is a constant independent of ϵ.

Proof. The lemma follows from that $P'(\xi) = i L^{1/2}(\xi)$ and equations (2.8),(2.14) and (2.27). □

Definition 2.20. Let $\mathbf{R}^- = \{\xi : Im\ \xi = 0, Re\ \xi < -\nu\}$.

Lemma 2.21. $\|N_1\|_{2,\mathbf{R}^-} < \infty$.

Proof. From (1.2), (2.2), (2.16), Lemmas 1.5 and 2.4, it follows that as $\xi \to -\infty$,

$$\frac{\bar{H}' H}{\bar{H}} - H' = O(\xi^{-3}),\ (F' + H)^{3/2} (\bar{F}' + \bar{H}')^{1/2}\ I = O(\xi^{-2-\tau}),$$
$$F[(F' + H)^{3/2} (\bar{F}' + \bar{H}')^{1/2} - H^{3/2} \bar{H}^{1/2}] = O(\xi^{-2-2\tau}),$$

and

$$\frac{\bar{F}'' + \bar{H}'}{\bar{F} + \bar{H}} (F' + H) - \frac{\bar{H}' H}{\bar{H}} = O(\xi^{-2-\tau}),\ L_1\ F = O(\xi^{-2-\tau})$$

Using these relations in (2.9) and (2.19) in expression for N_1, the lemma follows. □

Lemma 2.22. *Let $F(\xi)$ is a classical solution as in definition 1.7. If F can be analytically extended to $\mathcal{D}$, then F satisfies the following equation for $\xi \in \mathcal{D}$:*

$$(2.29)\quad F(\xi) = \beta g_1(\xi) + \mathcal{U} N_1(\xi);$$

where β is given by

$$\beta = g_1^{-1}(-\nu)\left(F(-\nu) - \frac{1}{\epsilon^2} g_2(-\nu)\int_{-\infty}^{-\nu} \frac{N_1(t)}{W(t)} g_1(t)dt\right) \tag{2.30}$$

Proof. First we consider $\xi \in \mathbf{R}^-$ on the boundary of $\mathcal{D}$. From continuity, (2.29) holds, where $I(\xi)$ occurring in $N_1(\xi)$ is understood as $\lim_{Im\ \xi\to 0^-} I(\xi)$. Using the method of variation of parameter for $\xi \in \mathbf{R}^-$, we have

$$F(\xi) = C_1 g_1 + C_2 g_2 + \mathcal{U} N_1(\xi) \tag{2.31}$$

Since $\|N_1\|_{2,\mathbf{R}^-} < \infty$, it follows on using Lemmas 2.16 and 2.17, restricted to $\mathbf{R}^-$ instead of $\mathcal{D}$, that $\|\mathcal{U}N_1\|_{0,\mathbf{R}^-} < \infty$. Since $g_1(-\infty) = 0$ and $g_2(-\infty) = +\infty$, it follows from $\sup_{\xi\in\mathbf{R}^-} |\xi - 2i|^\tau |F| < \infty$ that $C_2 = 0$. Using $C_2 = 0$ in (2.31) and evaluating it at $\xi = -\nu$, we obtain $F(-\nu) = C_1 g_1(-\nu) + \mathcal{U}N_1(-\nu)$. Hence $C_1 = \beta$ as given by (2.30). So (2.29) holds for $\xi \in \mathbf{R}^-$. By analytic continuation of each side of the equation, it follows that it must be valid in $\mathcal{D}$ as well. □

Definition 2.23.

$$n_1(\xi) = \mathcal{N}(f, I, \bar{F})[\xi] \ + \ \epsilon^2\ L_1(\xi) f(\xi);$$

We consider the following integral equation:

$$f(\xi) = \beta g_1(\xi) + \mathcal{U} n_1(\xi); \tag{2.32}$$

where β is still given as before by (2.30).

Lemma 2.24. *$\bar{F}' \in \mathbf{B}_1, \bar{F}'' \in \mathbf{B}_2$, with $\|\bar{F}'\|_1 \leq \delta_1$ and $\|\bar{F}''\|_2 \leq \delta_2$*

Proof. The lemma follows from Definition 2.1 and Lemma 1.5. □

Definition 2.25.

$$H_m \equiv \inf_{\xi\in\mathcal{D}} \left\{ |\xi - 2i||H(\xi)|, |\xi - 2i||\bar{H}(\xi)| \right\}$$

Lemma 2.26. *Define operator $\mathcal{G}_1$ so that*

$$\mathcal{G}_1(f')[t] = (f'(t) + H(t))^{3/2} (\bar{F}'(t) + \bar{H}(t))^{1/2};$$

Let $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$ and $\tilde{\delta}_1, \delta_1 < \frac{H_m}{2}$, where δ_1 is as defined in (1.8). Then, for $\xi \in \mathcal{D}$,

$$|\mathcal{G}_1(f')[\xi]| \leq C|\xi - 2i|^{-2} \tag{2.33}$$

where C is independent of ϵ .

Proof. From (1.2) and (2.2),

$$H_m|\xi - 2i|^{-1} \leq |H| \leq C_2|\xi - 2i|^{-1}; \tag{2.34}$$

$$H_m|\xi - 2i|^{-1} \leq |\bar{H}| \leq C_2|\xi - 2i|^{-1}; \tag{2.35}$$

where C_1, H_m are independent of independent of ϵ.

$$\begin{aligned} |\mathcal{G}_1(f')| &= |H^{3/2} \bar{H}^{1/2}| \left| \frac{f'}{H} + 1 \right|^{3/2} \left| \frac{\bar{F}'}{\bar{H}} + 1 \right|^{1/2} \\ &\leq C|\xi - 2i|^{-2}; \end{aligned}$$

□

Lemma 2.27. *Let $\mathcal{G}_2$ be an operator so that*

$$\mathcal{G}_2(f')[\xi] = \left[(\bar{F}'' + \bar{H}') \frac{f' + H}{\bar{F}' + \bar{H}} - \frac{\bar{H}' H}{\bar{H}} \right] (\xi); \tag{2.36}$$

If $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$,and $\delta_1, \tilde{\delta}_1 < \frac{H_m}{2}$, then for $\xi \in \mathcal{D}$,

$$|\mathcal{G}_2(f')[\xi]| \leq C|\xi - 2i|^{-2-\tau} [\delta_1 + \tilde{\delta}_1 + \delta_2] \tag{2.37}$$

where C is independent of ϵ and δ_1, δ_2 are as defined in (1.6).

Proof. Note from (2.2),

$$\bar{H}' = -\frac{(\xi - i\gamma)^2 + (\gamma^2 - 1)}{(\xi^2+1)^2} \;=\; O(\xi - 2i)^{-2}, \text{ for large } |\xi|; \tag{2.38}$$

$$\frac{\bar{H}'H}{\bar{H}} = -\frac{[(\xi - i\gamma)^2 + (\gamma^2 - 1)](\xi + i\gamma)}{(\xi^2+1)^2(\xi - i\gamma)} \;=\; O((\xi - 2i)^{-2}), \text{ for large } |\xi| \tag{2.39}$$

$$\begin{aligned} |\mathcal{G}_2(f')| &= \left| f' \frac{\bar{H}'}{\bar{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}} \frac{\bar{F}'}{\bar{F}' + \bar{H}} + \bar{F}'' \frac{f' + H}{\bar{F}' + \bar{H}} \right| \\ &\le C|\xi - 2i|^{-2-\tau}[\delta_1 + \tilde{\delta}_1 + \delta_2] \end{aligned}$$

□

Lemma 2.28. *We define operator $\mathcal{G}_3$ so that*

$$\mathcal{G}_3(f') = (f' + H)^{3/2}(\bar{F}' + \bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2}; \tag{2.40}$$

Assume that $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$ with $\delta_1, \tilde{\delta}_1 < H_m/2$. Then for $\xi \in \mathcal{D}$

$$|\mathcal{G}_3(f')[\xi]| \le C \; |\xi - 2i|^{-2-\tau}(\delta_1 + \tilde{\delta}_1) \tag{2.41}$$

where C is independent of ϵ.

Proof. Using(2.40):

$$\begin{aligned} |\mathcal{G}_3(f)| &\le |H^{3/2}\bar{H}^{1/2}| \left| \left(\frac{f'}{H} + 1\right)^{3/2} \left(\frac{\bar{F}'}{\bar{H}} + 1\right)^{1/2} - 1 \right| \\ &\le C|\xi - 2i|^{-2} \left\{ \left(\frac{|f'|}{|H|} + 1\right)^{3/2} \left(\frac{|\bar{F}|}{|\bar{H}|} + 1\right)^{1/2} - 1 \right\} \\ &\le C|\xi - 2i|^{-2-\tau}(\delta_1 + \tilde{\delta}_1) \end{aligned}$$

□

Lemma 2.29. *Let* $f \in \mathbf{B}_{0,\tilde{\delta}}$, $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$,*then* $n_1 \in \mathbf{B}_2$ *for* $\tilde{\delta}_1, \delta_1 < H_m/2$, *and*

$$\|n_1\|_2 \leq K_4(\epsilon^2(1+\delta_2) + \delta + \tilde{\delta}(\epsilon^2 + \tilde{\delta}_1 + \delta_1))$$

where K_4 *is independent of* ϵ.

Proof. Note that

(2.42)
$$n_1 = \mathcal{N}(f, \mathcal{I}(F), \bar{F}) = \epsilon^2 \left(\frac{\bar{H}' H}{\bar{H}} - H' \right) - i\epsilon^2 \mathcal{G}_1(f') I(F) + i f \mathcal{G}_3(f') + \epsilon^2 \mathcal{G}_2(f') + \epsilon^2 L_1 f;$$

$$\left| \epsilon^2 \left(\frac{\bar{H}' H}{\bar{H}} - H' \right) \right| = \epsilon^2 \frac{2\gamma}{|(\xi^2+1)(\xi - i\gamma)|} \leq C\epsilon^2 |\xi - 2i|^{-3}; \tag{2.43}$$

from Lemmas 2.4 and 2.26,

$$|\epsilon^2 \mathcal{G}_1(f')(\xi) \mathcal{I}(F)[\xi]| \leq C\delta |\xi - 2i|^{-2-\tau}. \tag{2.44}$$

Applying Lemma 2.26-Lemma 2.28 to get

$$|f||\mathcal{G}_3(f')| \leq C|\xi - 2i|^{-2-\tau} \tilde{\delta}(\delta_1 + \tilde{\delta}_1)$$

$$\epsilon^2 |\mathcal{G}_2(f')| \leq C\epsilon^2 |\xi - 2i|^{-2-\tau} (\tilde{\delta}_1 + \delta_1 + \delta_2)$$

From the expression of $L_1(\xi)$,we have:

$$|\epsilon^2 f||L_1(\xi)| \leq C\epsilon^2 |\xi - 2i|^{-2-\tau} \tilde{\delta}; \tag{2.45}$$

On using the expression for n_1 in (2.42), we have the proof by combining the above inequalities. It is to be noted that terms like $\epsilon^2 \tilde{\delta}$, $\epsilon^2 \delta_1$, etc. do not appear because they are smaller than terms explicitly appearing on the right hand side of the Lemma statement. Clearly, for suitable choice of K_4, such terms can be estimated away. □

Lemma 2.30. *Let $\mathcal{G}_1$ be as defined in Lemma 2.26. Let $f_j' \in \mathbf{B}_{1,\tilde{\delta}_1}, j = 1, 2$, then for $\delta_1, \tilde{\delta}_1 < H_m/2$,*

$$|\mathcal{G}_1(f_1')(\xi) - \mathcal{G}_1(f_2')(\xi)| \leq C|\xi - 2i|^{-2-\tau} \|f_1' - f_2'\|_1; \tag{2.46}$$

Where C is independent of ϵ .

Proof. By straightforward algebra:

$$\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') = \frac{(f_1' - f_2')(\bar{F}' + \bar{H})^{1/2}[(f_1' + H)^2 + (f_1' + H)(f_2' + H) + (f_2' + H)^2]}{(f_1' + H)^{3/2} + (f_2' + H)^{3/2}}$$

Lemma follows from above equation, on using upper and lower estimates for $|f_i' + H|$ and $|\bar{F}' + \bar{H}|$ as in preceding lemmas. □

Lemma 2.31. *Let $f_j' \in \mathbf{B}_{1,\tilde{\delta}_1}, j = 1, 2$. Let $\mathcal{G}_2(f')$ be defined as in Lemma 2.27, then for $\delta_1 < H_m/2$,*

$$\|\mathcal{G}_2(f_1') - \mathcal{G}_2(f_2')\|_2 \leq C(\delta_2 + 1)\|f_1' - f_2'\|_1; \tag{2.47}$$

Proof. We note

$$|(\bar{F}'' + \bar{H}')| \leq \frac{C}{|\xi - 2i|^2} + \frac{\delta_2}{|\xi - 2i|^{2+\tau}} \leq C|\xi - 2i|^{-2}(1 + \delta_2)$$

Also,

$$|(\bar{H} + \bar{F}')^{-1}| \leq \frac{4}{H_m}|\xi - 2i|$$

By straightforward algebra,

$$\mathcal{G}_2(f_1') - \mathcal{G}_2(f_2') = \frac{(\bar{F}_2'' + \bar{H}')}{(\bar{F}_1' + \bar{H})}(f_1' - f_2');$$

Using inequalities as above, we obtain the proof of the lemma. □

Lemma 2.32. *Let* $f_j \in \mathbf{B}_{0,\tilde{\delta}}, f'_j \in \mathbf{B}_{1,\tilde{\delta}_1}, j = 1, 2,$ *then for* $\tilde{\delta}_1, \delta_1 < H_m/2$,

$$\begin{aligned}(2.48)\qquad &\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F} + \epsilon^2 L_1(f_2 - f_1)\|_2 \\ &\le K_5 \left((\epsilon^2 + \delta_1 + \tilde{\delta}_1)\|f_1 - f_2\|_0 + (\epsilon^2 + \delta + \tilde{\delta} + \epsilon^2\delta_2)\|f'_1 - f'_2\|_1 \right)\end{aligned}$$

where K_5 *is independent of* ϵ.

Proof. From (2.42),

$$\begin{aligned}(2.49)\quad \mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) &= -i\epsilon^2 \mathcal{I}(F)\ \left(\mathcal{G}_1(f'_1) - \mathcal{G}_1(f'_2)\right) \\ &+ i(f_1 - f_2)\mathcal{G}_3(f'_1) + i f_2 \left(\mathcal{G}_1(f'_1) - \mathcal{G}_1(f'_2)\right) + \epsilon^2 \left(\mathcal{G}_2(f'_1) - \mathcal{G}_2(f'_2)\right);\end{aligned}$$

On using Lemmas 2.4, 2.28, 2.30, 2.31 and expression for $L_1(\xi)$, we obtain

$$\|\epsilon^2 I\ \left(\mathcal{G}_1(f'_1) - \mathcal{G}_1(f'_2)\right)\|_2 \le C\delta\|f'_1 - f'_2\|_1$$

$$\|(f_1 - f_2)\mathcal{G}_3(f'_1)\|_2 \le C(\delta_1 + \tilde{\delta}_1)\|f_1 - f_2\|_0$$

$$\|f_2 \left(\mathcal{G}_1(f'_1) - \mathcal{G}_1(f'_2)\right)\|_2 \le C\tilde{\delta}\|f'_1 - f'_2\|_1$$

$$\|\epsilon^2 \left(\mathcal{G}_2(f'_1) - \mathcal{G}_2(f'_2)\right)\|_2 \le C\epsilon^2(1 + \delta_2)\|f'_1 - f'_2\|_1$$

$$\|\epsilon^2 L_1(f_1 - f_2)\|_2 \le C\epsilon^2\|f_1 - f_2\|_0$$

Combining all the above inequalities, we get the proof. □

Lemma 2.33. *For sufficiently small* ϵ, *we have*

$$(2.50)\qquad \|\beta g_1\|_0 \le K_6(\epsilon^2 + \delta + \epsilon^2\delta_2);$$

where K_6 *is independent of* ϵ

Proof. For $|\xi| \leq R$, from (2.30),

$$|\beta g_1(\xi)| \leq |\beta g_1(-\nu)| \leq |F(-\nu)| + |\mathcal{U}N_1(-\nu)| \leq \delta + \|\mathcal{U}N_1\|_{2,\mathbf{R}^-};$$

but from(2.9) and (2.19) and using Lemma 2.29 with domain $\mathcal{D}$ replaced by $\mathcal{R}^-$, and f replaced by F (and hence $\tilde{\delta}$ by δ and $\tilde{\delta}_1$ by δ_1), we get

$$\|N_1\|_{2,\mathbf{R}^-} \leq C(\epsilon^2 + \delta + \epsilon^2\delta_2),$$

So $\|\mathcal{U}N_1\|_0 \leq C(K_1+K_2)(\epsilon^2+\delta+\epsilon^2\delta_2)$ from Lemma 2.16 and Lemma 2.17. Therefore $|\beta g_1(-\nu)| < \tilde{K}_6(\epsilon^2 + \delta + \epsilon^2\delta_2)$ for some $\tilde{K}_6$ independent of ϵ. For $|\xi| \geq R$, on using equations (2.12) and (2.22), which holds, $|g_1(\xi)g_1^{-1}(-\nu)| < C\left(|\xi|^{\frac{1}{2}-\frac{C_1}{\epsilon}}\right)$, where C, C_1 are independent of ϵ. For $\epsilon \leq \frac{C_1}{2}$, above $< C|\xi - 2i|^{-\tau}$ and the lemma follows.

□

Remark 2.34. The estimates in each of the Lemmas 2.16-2.33 generally depend on γ and therefore λ, as quantities such as H_m and upper bounds for $(\xi-2i)H$ or $(\xi-2i)\bar{H}$ have dependences on γ. If we consider λ in any fixed compact subset of the interval $(0,\ 1)$, *i.e.* for $\gamma = \frac{\lambda}{1-\lambda}$ in a compact subset of $(0,\infty)$, such dependences can be removed since H and $\bar{H}$ are continuous functions of γ in this interval.

We define spaces:

Definition 2.35.

$$\mathbf{E} := \mathbf{B} \oplus \mathbf{B}_1$$

For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}$,

$$\|\mathbf{e}\|_{\mathbf{E}} := \|u(\xi)\|_0 + \epsilon\|v(\xi)\|_1$$

It is easy to see that $\mathbf{E}$ is Banach space.We replace (f, f') by (u, v). Also we denote operator $\mathbf{n}$ so that $\mathbf{n}(u,v)(\xi) = n_1(\xi)$.

Definition 2.36. Let

$$\mathbf{O} : \mathbf{E} \longmapsto \mathbf{E}$$

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathbf{e}) = (\mathbf{O}_1(\mathbf{e}), \mathbf{O}_2(\mathbf{e}))$$

where

$$\mathbf{O}_1(\mathbf{e}) = \beta g_1 + \mathcal{U}\mathbf{n}(u, v); \tag{2.51}$$

$$\mathbf{O}_2(\mathbf{e}) = \beta h_1 g_1 + \mathcal{U}_1\mathbf{n}(u, v); \tag{2.52}$$

Definition 2.37. Let

$$\Delta = 8K(\delta + \epsilon^2(1 + \delta_2)) \tag{2.53}$$

where

$$K = \max\{K_6, (K_1 + K_2)K_4, K_3K_6, K_3K_4(K_1 + K_2)\}; \tag{2.54}$$

We define space $\mathbf{E}_\Delta = \{\mathbf{e} \in \mathbf{E} : \|\mathbf{e}\|_{\mathbf{E}} \leq \Delta\}$.

Lemma 2.38. *If $\mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E}_\Delta$, then for ϵ, δ_1 and $\epsilon \ln \frac{1}{\epsilon}\ \delta_2$ each sufficiently small (the latter two are part of Assumption (ii)), $\mathbf{O}(\mathbf{e}) \in \mathbf{E}_\Delta$.*

Proof. If $\mathbf{e} \in \mathbf{E}_\Delta$, it follows from the expression for Δ, that

$$\frac{\Delta}{\epsilon} \leq 8K[\epsilon + \frac{\delta}{\epsilon} + \epsilon\delta_2], \tag{2.55}$$

and this is small by assumption and remark 1.6. Thus, both $\|v\|_1$ (and therefore $\tilde{\delta}_1$) and δ_1 can be taken smaller than $\frac{H_m}{2}$ so as to apply Lemmas 2.29 and Lemma 2.33

which, together with Lemmas 2.16 and 2.17, gives:

$$\|\mathbf{O}_1(\mathbf{e})\|_0 \le \|\beta g_1\|_0 + \|\mathcal{U}\mathbf{n}(u,v)\|_0$$

$$\le K_6(\epsilon^2+\delta+\epsilon^2\delta_2) + (K_1+K_2)K_4[\epsilon^2(1+\delta_2)+\delta+\|u\|_0(\epsilon^2+\delta_1+\|v\|_1)]$$

$$\le 2K[\epsilon^2(1+\delta_2)+\delta] + K\|u\|_0(\epsilon^2+\delta_1+\|v\|_1))$$

Using $\|u\|_0 \le \Delta, \epsilon\|v\|_1 \le \Delta$ and (2.55), we get

$$K\|u\|_0(\epsilon^2+\delta_1+\|v\|_1)) \le \Delta[K(\epsilon^2+\delta_1) + K\frac{\Delta}{\epsilon}]$$

So

$$\|\mathbf{O}_1(\mathbf{e})\|_0 \le \Delta[\frac{1}{4} + K(\epsilon^2+\delta_1) + K\frac{\Delta}{\epsilon}]$$

From Lemma 2.19

$$\epsilon\|\mathbf{O}_2(\mathbf{e})\|_1 \le K_3[\|\beta g_1\|_0 + \|\mathcal{U}\mathbf{n}(u,v)\|_0]$$

$$\le \Delta[\frac{1}{4} + K(\epsilon^2+\delta_1) + K\frac{\Delta}{\epsilon}]$$

then, for sufficiently small ϵ, δ_1 and $\epsilon\ln\frac{1}{\epsilon}\ \delta_2$,

$$\|\mathbf{O}(\mathbf{e})\| = \|\mathbf{O}_1(\mathbf{e})\|_0 + \epsilon\|\mathbf{O}_2(\mathbf{e})\|_1$$

$$\le \Delta[\frac{1}{2} + 2K(\epsilon^2+\delta_1) + 2K\frac{\Delta}{\epsilon}] \le \Delta$$

□

Lemma 2.39. *If* $\mathbf{e}_j = (u(\xi), v(\xi)) \in \mathbf{E}_\Delta, j = 1,2,$ *then for* ϵ, δ_1 *and* $\epsilon\ln\frac{1}{\epsilon}\ \delta_2$ *small enough,*

$$\|\mathbf{O}(\mathbf{e}_1) - \mathbf{O}(\mathbf{e}_2)\| \le \Delta_1\|\mathbf{e}_1 - \mathbf{e}_2\|.$$

where

$$\Delta_1 = \tilde{K}\,[2\epsilon + \delta_1 + \epsilon\delta_2 + \Delta/\epsilon] \tag{2.56}$$

where $\tilde{K} = 2\max\{K_5(K_1+K_2), K_3K_5(K_1+K_2)\}$.

Proof. Since $(u_1, v_1), (u_2, v_2) \in \mathbf{E}_\Delta$, it follows that each of $\|u_1\|_0$, $\|u_2\|_0$, $\epsilon\|v_1\|_1$ and $\epsilon\|v_2\|_1$ are bounded by Δ and that we can assume each of $\|v_1\|_1$ and $\|v_2\|_1 < \frac{H_m}{2}$ so as to apply Lemmas 2.32, 2.16 and 2.17, which, on using $\tilde{\delta} \leq \Delta$ and $\tilde{\delta}_1 \leq \frac{\Delta}{\epsilon}$, gives:

$$\begin{aligned}
&\|\mathbf{O}_1(\mathbf{e}_1) - \mathbf{O}_1(\mathbf{e}_2)\|_0 \\
&\leq (K_1 + K_2)K_5\Big\{(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon})\|u_1 - u_2\|_0 + (\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon})\epsilon\|v_1 - v_2\|_1\Big\} \\
&\leq \frac{\Delta_1}{2}\|\mathbf{e}_1 - \mathbf{e}_2\|
\end{aligned}$$

$$\begin{aligned}
&\epsilon\|\mathbf{O}_2(\mathbf{e}_1 - \mathbf{O}_1(\mathbf{e}_2)\|_1 \\
&\leq K_3(K_1 + K_2)K_5\Big\{(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon})\|u_1 - u_2\|_0 + (\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon})\epsilon\|v_1 - v_2\|_1\Big\} \\
&\leq \frac{\Delta_1}{2}\|\mathbf{e}_1 - \mathbf{e}_2\|
\end{aligned}$$

So, proof of Lemma follows by combining the above. □

Theorem 2.40. *For sufficiently small $\delta_1, \epsilon \ln \frac{1}{\epsilon}\ \delta_2$ and ϵ, the operator $\mathbf{O}$ is a contraction mapping from $\mathbf{E}_\Delta$ to $\mathbf{E}_\Delta$. Therefore, there exists unique solution $(u(\xi), v(\xi)) \in \mathbf{E}_\Delta$ to $\mathbf{e} = \mathbf{O}(\mathbf{e})$ and hence to the integral equation (2.32), where $f = u$ and $f' = v$.*

Proof. From assumptions and remark 1.6, we know that $\Delta_1 < 1$. The theorem follows from Lemmas 2.38 and 2.39. □

Lemma 2.41. *If f is the solution in Theorem 2.40 and F is a classical solution as defined earlier. Then $f(\xi) \equiv F(\xi)$ for $\xi \in (-\infty, -\nu]$ for small enough ϵ, δ_1 and $\epsilon \ln\ \frac{1}{\epsilon}\ \delta_2$.*

Proof. Let $u = f - F, v = f' - F'$. From (2.29), (2.32), u and v satisfy the following equations :

$$u = \mathcal{U}(n_1 - N_1), v = \mathcal{U}_1(n_1 - N_1);$$

By Lemma 2.32 restricted to domain $\mathbf{R}^-$, with $f_1 = f$ and $f_2 = F$ and using $\|\epsilon^2 L_1 u\|_{0,\mathbf{R}^-} \leq C\epsilon^2 \|u\|_{0,\mathbf{R}^-}$

$$\|n_1 - N_1\|_{2,\mathbf{R}^-} \leq C\left[(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon})\|u\|_{0,\mathbf{R}^-} + (\epsilon^2 + \epsilon^2\delta_2 + \delta + \Delta)\|v\|_{1,\mathbf{R}^-}\right]$$

So, from using Lemmas 2.16 and 2.17, restricted to domain $\mathbf{R}^-$,

$$\|u\|_{0,\mathbf{R}^-} \leq C\left[(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon})\|u\|_{0,\mathbf{R}^-} + (\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon})\epsilon\|v\|_{1,\mathbf{R}^-}\right]$$

$$\epsilon\|v\|_{1,\mathbf{R}^-} \leq C\left[(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon})\|u\|_{0,\mathbf{R}^-} + (\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon})\epsilon\|v\|_{1,\mathbf{R}^-}\right]$$

where C is a constant independent of ϵ. So, combining the above,

$$\|u\|_{0,\mathbf{R}^-} + \epsilon\|v\|_{1,\mathbf{R}^-} \leq C(\epsilon + \delta_1 + \frac{\Delta}{\epsilon} + \epsilon\delta_2 + \frac{\delta}{\epsilon})(\|u\|_{0,\mathbf{R}^-} + \epsilon\|v\|_{1,\mathbf{R}^-})$$

Since the constant C is independent of ϵ in the estimate on the right side of the above equation. It follows that for small ϵ, $\epsilon \ln \frac{1}{\epsilon}$ δ_2 and δ_1 (and hence small $\frac{\Delta}{\epsilon}$ because of Remark 1.6), $(u, v) \equiv \mathbf{0}$. Hence, the Lemma follows. □

Theorem 2.42. *If F is a classical solution satisfying assumptions (i) and (ii), then for small enough ϵ, then $F \in \mathbf{B}_{0,\Delta}$ and $F' \in \mathbf{B}_{1,\frac{\Delta}{\epsilon}}$*

Proof. The theorem follows from Theorem 2.40 and Lemma 2.41. □

3. Analyticity in the triangular region

Let $\mathcal{S} = \{\xi : Re\ \xi = -a, -\nu + a \leq Im\ \xi \leq 0\}$ where $0 \leq a < \nu$ be a vertical straight line segment in the triangular region $\mathcal{T}$ bounded by negative real axis,negative imaginary axis and line segment $\{\xi : \xi = -\nu + se^{-\pi i/4}, 0 \leq s \leq \sqrt{2}\nu\}$. This is the triangular region (See Fig. 1), which is the complement of the region $\mathcal{D}$ in the third-quadrant. It is to be noted that in the triangular region $\mathcal{T}$, $P(\xi) = P(0) + i\gamma\xi + O(\nu^2)$

and so on $\mathcal{S}$ when $\xi = -a - is$, $Re\ P$ increases monotonically with s such that $\frac{d}{ds}\ Re\ P(\xi(s)) > C > 0$, where C is independent of ϵ and ν for sufficiently small ν.

We consider the following boundary value problem on the line segment $\mathcal{S}$

$$\begin{aligned} &\epsilon^2 f'' + (L(\xi) + \epsilon^2 L_1(\xi)) f = \mathcal{N}(f, I(F), \bar{F})[\xi] + \epsilon^2 L_1 f(\xi) \equiv n_1(\xi); \\ &f(-a) = F(-a), f(-a_1) = F(-a_1) \end{aligned} \tag{3.1}$$

where $a_1 = a + i(\nu - a)$.

Lemma 3.1. *$f \in C^2(\mathcal{S})$ is a solution of boundary value problem (3.1) if only if f is a solution of the following integral equation:*

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \mathcal{U}_3 n_1 \tag{3.2}$$

where

$$\mathcal{U}_3 n_1 = -\frac{1}{\epsilon^2} g_1 \int_{-a}^{\xi} \frac{n_1(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} g_2 \int_{-a_1}^{\xi} \frac{n_1(t)}{W(t)} g_1(t) dt \tag{3.3}$$

$$\alpha_1 = \frac{\gamma_1 g_2(-a_1) - \gamma_2 g_2(-a)}{g_1(-a) g_2(-a_1) - g_1(-a_1) g_2(-a)} \tag{3.4}$$

$$\alpha_2 = \frac{\gamma_1 g_1(-a_1) - \gamma_2 g_1(-a)}{g_1(-a) g_2(-a_1) - g_1(-a_1) g_2(-a)} \tag{3.5}$$

where

$$\gamma_1 = F(-a) - \frac{1}{\epsilon^2} g_2(-a) \int_{-a_1}^{-a} \frac{n_1(t)}{W(t)} g_1(t) dt \tag{3.6}$$

$$\gamma_2 = F(-a_1) + \frac{1}{\epsilon^2} g_1(-a_1) \int_{-a}^{-a_1} \frac{n_1(t)}{W(t)} g_2(t) dt \tag{3.7}$$

Proof. If $f \in C^2(\mathcal{S})$ is a solution of boundary problem (3.1), then by variation of parameters, we have

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \mathcal{U}_3 n_1 \tag{3.8}$$

for some α_1 and α_2. Plugging the boundary conditions in (3.1), and solving for α_1 and α_2, we have (3.4) and (3.5). By straight forward computation, we get that a solution of (3.2) is a solution of the boundary problem (3.1). □

Remark 3.2. γ_1 and γ_2 depend on f, f' through n_1, γ_1 and γ_2 are functionals of f, f', so are α_1 and α_2.We use notation $\alpha_j(f, f')$ to indicate the dependence on f, f'. The norm $\|\cdot\|$ mean maximum norm $\|\cdot\|_\infty$ in this section.

Lemma 3.3. *If $\tilde{n} \in C(\mathcal{S})$, let $\tilde{f}_1(\xi) = \frac{1}{\epsilon^2} g_2(\xi) \int_{-a_1}^{\xi} \frac{\tilde{n}(t)}{W(t)} g_1(t) dt$,then $\tilde{f}_1 \in C(\mathcal{S})$ and $\|\tilde{f}_1\| \leq K_1 \|\tilde{n}\|$ for constant K_1 independent of ϵ.*

Proof. Using monotonicity of Re P on $\mathcal{S}$ with s, as noted before,

$$\begin{aligned}
|\tilde{f}_1(\xi)| &= \left| \frac{1}{2i\epsilon} \int_{-a_1}^{\xi} L^{-1/4}(\xi) L^{-1/4}(t) \tilde{n}(t) \exp\{-\frac{1}{\epsilon}(P(t) - P(\xi))\} dt \right| \\
&\leq C \int_{\exp\{-\frac{1}{\epsilon}(P(-a_1) - P(\xi))\}}^{1} \frac{|L^{-1/4}(\xi) L^{-1/4}(t) \tilde{n}(t)|}{\frac{d}{ds} Re\ P(t(s))}\ d \left[\exp\{-\frac{1}{\epsilon}(P(t) - P(\xi))\} \right] \\
&\leq K_1 \|\tilde{n}\|
\end{aligned}$$

□

Lemma 3.4. *If $\tilde{n} \in C(\mathcal{S})$, let $\tilde{f}_2 = \frac{1}{\epsilon^2} g_1(\xi) \int_{-a}^{\xi} \frac{\tilde{n}(t)}{W(t)} g_2(t) dt$,then $\tilde{f}_2 \in C(\mathcal{S})$ and $\|\tilde{f}_2\| \leq K_2 \|f\|$.*

Proof. The proof is very similar to Lemma 3.3. □

Lemma 3.5. *Let $f_j \in C(\mathcal{S}), f_j' \in C(\mathcal{S}), j = 1, 2,$ then*

$$\begin{aligned} &\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2)\| \\ &\leq K_5 \left((\epsilon^2 + \delta_1 + \|f_1'\|)\|f_1 - f_2\| + (\epsilon^2 + \delta + \|f_2\| + \epsilon^2\delta_2)\|f_1' - f_2'\| \right) \end{aligned} \tag{3.9}$$

Proof. The proof parallels that of lemma 2.32, except that the domain is $\mathcal{S}$ instead of $\mathcal{D}$ and the norm is the max norm. □

Lemma 3.6. *If $f' \in C(\mathcal{S})$ then $\alpha_j g_j \in C(\mathcal{S})$ for $j = 1, 2$ and*

$$\|\alpha_j g_j\| \leq k_1 \left(|F(-a)| + |F(-a_1)| + \|n_1\|\right), \text{where } k_1 \text{ is independent of } \epsilon \tag{3.10}$$

Proof. Let

$$D = g_1(-a)g_2(-a_1) - g_2(-a)g_1(-a_1) \tag{3.11}$$

Using (2.12), we have:

$$D = L^{-1/4}(-a)L^{-1/4}(-a_1)\exp\{\frac{1}{\epsilon}\left(P(-a_1) - P(-a)\right)\}\left[1 - \exp\{\frac{2}{\epsilon}\left(P(-a) - P(-a_1)\right)\}\right] \tag{3.12}$$

Since $Re\ P(-a_1) > Re\ P(-a)$, D^{-1} is exponentially small in ϵ, $Re\ P(\xi) \geq Re\ P(-a)$ for $\xi \in \mathcal{S}$. We also have

$$\left|\frac{g_2(-a)g_1(\xi)}{D}\right| = \left|\frac{L^{-1/4}(\xi)}{L^{-1/4}(-a_1)}\right| \left|\frac{\exp\{\frac{1}{\epsilon}\left(2P(-a) - P(\xi) - P(-a_1 i)\right)\}}{1 - \exp\{\frac{2}{\epsilon}\left(P(-a) - P(-a_1)\right)\}}\right| \leq C \tag{3.13}$$

with C independent of ϵ. Also,

$$\left|\frac{g_2(-a_1)g_1(\xi)}{D}\right| = \left|\frac{L^{-1/4}(\xi)}{L^{-1/4}(-a)}\right| \left|\frac{\exp\{-\frac{1}{\epsilon}\left(P(\xi) - P(-a)\right)\}}{1 - \exp\{\frac{2}{\epsilon}\left(P(-a) - P(-a_1)\right)\}}\right| \leq C \tag{3.14}$$

Similarly, we get constant upper bounds for $\frac{g_1(-a_1)g_2(\xi)}{D}$ and $\frac{g_1(-a)g_2(\xi)}{D}$. Using lemmas 3.3 and 3.4 in (3.6) and (3.7), we have

$$|\gamma_1| \leq \left(|F(-a)| + K_1\|n_1\|\right) \tag{3.15}$$

$$|\gamma_2| \leq (|F(-a_1)| + K_2\|n_1\|) \tag{3.16}$$

Using (3.13), (3.14) and similar bounds, in (3.4) and (3.5), we get the lemma. ☐

Lemma 3.7. *If $f'_j \in C(\mathcal{S}), j = 1, 2,$ then $(\alpha_j(f_1, f'_1) - \alpha_j(f_2, f'_2))\, g_j \in C(\mathcal{S})$ and*

$$\|(\alpha_j(f_1, f'_1) - \alpha_j(f_2, f'_2))\, g_j\| \\ \leq C(\epsilon^2 + \delta_1 + \|f'_1\|)\|f_1 - f_2\| + (\epsilon^2 + \|f_2\| + \delta + \epsilon^2\delta_2)\|f'_1 - f'_2\|. \tag{3.17}$$

Proof.

$$|(\alpha_1(f_1, f'_1) - \alpha_1(f_2, f'_2))\, g_1| \\ \leq |\gamma_1(f_1, f'_1) - \gamma_1(f_2, f'_2)| \left|\frac{g_2(-a)g_1(\xi)}{D}\right| + |\gamma_2(f_1, f'_1) - \gamma_2(f_2, f'_2)| \left|\frac{g_2(-a_1)g_1(\xi)}{D}\right| \tag{3.18}$$

Using (3.6),(3.7), Lemmas 3.3 and 3.4:

$$|\gamma_1(f_1, f'_1) - \gamma_1(f_2, f'_2)| \leq C\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2)\|, \tag{3.19}$$

$$|\gamma_2(f_1, f'_1) - \gamma_2(f_2, f'_2)| \leq C\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2)\|, \tag{3.20}$$

The lemma follows from (3.13), (3.14),(3.18) and Lemma 3.5. Similar proof for $j = 2$. ☐

We consider the following integral equations:

$$f(\xi) = \mathbf{o}_3(f, f') := \alpha_1 g_1(\xi) + \alpha_2 g_2(\xi) + \mathcal{U}_3 n_1(\xi); \tag{3.21}$$

$$f'(\xi) = \mathbf{o}_4(f, f') := \alpha_1 h_1(\xi) g_1(\xi) + \alpha_2 h_2(\xi) g_2(\xi) + \mathcal{U}_4 n_1(\xi); \tag{3.22}$$

where

$$\mathcal{U}_4 n_1 = -\frac{1}{\epsilon^2} h_1(\xi) g_1 \int_{-a}^{\xi} \frac{n_1(t)}{W(t)} g_2(t) dt + \frac{1}{\epsilon^2} h_2(\xi) g_2 \int_{-a_1}^{\xi} \frac{n_1(t)}{W(t)} g_1(t) dt \tag{3.23}$$

We define spaces:

Definition 3.8.

$$\mathbf{E}(\mathcal{S}) := C(\mathcal{S}) \oplus C(\mathcal{S})$$

For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}(\mathcal{S})$,

$$\|\mathbf{e}\|_{\mathbf{E}(\mathcal{S})} := \|u(\xi)\|_\infty + \epsilon \|v(\xi)\|_\infty$$

It is easy to see that $\mathbf{E}(\mathcal{S})$ is Banach space.

Definition 3.9. We define k_3 independent of ϵ so that

$$k_3 \geq \sup_{\xi \in \mathcal{T}} \{\epsilon |h_1(\xi)|, \epsilon |h_2(\xi)|\}$$

Definition 3.10.

$$\mathbf{E}_{\Delta,\mathcal{S}} := \{\mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E}(\mathcal{S}) : \|u(\xi)\| \leq 8k_1\Delta, \|v(\xi)\| \leq 8k_1k_3\frac{\Delta}{\epsilon}\}. \tag{3.24}$$

where k_1 and k_3 are $O(1)$ constants, as defined in Lemma 3.6 and definition 3.9 and Δ is as defined in (2.53).

Definition 3.11. Let

$$\mathbf{O}(\mathcal{S}) : \mathbf{E}(\mathcal{S}) \longmapsto \mathbf{E}(\mathcal{S})$$

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathcal{S})(\mathbf{e}) = (\mathbf{O}_3(\mathbf{e}), \mathbf{O}_4(\mathbf{e}))$$

Theorem 3.12. *For sufficiently small δ_1, $\epsilon \ln \frac{1}{\epsilon}$ δ_2 and ϵ, the operator $\mathbf{O}(\mathcal{S})$ is a contraction mapping from $\mathbf{E}_{\Delta,\mathcal{S}}$ to $\mathbf{E}_{\Delta,\mathcal{S}}$. Therefore there exists unique solution $(u(\xi), v(\xi)) \in \mathbf{E}_{\Delta,\mathcal{S}}$ to the equations (3.21), (3.22).*

Proof. Replacing space $\mathbf{B}_j$ with $C(\mathcal{S})$, the proof is parallel to that of Theorem 2.40. □

Theorem 3.13. *Let F be the classical solution in Theorem 2.40, then F is analytic inside the triangular region $\mathcal{T}$.*

Proof. Let f be the solution in Theorem 3.12, then f satisfies the boundary value problem (3.1). Since all the coefficients in equation (3.1) are analytic in a neighborhood of $\mathcal{S}$, it follows from the classical local theory of ordinary differential equations that f must be analytic in a neighborhood of $\mathcal{S}$. Since a is arbitrary in interval $(0, \nu)$, f is analytic in $\mathcal{T}$ and continuous on the closure of $\mathcal{T}$. From boundary conditons in (3.1), f equals analytic function F on $(-\nu, 0) \cup \{\xi : \xi = -\nu + se^{-\pi i/4}, 0 \leq s \leq \sqrt{2}\nu\}$. From properties of analytic continuation, f must be analytic continuation of F across $(-\nu, 0) \cup \{\xi : \xi = -\nu + se^{-\pi i/4}\}$ in the region $\mathcal{T}$. Therefore, the theorem follows. □

Lemma 3.14. *Let F be the classical solution in Theorem 2.40, then F is analytic on the line segment on imaginary axis $\mathcal{S}_0 = \{\xi : \; Re\ \xi = 0, -b \leq \; Im\ \xi \leq 0\}$.*

Proof. Considering the boundary problem for $\xi \in \mathcal{S}_0$:

$$\begin{aligned} &\epsilon^2 f'' + (L(\xi) + \epsilon^2 L_1(\xi)) f = \mathcal{N}(f, I(F), \bar{F})(\xi) + \epsilon^2 L_1 f(\xi) \equiv n_1(\xi); \\ &f(-a) = F(-a), f(-bi) = F(-bi) \end{aligned} \tag{3.25}$$

It follows from a variation of the proof of Theorem 3.12 that there exists an unique solution f in $\mathbf{E}_{\Delta,\mathcal{S}_0}$ to the above boundary problem. Since the coefficients of (3.25) are all analytic in a neighborhood of $\mathcal{S}_0$, the solution must be analytic on $\mathcal{S}_0$ from classical theory of differential equations. On the other hand, from Theorem 3.13, F satisfies equation (3.25) in $\mathcal{D} \cup \mathcal{T}$, since F and F' are continuous upto the closure of $\mathcal{D} \cup \mathcal{T}$. From continuity, F restricted on $\mathcal{S}_0$ satisfies the boundary problem (3.25) and $F \in \mathbf{E}_{\Delta,\mathcal{S}_0}$. By uniqueness, $F \equiv f$, therefore theorem follows. □

Definition 3.15.

$$k_2 = \sup_{\xi \in \mathcal{T}} \left\{ |\xi - 2i|^{\tau}, |\xi - 2i|^{\tau+1} \right\} \tag{3.26}$$

Remark 3.16. It is to be noted that

$$\sup_{\xi \in \mathcal{T}} |\xi - 2i|^{\tau} |F(\xi)| \leq k_2 \sup_{\xi \in \mathcal{T}} |F(\xi)|$$

$$\sup_{\xi \in \mathcal{T}} |\xi - 2i|^{\tau+1} |F'(\xi)| \leq k_2 \sup_{\xi \in \mathcal{T}} |F'(\xi)|$$

Definition 3.17.

$$\hat{\Delta} = max \, \{\Delta, 8k_1 k_2 \Delta, 8k_1 k_2 k_3 \Delta\} \tag{3.27}$$

Theorem 3.18. *If F is a classical solution as in definition 1.7, then F is analytic in $\mathcal{R} \cup \bar{\mathbf{Z}}^+$ and $F \in \mathbf{A}_{0,\hat{\Delta}}$, $F' \in \mathbf{A}_{1,\hat{\Delta}/\epsilon}$.*

Proof. Combining Theorems 2.42, 3.12 and 3.13 F is analytic in the domain $\mathcal{R}^-$, as defined in definition 1.1, with

$$\sup_{\xi \in \mathcal{R}^-} |\xi - 2i|^{\tau} |F(\xi)| \leq \hat{\Delta}$$

and

$$\sup_{\xi \in \mathcal{R}^-} |\xi - 2i|^{\tau+1} |F'(\xi)| \leq \frac{\hat{\Delta}}{\epsilon}$$

Since F is analytic in $\mathbf{Z}^+$ and from Lemma 3.14, analytic on the line segment $\mathcal{S}_0$ on the imaginary axis, from condition (iii) and successive Taylor expansions of F on the imaginary ξ-axis, starting at $\xi = 0$, implies $Im \; F = 0$ on $\mathcal{S}_0$. From Schwartz reflection principle for $\xi \in \mathcal{R}^+$, $F(\xi) = [F(-\xi^*)]^*$ provides the analytic extension to $Re \; \xi > 0$. Thus F is analytic in $\mathcal{R}$ and continuous upto its boundary, including the real axis. Thus, F must be analytic in $\mathcal{R} \cup \bar{\mathbf{Z}}^+$. Since from reflection, $\|F\|_{0,\mathcal{R}} = \|F\|_{0,\mathcal{R}^-}$, $\|F'\|_{1,\mathcal{R}} = \|F'\|_{1,\mathcal{R}^-}$, the proof of theorem is complete. □

Lemma 3.19. *If F is a classical solution as in definition 1.7, then $\delta, \delta_1, \delta_2$, as defined in (1.5),(1.6), equals $O(\epsilon^2)$. Therefore, in the domain $\mathcal{R}$, $\|F\|_0 = O(\epsilon^2)$, $\|F'\|_1 = O(\epsilon)$.*

Proof. Since F is analytic in $\mathcal{R} \cup \mathbf{Z}^+$ and decays algebraically at ∞ in this region, it follows from Cauchy's formula that for real $\xi \in (-\infty, \infty)$,

$$F^{(j)}(\xi) = \frac{j!}{2\pi i} \int_{l_1 \cup l_2} \frac{F(t)}{(t-\xi)^{j+1}} dt$$

Using Lemma 2.11 in [1], it follows that

$$\delta_j \equiv \sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{j+\tau} |F^{(j)}(\xi)| \le C_j \, \|F\|_0 = O(\hat{\Delta})$$

Therefore, using (1.1), on the real axis, $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^\tau \, |Re \; F(\xi)| = O(\epsilon^2)$. Further, on taking derviative of (1.1) with respect to ξ, and using $O(\hat{\Delta})$ *a priori* bounds on F', F'' and F''' on the real axis as above, it follows that $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau+1} \, |Re \; F'(\xi)| = O(\epsilon^2)$. Using Hilbert Transform property as in Appendix A (Lemma **A.1** for $k = \frac{1}{2}$), with $g(\xi) = Re \; F(\xi)$ and using $Im \; F(\xi) = \mathcal{H}(Re \; F)[\xi]$, it follows that $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^\tau \, |Im \; F(\xi)| = O(\epsilon^2)$. Therefore, $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^\tau \, |F(\xi)| = O(\epsilon^2)$. Hence $\delta = O(\epsilon^2)$. By taking upto third derivative of (1.1) and using *a priori* bounds on all derivatives of F for real ξ occurring on the right of (1.1), we get $O(\epsilon^2)$ upper bounds for $|\xi - 2i|^{\tau+1}|g'(\xi)|$, $|\xi - 2i|^{\tau+2}|g''(\xi)|$ and $|\xi - 2i|^{\tau+3}|g'''(\xi)|$ where $g(\xi) = Re \; F(\xi)$, as before. Using properties of Hilbert transform (Lemmas A.2 and A.3 in the appendix), it follows that $|\xi - 2i|^{\tau+1}|Im \; F'(\xi)|$, $|\xi - 2i|^{\tau+2}|Im \; F''(\xi)|$ also have $O(\epsilon^2)$ upper bounds. Hence $\delta_1, \delta_2 = O(\epsilon^2)$. Therefore, $\hat{\Delta} = O(\epsilon^2)$, where $\hat{\Delta}$ is as defined in definition 3.17. From previous theorem, in the domain $\mathcal{R}$, $\|F\|_0 = O(\epsilon^2)$ and $\|F'\|_1 = O(\epsilon)$. □

Proof of Theorem 1.3 follows from Theorem 3.18, after using the Lemma 3.19.

4. Nonexistence of solution for $\lambda < \frac{1}{2}$

Rewriting (2.11), we have

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i(F'(\xi)+H)^{1/2}(\bar{F}'(\xi)+\bar{H})^{1/2}} \left[\frac{F''(\xi)+H'}{F'(\xi)+H} - \frac{\bar{F}''(\xi)+\bar{H}'}{\bar{F}'(\xi)+\bar{H}} \right]; \tag{4.1}$$

On multiplying (4.1) by $(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2}$ and introducing change of variable:

$$\xi + i\gamma = i\tilde{k}_1 \epsilon^{4/7}\chi, \text{where } \ \tilde{k}_1 = (1-\gamma^2)^{3/7}[i\bar{F}'(-i\gamma) + i\bar{H}(-i\gamma)]^{-1/7} \tag{4.2}$$

$$F(\xi(\chi)) = \frac{\tilde{k}_1^2 \epsilon^{8/7} G(\chi)}{(1-\gamma^2)} \tag{4.3}$$

(4.1) becomes:

$$\begin{aligned} G'' - 1 - \chi^{3/2}(1-\frac{G'}{\chi})^{3/2} G &= \epsilon^{4/7}\chi\tilde{A}_1(\epsilon^{4/7}\chi)[1-\frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)] \\ &+ \left[\left(1-\frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right)^{3/2} - \left(1-\frac{G'}{\chi}\right)^{3/2} \right] G\chi^{3/2} \\ &+ \epsilon^{4/7}\chi\tilde{A}_3(\epsilon^{4/7}\chi)[1-\frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)]^{3/2}\left[G\chi^{3/2} + (\epsilon^{4/7}\chi)^{3/2}\tilde{A}_4(\epsilon^{4/7}\chi)\right] \\ &+ (\epsilon^{4/7}\chi)^{3/2}\tilde{A}_4(\epsilon^{4/7}\chi)\left[1-\frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right]^{3/2} + \epsilon^{4/7}\chi\tilde{A}_5(\epsilon^{4/7}\chi) \end{aligned} \tag{4.4}$$

where $\tilde{A}_j(\epsilon^{4/7}\chi)$ are analytic functions in $\epsilon^{4/7}\chi$.

A further change of variable

$$\chi = (\frac{7}{4}\eta)^{4/7}, \ \chi^{3/2}G(\chi) = -\eta\phi(\eta) \tag{4.5}$$

leads to

$$\begin{aligned} \mathcal{L}\phi \equiv \frac{d^2\phi}{d\eta^2} + \frac{5}{7\eta}\frac{d\phi}{d\eta} - (1+\frac{45}{196\eta^2})\phi &= -\frac{1}{\eta} - \frac{33}{196\eta^2}\phi \\ &+ \phi\{[1+\frac{4}{49\eta}\phi + \frac{4}{7}\phi']^{3/2} - 1\} + \frac{(\epsilon\eta)^{4/7}}{\eta} \ E((\epsilon\eta)^{2/7}, \phi, \phi', \eta^{-1}), \end{aligned} \tag{4.6}$$

Remark 4.1. It is to be noted that E has a convergent series in ϕ, ϕ':

$$E = \Sigma_{j_1,j_2\geq 0}^{\infty} E_{j_1,j_2}((\epsilon\eta)^{2/7}, \frac{1}{\eta})\phi^{j_1}(\phi')^{j_2} \tag{4.7}$$

where we can choose ρ, C independent of ϵ and η so that

$$|E_{j_1,j_2}| \ < \ C \ \rho^{-j_1-j_2}$$

in the domain $\frac{q_1}{\epsilon} \geq |\eta| \geq R$, for R sufficiently large and ϵ small for some q_1 independent of ϵ.

Theorem 4.2. *Let $F(\xi)$ be the solution to (4.1) as ensured by Theorem 1.3. After change of variable (4.2),(4.3) and (4.5), $\phi(\eta, \epsilon, a)$ satisfies (4.6) for $q_0\epsilon^{-1} \leq |\eta| \leq q_1\epsilon^{-1}$, at least for $0 \leq \arg\eta \leq \frac{5\pi}{8}$ (where q_0, $q_1 = O(\gamma-b)^{7/4}$ but independent of ϵ). In that domain, $\phi(\eta,\epsilon), \phi'(\eta,\epsilon) = O(\epsilon)$ as $\epsilon \to 0^+$. Further, in this domain $|\eta\phi| = O((\gamma-b)^{3/2}), |\eta\phi'| = O((\gamma-b)^{3/4}, \frac{\epsilon}{(\gamma-b)^{1/4}})$. Also, on the positive real η axis in the interval $q_0\epsilon^{-1} \leq \eta \leq q_1\epsilon^{-1}$, $Im\ \phi = 0$.*

Proof. Notice from transformation (4.2),(4.5), if $\eta = O(\epsilon^{-1}), \xi + i\gamma = O(1)$ and if $0 \leq \arg\eta \leq \frac{5\pi}{8}$, then $0 \leq \arg(\gamma + i\xi) \leq \frac{5\pi}{14}$ and for suitable $q_0, q_1 = O(\gamma-b)^{7/4}$, this corresponds to $\xi \in \mathcal{R}^-$ close to $\xi = -ib$, where F is known to satisfy (4.1) with $\|F\|_0 = O(\epsilon^2), \|F'\|_1 = O(\epsilon)$. Hence $\phi(\eta,\epsilon)$ must satisfy transformed equation (4.6).Also from (4.2),(4.3), (4.5), it is clear that as $\epsilon \to 0^+, \phi(\eta,\epsilon), \phi'(\eta,\epsilon) = O(\epsilon)$ and that $\eta\phi = O(\gamma-b)^{3/2}$ and $\eta\phi' = O((\gamma-b)^{3/4}, \epsilon(\gamma-b)^{-1/4})$. Since $F(\xi)$ is real at least on the imaginary ξ axis segment $[-ib, 0]$, it follows from (4.2) that for suitable q_0, q_1, Im $\phi = 0$ for η real and positive, at least when $\frac{q_0}{\epsilon} \leq \eta \leq \frac{q_1}{\epsilon}$. □

Definition 4.3. $\mathcal{R}_{2,R} = \{\eta : R < Im\ \eta + Re\ \eta < \tilde{k}_0\epsilon^{-1}, \arg\eta \in [0, \frac{5\pi}{8}); -Im\ \eta + R < Re\ \eta < Im\ \eta + \tilde{k}_0\epsilon^{-1}, \arg\eta \in (-\frac{\pi}{8}, 0]\}$, where $q_0 < \tilde{k}_0 < q_1$.

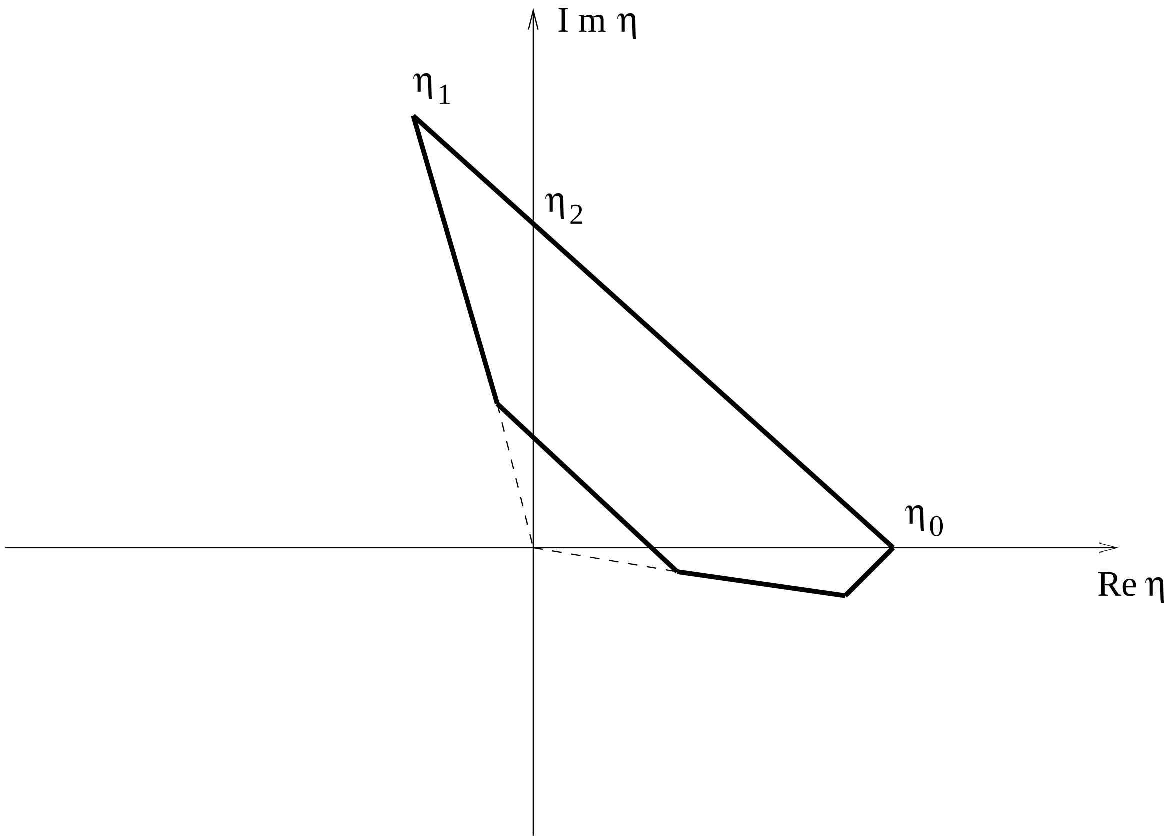

FIGURE 2. Region $\mathcal{R}_{2,R}$.

Definition 4.4. We define $\tilde{\phi}(\eta)$ (suppressing the ϵ dependence) as the solution $\phi(\eta, \epsilon)$ in Theorem 4.1

Definition 4.5.

$$\phi_1(\eta) = \eta^{-5/14} e^{-\eta}, \phi_2(\eta) = \eta^{-5/14} e^{\eta}; \tag{4.8}$$

$\phi_1(\eta), \phi_2(\eta)$ satisfy the following equation exactly:

$$\mathcal{L}\phi \equiv \frac{d^2\phi}{d\eta^2} + \frac{5}{7\eta}\frac{d\phi}{d\eta} - (1 + \frac{45}{196\eta^2})\phi = 0$$

The Wronskian of ϕ_1 and ϕ_2

$$\mathcal{W}(\phi_1, \phi_2)(\eta) = 2\eta^{-5/7} \tag{4.9}$$

Equation (4.6) can be rewritten as

$$\mathcal{L}\phi = \mathcal{N}_1(\phi, \phi', \epsilon) \tag{4.10}$$

where the operator $\mathcal{N}_1$ is defined by

(4.11)

$$\mathcal{N}_1(\phi, \phi', \epsilon)[\eta] = -\frac{1}{\eta} - \frac{33}{196\eta^2}\phi + \phi\{[1 + \frac{4}{49\eta}\phi + \frac{4}{7}\phi']^{3/2} - 1\} + \frac{(\epsilon\eta)^{4/7}}{\eta} E((\epsilon\eta)^{2/7}, \phi, \phi', \eta^{-1}),$$

Definition 4.6.

$$\eta_0 = \tilde{k}_0 \epsilon^{-1}, \ \eta_1 = \tilde{k}_0 \epsilon^{-1} \frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}} e^{\frac{5i\pi}{8}}, \ \eta_2 = i\tilde{k}_0\epsilon^{-1}; \tag{4.12}$$

Lemma 4.7. *The solution $\tilde{\phi}(\eta)$ as defined earlier satisfies the following integral equation:*

(4.13)

$$\tilde{\phi} = -\phi_1(\eta) \int_{\eta_1}^{\eta} \frac{\phi_2(t)}{2t^{-5/7}} \mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[t]dt + \phi_2(\eta) \int_{\eta_0}^{\eta} \frac{\phi_1(t)}{2t^{-5/7}} \mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[t]dt$$
$$- \phi_1(\eta) \frac{\Big(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1)\Big)}{2\eta_1^{-5/7}} + \phi_2(\eta) \frac{\Big(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0)\Big)}{2\eta_0^{-5/7}};$$

Proof. By using variation of parameters on (4.6)(with $\phi(\eta, \epsilon)$ replaced by $\tilde{\phi}(\eta)$, we get

(4.14)

$$\tilde{\phi}(t) = -\phi_1(t) \int_{\eta_2}^{t} \frac{\phi_2(s)}{2s^{-5/7}} \mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[s]ds + \phi_2(t) \int_{\eta_2}^{t} \frac{\phi_1(s)}{2s^{-5/7}} \mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[s]ds + A_1\phi_1(t) + A_2\phi_2(t);$$

Evaluating (4.14) and its derivative at $t=\eta_2$ and solve for A_1, A_2, we have

$$
\begin{aligned}
A_1 &= \frac{\tilde{\phi}(\eta_2)\phi_2'(\eta_2)-\tilde{\phi}'(\eta_2)\phi_2(\eta_2)}{2\eta_2^{-5/7}};\\
A_2 &= \frac{\tilde{\phi}(\eta_2)\phi_1'(\eta_2)-\tilde{\phi}'(\eta_2)\phi_1(\eta_2)}{2\eta_2^{-5/7}};
\end{aligned} \tag{4.15}
$$

However,on using integration by parts twice:

$$
\begin{aligned}
&-\phi_1(t)\int_{\eta_2}^{\eta_1}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{L}\tilde{\phi}(t)dt+\phi_2(t)\int_{\eta_2}^{\eta_0}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{L}\tilde{\phi}(t)dt\\
&=-\phi_1(t)\frac{\Big(\phi_2(\eta_1)\tilde{\phi}'(\eta_1)-\phi_2'(\eta_1)\tilde{\phi}(\eta_1)\Big)}{2\eta_1^{-5/7}}+\phi_2(t)\frac{\Big(\phi_1(\eta_0)\tilde{\phi}'(\eta_0)-\phi_1'(\eta_0)\tilde{\phi}(\eta_0)\Big)}{2\eta_0^{-5/7}}\\
&-A_1\phi_1(t)-A_2\phi_2(t);
\end{aligned} \tag{4.16}
$$

Using $\mathcal{L}\tilde{\phi}=\mathcal{N}_1(\tilde{\phi},\tilde{\phi}',\epsilon)$ in (4.16) and using this expression in (4.14), we get (4.13) and hence the lemma follows. □

Definition 4.8.

$$
\mathbf{W}=\{\phi:\phi(\eta)\text{ is analytic in }\mathcal{R}_{2,R}\text{ and continuous in its closure, with } \|\phi\|:=\sup_{\mathcal{R}_{2,R}}|\eta\phi(\eta)|<\infty;\} \tag{4.17}
$$

Lemma 4.9. *Let $N\in\mathbf{W}$,define*

$$\psi_1(\eta):=\phi_1(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}N(t)dt,\psi_2(\eta):=\phi_2(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}N(t)dt;$$

$$\psi_3(\eta):=\phi_1'(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}N(t)dt,\psi_4(\eta):=\phi_2'(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}N(t)dt;$$

Then $\|\psi_1\|\le K\|N(t)\|,\|\psi_2\|\le K\|N(t)\|;\|\psi_3\|\le K\|N(t)\|,\|\psi_4\|\le K\|N(t)\|$ where K is some constant independent of R,ϵ.

Proof. It is clear from the nature of the domain $\mathcal{R}_{2,R}$ that any point $\eta \in \mathcal{R}_{2,R}$ can be connected to η_0 by a straight line entirely within $\mathcal{R}_{2,R}$ so on line $t(s)$, parametrized by arclength s, Re $(t(s)-\eta)$ increases from η to η_0 so that on the segment $\frac{d}{ds} Re\ (t(s)-\eta) > C_1 > 0$, where C_1 is a constant independent of ϵ. Further, on this straight line $0 < C_2 < |t/\eta|$ where C_2 is independent of ϵ. Then

$$\begin{aligned} |\psi_2(\eta)| &= \left| \int_{\eta_0}^{\eta} t^{-9/14} \eta^{-5/14} e^{-(t-\eta)} (tN(t)) dt \right| \\ &\leq C_2^{-9/14} |\eta|^{-1} \|N\| \int_0^1 \frac{d\{e^{-Re\ (t(s)-\eta)}\}}{\frac{d}{ds} Re\ t(s)} \\ &\leq C_1^{-1} C_2^{-9/14} |\eta|^{-1} \|N\| \end{aligned}$$

So $\|\psi_2\| \leq K \|N\|$. Similarly for ψ_4 since $\frac{|\phi_2'|}{|\phi_2|} \leq C$.

Also, it is clear that any point η can be connected to η_1 by a straight line entirely within $\mathcal{R}_{2,R}$ and on such a particular path $t(s)$, $\frac{d}{ds} Re(\eta - t(s)) > C_1 > 0$,where C_1 is independent of ϵ. Further on this straight line $|t/\eta| > C_2 > 0$, C_2 independent of ϵ. So

$$\begin{aligned} |\psi_1(\eta)| &= \left| \int_{\eta_1}^{\eta} t^{-9/14} \eta^{-5/14} e^{-(\eta-t)} tN(t) dt \right| \\ &\leq C_2^{-9/14} C_1^{-1} |\eta|^{-1} \|N\| \leq K |\eta|^{-1} \|N\| \end{aligned}$$

Similarly for $\|\psi_3\|$ since $|\frac{\phi_1'}{\phi_1}| \leq C$. □

Definition 4.10. We define ϕ_3 and ϕ_4 so that

$$\phi_3(\eta) = -\phi_1(\eta) \frac{\Big(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1)\Big)}{2\eta_1^{-5/7}} \tag{4.18}$$

$$\phi_4(\eta) = \phi_2(\eta) \frac{\Big(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0)\Big)}{2\eta_0^{-5/7}}; \tag{4.19}$$

Lemma 4.11.

$$\|\phi_3'\|, \|\phi_3\| \le C_1 \left[|\eta_1 \tilde{\phi}(\eta_1)| + |\tilde{\phi}'(\eta_1)\eta_1| \right] ;$$

$$\|\phi_4'\|, \|\phi_4\| \le C_1 \ \left[|\eta_0 \tilde{\phi}(\eta_0)| + |\tilde{\phi}'(\eta_0)\eta_0| \right] ;$$

where C_1 is independent of ϵ.

Proof. Since $|\eta_1| \ > \ |\eta|$, it follows that

$$\left| \frac{\phi_1(\eta)\phi_2(\eta_1)}{\eta_1(\eta_1^{-5/7})} \right| = \exp[Re\ \eta_1 - Re\ \eta] |\eta_1|^{-9/14} |\eta|^{-5/14} \le C_1 |\eta|^{-1}$$

Also, since $|\eta| \ < \ C\ |\eta_0|$, for constant C independent of ϵ,

$$\left| \frac{\phi_2(\eta)\phi_1(\eta_0)}{\eta_0(\eta_0^{-5/7})} \right| = \exp[Re\ \eta - Re\ \eta_0] |\eta_0|^{-9/14} |\eta|^{-5/14} \le C_2 |\eta|^{-1}$$

Since $\phi_2'(\eta) = \phi_2(\eta)[1 - \frac{5}{14\eta}]$ and $\phi_1'(\eta) = \phi_1(\eta)[-1 - \frac{5}{14\eta}]$, the same arguments as above show that $\left| \frac{\phi_1(\eta)\phi_2'(\eta_1)}{\eta_1(\eta_1^{-5/7})} \right| \le C_2 |\eta|^{-1}$ and $\left| \frac{\phi_2(\eta)\phi_1'(\eta_0)}{\eta_0(\eta_0^{-5/7})} \right| \le C_1 |\eta|^{-1}$. Hence, the lemma follows from definition of ϕ_3 and ϕ_4 in definition 4.10. □

Definition 4.12.

$$\mathbf{W}_\sigma \equiv \{ \phi \in \mathbf{W}, \|\phi\| < \sigma \}$$

Definition 4.13.

$$D(\phi, \phi', \frac{1}{\eta}) \equiv \phi \{ (1 + \frac{4}{49\eta} \frac{\phi}{\eta} + \frac{4}{7} \phi')^{3/2} - 1 \} \ = \ \sum_{j_1 + j_2 \ \ge 2} \tilde{A}_{j_1, j_2}(\eta) \ \phi^{j_1} (\phi')^{j_2},$$

We define constant $\tilde{\rho}$, independent of η and ϵ in the domain $\mathcal{R}_{2,R}$ so that

$$|\tilde{A}_{j_1, j_2}(\eta)| \ < \ C\ \tilde{\rho}^{-j_1 - j_2}$$

Lemma 4.14. *If $\phi, \phi' \in \mathbf{W}_\sigma$, then $\mathcal{N}_1(\phi, \phi', \epsilon) \in \mathbf{W}_\sigma$ and*

$$\|\mathcal{N}_1(\phi, \phi', \epsilon)\| < 1 + C(\frac{1}{R^3}\sigma + \tilde{k}_0^{4/7} + \frac{\sigma^2}{R\rho^2}),$$

for R large enough so that $\frac{\sigma}{\tilde{\rho}R}, \frac{\sigma}{\rho R} \leq \frac{1}{2}$, where ρ and $\tilde{\rho}$ are as in remark 4.1 and definition 4.13.

Proof. In (4.11),$\|\frac{-33}{196\eta^2}\phi\| < \frac{c}{R^3}\|\phi\|$, $\|1/\eta\| \leq 1$.

The norm of the first nonlinear term in (4.11) can be estimated by noting

$$\begin{aligned}|\eta D(\phi, \phi', \frac{1}{\eta})| &= \left|\sum_{j_1+j_2\geq 2} \tilde{A}_{j_1,j_2}(\eta)\ \phi^{j_1}\phi'^{j_2}\eta\right| \\ &\leq C \sum_{j_1+j_2\geq 2} \frac{1}{|\eta|^{j_1+j_2-1}} \frac{1}{\tilde{\rho}^{j_1+j_2}} |\eta\phi|^{j_1}|\eta\phi'|^{j_2} \\ &\leq C\frac{\sigma^2}{R\tilde{\rho}^2}\end{aligned}$$

The norm of the second nonlinear term in (4.11) can be estimated by using (4.7) and noting

$$\begin{aligned}|(\epsilon\eta)^{4/7}E| &\leq (\epsilon\eta_0)^{4/7}\left|\Sigma_{j_1,j_2\geq 0}^{\infty} E_{j_1,j_2}((\epsilon\eta)^{2/7}, \frac{1}{\eta})\phi^{j_1}(\phi')^{j_2}\right| \\ &\leq C(\epsilon\eta_0)^{4/7}\left(\Sigma_{j_1,j_2\geq 0}^{\infty}\frac{1}{R^{j_1+j_2}}\frac{1}{\rho^{j_1+j_2}}\sigma^{j_1+j_2}\right) \leq C\tilde{k}_0^{4/7} \ = \ O(\gamma - b)\end{aligned}$$

where ρ is as defined in remark 4.1. The lemma follows on combining above results.

□

Lemma 4.15. *If $\phi \in \mathbf{W}_\sigma, \psi \in \mathbf{W}_\sigma, \phi' \in \mathbf{W}_\sigma, \psi' \in \mathbf{W}_\sigma$, then for $R > \{\frac{2\sigma}{\tilde{\rho}}, \frac{2\sigma}{\rho}\}$*

$$\|\mathcal{N}_1(\phi, \phi', \epsilon) - \mathcal{N}_1(\psi, \psi', \epsilon)\| \leq C[\frac{1}{R^3} + \frac{\sigma}{\tilde{\rho}^2 R} + \tilde{k}_0^{4/7}]\,(\|\phi - \psi\| + \|\phi' - \psi'\|)$$

where $\rho, \tilde{\rho}$ are as in remark 4.1 and definition 4.13 and C is independent of ϕ, ψ and ϵ.

Proof. $\|\frac{-33}{196\eta^2}(\phi-\psi)\| \leq \frac{C}{R^3}\|(\phi-\psi)\|$, Note that

(4.20)

$$\begin{aligned}
&|\eta(\phi^{j_1}\psi^{j_2}-\psi^{j_1}\phi^{j_2})| = |\eta\phi^{j_1}(\phi'^{j_2}-\psi'^{j_2})+\eta\psi'^{j_2}(\phi^{j_1}-\psi^{j_1})| \\
&\leq \|\phi\|^{j_1}\frac{1}{R^{j_1+j_2-1}}j_2(\|\phi'\|+\|\psi'\|)^{j_2-1}\|\psi'-\phi'\|+\frac{\|\psi\|^{j_2}}{R^{j_1+j_2-1}}j_1(\|\psi\|+\|\phi\|)^{j_1-1}\|\psi-\phi\| \\
&\leq (\frac{\sigma}{R})^{j_1+j_2-1}[j_2\|\psi'-\phi'\|+j_1\|\phi-\psi\|]
\end{aligned}$$

So in (4.11),

$$\begin{aligned}
&|\eta D(\phi,\phi',\frac{1}{\eta})-\eta D(\psi,\psi',\frac{1}{\eta})| \leq \sum_{j_1+j_2\geq 2}|\tilde{A}_{j_1,j_2}(\eta)||\eta(\phi^{j_1}\phi'^{j_2}-\psi^{j_1}\psi'^{j_2})| \\
&\leq C\sum_{j_1+j_2\geq 2}\frac{1}{\tilde{\rho}^{j_1+j_2}}(\frac{\sigma}{R})^{j_1+j_2-1}\{j_2\|\psi'-\phi'\|+j_1\|\psi-\phi\|\} \\
&\leq \frac{C}{\tilde{\rho}^2}\frac{\sigma}{R}[\|\psi'-\phi'\|+\|\psi-\phi\|], \text{ for } R>\frac{2\sigma}{\tilde{\rho}}
\end{aligned}$$

From (4.7) and (4.20)

$$\begin{aligned}
&\left|(\epsilon\eta)^{4/7}\left[E(\epsilon\eta)^{4/7},\frac{1}{\eta},\phi,\phi')-E((\epsilon\eta)^{2/7},\frac{1}{\eta},\psi,\psi')\right]\right| \\
&\leq C|\epsilon\eta_0|^{4/7}\Sigma_{j_1,j_2\geq 0}\frac{1}{\rho^{j_1+j_2}}(\frac{\sigma}{R})^{j_1+j_2-1}\{j_2\|\psi'-\phi'\|+j_1\|\psi-\phi\|\} \\
&\leq C\tilde{k}_0^{4/7}[\|\psi'-\phi'\|+\|\psi-\phi\|] \text{ for } R>\frac{2\sigma}{\tilde{\rho}}
\end{aligned}$$

□

Consider the integral equation in the domain $\mathcal{R}_{2,R}$:

(4.21)
$$\begin{aligned}
\phi(\eta) = \mathcal{L}_1\phi(\eta)-\phi_1(\eta)\frac{\Big(\phi_2(\eta_1)\tilde{\phi}'(\eta_1)-\phi_2'(\eta_1)\tilde{\phi}(\eta_1)\Big)}{2\eta_1^{-5/7}} \\
+\phi_2(\eta)\frac{\Big(\phi_1(\eta_0)\tilde{\phi}'(\eta_0)-\phi_1'(\eta_0)\tilde{\phi}(\eta_0)\Big)}{2\eta_0^{-5/7}};
\end{aligned}$$

Where

$$\mathcal{L}_1\phi \equiv -\phi_1(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)[t]dt + \phi_2(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)[t]dt \tag{4.22}$$

Definition 4.16.

$$\mathbf{E}^1 := \mathbf{W}\oplus\mathbf{W}, \|(\phi,\phi')\|_{\mathbf{E}^1} = \|\phi\| + \|\phi'\|; \tag{4.23}$$

This is clearly a Banach space. Similarly, $\mathbf{E}^1_\sigma = \{(\phi,\phi') \in \mathbf{E}^1 \text{ with } \|(\phi,\phi')\| \le \sigma\}$.

Define

$$\mathcal{M} : \mathbf{E}^1 \to \mathbf{E}^1, \mathcal{M}(\phi,\phi') = (\mathcal{M}_1(\phi,\phi'), \mathcal{M}_2(\phi,\phi'))$$

where

$$\mathcal{M}_1(\phi,\phi') = \mathcal{L}_1\phi(\eta) + \phi_3(\eta) + \phi_4(\eta) \tag{4.24}$$

$$\mathcal{M}_2(\phi,\phi') = \mathcal{L}_2\phi(\eta) + \phi_3'(\eta) + \phi_4'(\eta) \tag{4.25}$$

where

$$\mathcal{L}_2\phi \equiv -\phi_1'(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)dt + \phi_2'(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)(t)dt \tag{4.26}$$

Theorem 4.17. *For fixed $\sigma \ge 4K$, where K is as defined in Lemma 4.9, there exists $\tilde{k}_0$ small enough but independent of ϵ (i.e. b chosen so that $\gamma - b$ is small but independent of ϵ), and R large enough so that for any ϵ small enough, $\mathcal{M}$ is a contraction mapping from $\mathbf{E}^1_\sigma$ to $\mathbf{E}^1_\sigma$.*

Proof. Using Lemmas 4.9,4.11,4.14 in (4.24) and (4.25), it follows that

$$\begin{aligned}
&\|\mathcal{M}(\phi,\phi')\| = \|\mathcal{M}_1(\phi,\phi')\| + \|\mathcal{M}_2(\phi,\phi')\| \\
&\le 2K\left[1 + C\left(\frac{\sigma}{R^3} + \tilde{k}_0^{4/7} + \frac{\sigma^2}{R\tilde{\rho}^2}\right)\right] \\
&+ C_1[|\eta_0\tilde{\phi}(\eta_0)| + |\tilde{\phi}'(\eta_0)\eta_0| + |\eta_1\tilde{\phi}(\eta_1)| + |\eta_1\tilde{\phi}'(\eta_1)|]
\end{aligned} \tag{4.27}$$

From Theorem 4.2, $\eta\tilde{\phi}(\eta), \eta\tilde{\phi}'(\eta) = O((\gamma - b)^{3/4}, \frac{\epsilon}{(\gamma-b)^{1/4}})$ for $\eta = \eta_0$ or η_1. But since $(\gamma - b)^{7/4} = O(\tilde{k}_0)$ it follows that $\tilde{k}_0$ can be chosen small enough (but independent of ϵ) and R can be chosen large enough so that the the right hand side of (4.27) is less than $4K$ for small enough ϵ.

Further from Lemma 4.9 and Lemma 4.15

$$\|\mathcal{M}_{1,2}(\phi_1, \phi_2) - \mathcal{M}_{1,2}(\phi_2, \phi_2')\| \leq K\|\mathcal{N}(\phi_1, \phi_1', \epsilon) - \mathcal{N}(\phi_2, \phi_2', \epsilon)\|$$
$$\leq KC[\|\phi_1 - \phi_2\| + \|\phi_1' - \phi_1'\|][\frac{1}{R^3} + \frac{\sigma}{\tilde{\rho}^2 R} + \tilde{k}_0^{4/7}]$$

So

$$\|\mathcal{M}(\phi_1, \phi_1') - \mathcal{M}(\phi_2, \phi_2')\| \leq 2KC[\frac{1}{R^3} + \frac{\sigma}{\tilde{\rho}^2 R} + \tilde{k}_0^{4/7}]\|(\phi_1 - \phi_2, \phi_1' - \phi_2')\|$$

which is a contraction for $\tilde{k}_0$ small and R large. □

Remark 4.18. Note that R can be chosen large enough and $\tilde{k}_0$ small enough once for all and Theorem holds for all small ϵ. In otherwords, choice of R, $\tilde{k}_0$ can be made independent of ϵ, though the Theorem also holds if $R = O(\frac{1}{\epsilon})$ for sufficiently small ϵ and $\tilde{k}_0$.

Corollary 4.19. The integral equation(4.21) has the unique analytic solution $\phi(\eta)$ and $\phi(\eta) = \tilde{\phi}(\eta)$ in the domain $\mathcal{R}_{2,R}$.

Proof. Unique solution ϕ follows from Thm 4.17 using contraction mapping theorem. If we choose $R = O(\frac{1}{\epsilon})$ suitably, then Theorem 4.2 applies to domain $\mathcal{R}_{2,R}$ and from Lemma 4.7, $\phi = \tilde{\phi}$. From analytic continuation, $\phi - \tilde{\phi} = 0$ everywhere on $\mathcal{R}_{2,R}$ even when R is independent of ϵ but large. □

Lemma 4.20. *The solution $\tilde{\phi}(\eta)$ satisfies $Im\ \tilde{\phi}(\eta) = 0$ for $R < \eta < \frac{q_1}{\epsilon}$ for sufficiently large R and small enough ϵ, for R independent of ϵ.*

Proof. From Corollary 4.19, it follows that $\tilde{\phi}(\eta)$ is analytic in particular on the real axis for $R < \eta < \frac{\tilde{k}_0}{\epsilon}$. However, from Theorem 4.2, $Im\ \tilde{\phi} = 0$ for $\frac{q_0}{\epsilon} \leq \eta \leq \frac{q_1}{\epsilon}$. Since $\tilde{k}_0 > q_0$, the lemma follows. □

Lemma 4.21. *For any fixed* η *in the domain* $\{\eta : Re\ \eta + Im\ \eta > R, \frac{5\pi}{8} > \arg\eta > -\frac{\pi}{8}\}$, $\lim_{\epsilon\to 0}\tilde{\phi}(\eta,\epsilon) = \phi_0(\eta)$, *where* $\phi_0(\eta)$ *satisfies*

$$\phi_0(\eta) = -\phi_2(\eta)\int_{\infty e^{5i\pi/8}}^{\eta}\frac{\mathcal{N}_1(\phi_0,\phi_0',0)[t]}{2t^{-5/7}}dt + \phi_1(\eta)\int_{\infty}^{\eta}\frac{\mathcal{N}_1(\phi_0,\phi_0',0)[t]}{2t^{-5/7}}dt \tag{4.28}$$

Proof. Follows from (4.13) by taking limit $\epsilon \to 0$ and using Theorem 4.2, $\tilde{\phi}(\eta_1), \tilde{\phi}'(\eta_1), \tilde{\phi}(\eta_0)$ and $\phi'(\eta_0)$ all tend to 0 while $\frac{\phi_1(\eta)\phi_2(\eta_1)}{2\eta_1^{-5/7}} \to 0$ and $\frac{\phi_2(\eta)\phi_1(\eta_0)}{2\eta_0^{-5/7}} \to 0$ as $\epsilon \to 0$ since $\eta_1, \eta_0 \to \infty$. □

Corollary 4.22. $\phi_0(\eta)$ satisfies differential equation

$$\mathcal{L}_1\phi_0 = \mathcal{N}_1(\phi_0,\phi_0',0) = \frac{-33}{196\eta^2} - \frac{1}{\eta} + \phi_0\left\{(1 + \frac{4}{49}\frac{\phi_0}{\eta} + \frac{4}{7}\phi_0')^{3/2} - 1\right\} \tag{4.29}$$

with $\eta\phi_0(\eta)$ finite as $\eta \to \infty$, at least for $\arg\eta \in (-\frac{\pi}{8}, \frac{5\pi}{8})$.

Proof. $\mathcal{L}_1\phi_0 = \mathcal{N}_1(\phi_0,\phi_0',0)$ follows simply from applying $\mathcal{L}_1$ to (4.28). Since $|\eta\phi(\eta)|$ was bounded independent of ϵ in the domain $\mathcal{R}_{2,R}$, it follows that as $\epsilon \to 0$, $|\eta\phi_0|$ is also bounded at least for $\arg\eta \in (-\frac{\pi}{8}, \frac{5\pi}{8})$. □

Remark 4.23. It is known from general theory worked out by Costin [30] that (4.29) has unique solution with asymptotic expansion

$$\phi_0 \sim \Sigma_{j=1}^{\infty}\frac{a_j}{\eta^j},\ \text{valid } for -\frac{\pi}{2} < \arg\eta < \pi$$

and that on the positive real axis

$$Im\ \phi_0 \sim S\eta^{-5/14}e^{-\eta} \tag{4.30}$$

for some Stokes constant S (which is a pure number) that can be computed.

However, applying transformation (4.2),(4.3),(4.5) and going back to variable χ and G it is clear that $\lim_{\epsilon \to 0} G(\chi(\eta)) = G_0(\chi(\eta))$ and that $G_0(\chi)$ satisfies

$$G_0'' = 1 + (\chi - G_0')^{3/2} G_0 \tag{4.31}$$

If we use transformation

$$V_0(\chi) = (\chi - G_0')^{-1/2} \tag{4.32}$$

then it follows from (4.31) that $V_0(\chi)$ satisfies

$$2V_0''(\chi) = \chi - V_o^{-2} \tag{4.33}$$

with $V_0(\chi) \to \chi^{-1/2}$ as $\chi \to \infty$, at least for $\frac{5\pi}{14} \geq \arg \chi \geq 0$.

Combescot *et al* [18] considered (4.33) and by computing many terms in the asymptotic expansion for large χ, was able to use a Borel summation procedure to compute the constant $\tilde{S}$ in the asymptotic expression

$$Im\ V_0(\chi) \sim \tilde{S} \chi^{-3/8} e^{-\frac{4}{7}\chi^{7/4}} \tag{4.34}$$

for large positive χ. The number $\tilde{S}$ was found to be nonzero. Using the transformation from χ to η, it follows that S in (4.31) must also be nonzero.

Lemma 4.24. *For all sufficiently small* ϵ

$$Im\ \tilde{\phi}(\eta, \epsilon) \neq 0 \ \textit{for any}\ \eta \in (R, \frac{q_1}{\epsilon}) \tag{4.35}$$

Proof. Since $\lim_{\epsilon \to 0} \tilde{\phi}(\eta) = \phi_0(\eta)$, $\lim_{\epsilon \to 0} Im\ \tilde{\phi}(\eta, \epsilon) = Im\ \phi_0(\eta) \neq 0$ from (4.30), since S is nonzero. □

Corollary 4.25. *$Im\ F \neq 0$ on some imaginary ξ axis segment $[-ib, -ib']$ for some $b' < b$.*

Proof. On using transformation (4.2), (4.3) and (4.5), the interval $(R, \frac{q_1}{\epsilon})$ in η corresponds to an $Im\ \xi$ axis interval that includes $[-ib, -ib']$ for some suitably chosen $b' < b$. So, at least on this segment, $Im\ F(\xi) = Im\ \phi(\eta(\xi)) \neq 0$. □

Proof of Theorem 1.8 : We have shown any classical solution $F(\xi)$, if it exists, is analytic in $\mathcal{R} \cup \bar{\mathbf{Z}}^+$ and belongs to $\mathbf{A}_0$. It is also analytic in the Im ξ axis segment $[-ib, i\infty)$. From successive Taylor expansions on the imaginary ξ axis, starting at $\xi = 0$, if follows that the symmetry condition (iii) implies $Im\ F = 0$ for $\xi \in [-ib, i\infty)$. But this contradicts the previous corollary for all sufficiently small ϵ. Hence proof of Theorem 1.8 follows.

Acknowledgements: The authors acknowlege very helpful discussions with Ovidiu Costin of Rutgers University. The research was supported in part by National Science Foundation and NASA. XX also acknowleges support from the Ohio State University Math Research Institute.

Appendix A. Proof of some lemmas

Lemma A.1. *Let $g \in \mathcal{C}^1(-\infty, \infty)$ such that $\|(\xi - 2i)^\tau g\|_\infty < \infty$ for some $0 < \tau < 1$ and let $\||\xi - 2i|^{\tau+1} g'\|_\infty < \infty$ as well. Then, for any $k \in (0, \frac{1}{2}]$,*

$$\|(\xi - 2i)^\tau \mathcal{H}(g)\|_\infty \leq C_1 \ \ln \frac{1}{k} \ \|(\xi - 2i)^\tau g\|_\infty + C_2 k \|(\xi - 2i)^{\tau+1} g'\|_\infty \tag{A.1}$$

where C_1 and C_2 are independent of k and $\mathcal{H}$ is the Hilbert transform operater defined as

$$\mathcal{H}(g)[\xi] \equiv \frac{1}{\pi}(P) \int_{-\infty}^{\infty} \frac{g(\xi + \xi')}{\xi'} d\xi' \tag{A.2}$$

Proof. We first take $\xi \geq 1$. Denote $k' = 2 - k$, clearly $\frac{3}{2} \leq k' < 2$. We break up the integral in (A.2) into four parts,

$$\int_{-\infty}^{\infty} = \int_{-k\xi}^{k\xi} + \int_{-k'\xi}^{-k\xi} + \int_{-\infty}^{-k'\xi} + \int_{k\xi}^{\infty} \frac{1}{\pi} \frac{g(\xi + \xi')}{\xi'} d\xi', \tag{A.3}$$

Consider the first term:

$$\left| \frac{1}{\pi}(P) \int_{-k\xi}^{k\xi} \frac{g(\xi + \xi') - g(\xi)}{\xi'} d\xi' \right| \leq \left| \frac{1}{\pi} \int_{-k\xi}^{k\xi} g'(\bar{\xi} + \xi) d\xi' \right|$$
$$\leq \frac{1}{\pi} \|(\xi - 2i)^{1+\tau} g'\|_\infty \int_{-k\xi}^{k\xi} |\bar{\xi} + \xi - 2i|^{-\tau-1} d\xi'$$

where $\bar{\xi} \in (-k\xi, k\xi)$.

But

$$\int_{-k\xi}^{k\xi} |\bar{\xi} + \xi - 2i|^{-\tau-1} d\xi' \leq |\xi(1-k) - 2i|^{-1-\tau} 2k\xi \leq C_2 k |\xi - 2i|^{-\tau}$$

where C_2 can be made independent of $k \in (0, \frac{1}{2}]$. Hence

$$\left| \frac{1}{\pi} \int_{-k\xi}^{k\xi} \frac{g(\xi + \xi') - g(\xi)}{\xi'} d\xi' \right| \leq k C_2 |\xi - 2i|^{-\tau} \|\xi - 2i)^{1+\tau} g'\|_\infty$$

Consider the second term: on change of variable $\xi' + \xi = \xi''$ and let $L = (1-k)\xi$, we get

$$\frac{1}{\pi} \int_{-L}^{L} \frac{g(\xi'')}{(\xi'' - \xi)} d\xi''$$

We write this integral as

$$\frac{1}{\pi} \int_{-L}^{L} \frac{g(\xi'')}{(\xi'' - \xi)} d\xi'' = \frac{1}{\pi} \int_{-L}^{L} \left[\frac{g(\xi'')}{(\xi'' - \xi)} + \frac{g(\xi'')}{\xi} \right] d\xi'' - \frac{1}{\pi\xi} \int_{-L}^{L} g(\xi'') d\xi'' \tag{A.4}$$

and estimate each term separately. The second term on the right hand side above can be estimated as

(A.5)
$$\left| \frac{1}{\pi\xi} \int_{L}^{L} |g(\xi')(\xi' - 2i)^\tau| |\xi' - 2i|^{-\tau} d\xi' \right| \leq C \frac{L^{1-\tau}}{\xi} \|(\xi - 2i)^\tau g\|_\infty \leq C \xi^{-\tau} \ \|(\xi - 2i) g\|_\infty$$

where C can be made independent of k. Now consider the first term in (A.4):

$$\begin{aligned}\left|\frac{1}{\pi}\int_{-L}^{L}\left[\frac{g(\xi'')}{\xi''-\xi}+\frac{g(\xi'')}{\xi}\right]d\xi''\right| &= \left|\frac{1}{\pi}\int_{-L}^{L}\frac{g(\xi'')\xi''}{(\xi-\xi'')\xi}d\xi''\right| \\ &\le C\|(\xi-2i)^\tau g\|_\infty \int_{-L}^{L}\frac{|\xi''|^{1-\tau}}{(\xi-\xi'')\xi}d\xi'' \\ &\le C\|(\xi-2i)^\tau g\|_\infty \xi^{-\tau}\left[\int_{(1-k)}^{1-k}\frac{|\hat{\xi}|^{1-\tau}}{(1-\hat{\xi})}d\hat{\xi}\right] \\ &\le C_1 \ln\frac{1}{k}\; |\xi-2i|^{-\tau}\; \|(\xi-2i)^\tau g\|_\infty\end{aligned} \tag{A.6}$$

where C_1 is independent of k, and $\ln\frac{1}{k}$ term accounts for the behavior of the estimate on the right hand side as $k \to 0^+$. We now estimate the third term in (A.3):

$$\left|\frac{1}{\pi}\int_{-\infty}^{-k'\xi}\frac{g(\xi'+\xi)}{\xi'}d\xi'\right| = \left|\frac{1}{\pi}\int_{k'\xi}^{\infty}\frac{g(-\xi'+\xi)}{\xi'}d\xi'\right|$$
$$\le \|(\xi-2i)^\tau g\|_\infty \frac{1}{\pi}\int_{k'\xi}^{\infty}\frac{(\xi'-\xi)^{-\tau}}{\xi'}d\xi' \le C\|(\xi-2i)^\tau g\|_\infty \xi^{-\tau}$$

where C above can be chosen independent of k. Consider the 4th term in (A.3):

$$\left|\frac{1}{\pi}\int_{k\xi}^{\infty}\frac{g(\xi'+\xi)}{\xi'}d\xi'\right| \le \|(\xi-2i)^\tau g\|_\infty \frac{1}{\pi}\int_{k\xi}^{\infty}\frac{(\xi'+\xi)^{-\tau}}{\xi'}d\xi'$$
$$\le \|(\xi-2i)^\tau g\|_\infty \xi^{-\tau}\int_{k}^{\infty}\frac{(1+\hat{\xi})^{-\tau}}{\hat{\xi}}d\hat{\xi} \le C_1\;\ln\frac{1}{k}\;\|\xi^\tau g\|_\infty \xi^{-\tau}$$

where C_1 is chosen independent of k and $\ln\frac{1}{k}$ accounts for the asymptotic behavior of the integral on the right hand side, as $k \to 0^+$. Combinining all the terms above, we obtain the proof of the Lemma for $\xi > 1$. Now for $0 \le \xi \le 1$, we split the integral in (A.2) into :

$$\frac{1}{\pi}\int_{-k}^{k}\frac{g(\xi'+\xi)-g(\xi)}{\xi'}d\xi' + \frac{1}{\pi}\int_{k}^{\infty}\frac{g(\xi'+\xi)}{\xi'}d\xi' + \frac{1}{\pi}\int_{-\infty}^{-k}\frac{g(\xi'+\xi)}{\xi'}d\xi'$$

First term:

$$\left|\frac{1}{\pi}\int_{-k}^{k}\frac{g(\xi'+\xi)-g(\xi)}{\xi'}d\xi'\right| \le \frac{1}{\pi}\int_{-k}^{k}|g'(\hat{\xi})d\xi'| \le C_2 k\|(\xi-2i)^{1+\tau}g'\|_\infty$$

where C_2 is independent of ϵ. Second term:

$$\frac{1}{\pi}\int_k^\infty \frac{g(\xi'+\xi)}{\xi'}d\xi' \le C\|(\xi-2i)^\tau g\|_\infty \int_k^\infty \frac{|\xi'+\xi-2i|^{-\tau}}{\xi'}d\xi'$$
$$\le C\|(\xi-2i)^\tau g\|_\infty \int_k^\infty \frac{(\xi'^2+4)^{-\tau/2}}{\xi'}\,d\xi' \;\le\; C_1 \;\ln\frac{1}{k}\;\|(\xi-2i)^\tau g\|_\infty$$

where C_1 is independent of k and $\ln\frac{1}{k}$ accounts for the asymptotic behavior of the divergence on the righthand side estimate as $k \to 0^+$. Consider the third term:

$$\left|\frac{1}{\pi}\int_{-\infty}^{-k} \frac{g(\xi'+\xi)}{\xi'}d\xi'\right| \le \left|\frac{1}{\pi}\int_k^\infty \frac{g(-\xi'+\xi)}{\xi'}d\xi'\right| \le \|(\xi-2i)^\tau g\|_\infty \frac{1}{\pi}\int_k^\infty \frac{|\xi'-\xi-2i|^{-\tau}}{\xi'}d\xi'$$
$$\le C_1 \;\ln\frac{1}{k}\;\|(\xi-2i)^\tau g\|_\infty$$

where C_1 is made independent of k by accounting for the asymptotic behavior of the divergence of the right hand side estimate, as $k \to 0^+$.

For $\xi < 0$, we note that

$$\mathcal{H}(g)[\xi] = \frac{1}{\pi}\int_{-\infty}^\infty g(\xi')\frac{d\xi'}{\xi'-\xi} \;=\; -\frac{1}{\pi}\int_{-\infty}^\infty \frac{g(-\xi')d\xi'}{\xi'-(-\xi)}$$

which is negative of the Hilbert transform of the function $g(-\xi)$, evaluated at the point $-\xi > 0$. Since $g(-\xi)$ satisfies the same conditions as those given for $g(\xi)$ in this Lemma, it follows all bounds also hold for $\xi < 0$. $\square$

Lemma A.2. *Let $g \in \mathcal{C}^2(-\infty,\infty)$ such that $\|(\xi-2i)^\tau g\|_\infty$, and $\|(\xi-2i)^{\tau+2}g''\|_\infty$ are each bounded for some $\tau \in (0,1)$. Then,*

$$\|(\xi-2i)^{\tau+1}\mathcal{H}(g')\|_\infty \;\le\; C_2\|(\xi-2i)^{\tau+2}g''\|_\infty \;+\; C_0\;\|(\xi-2i)^\tau g\|_\infty \tag{A.7}$$

Proof. First, we consider the case $\xi > 1$. Then, we decompose

$$\begin{aligned}\mathcal{H}(g')[\xi] = &\frac{1}{\pi}\int_{-\xi/2}^{\xi/2}\left[\frac{g'(\xi+\xi')-g'(\xi)}{\xi'}\right]d\xi' - \frac{2}{\pi\xi}\left(g(\frac{3}{2}\xi)+g(\frac{\xi}{2})\right)\\ &+\frac{1}{\pi}\left(\int_{\xi/2}^\infty + \int_{-\infty}^{-\frac{3}{2}\xi}\right)\frac{g(\xi+\xi')}{\xi'^2}d\xi' + \frac{1}{\pi}\int_{-\frac{3}{2}\xi}^{-\frac{\xi}{2}}\frac{g(\xi+\xi')}{\xi'^2}d\xi'\end{aligned} \tag{A.8}$$

Using arguments similar to Lemma A.1 for $k = \frac{1}{2}$, it is clear that the first term on the right of (A.8) is bounded by

$$|\frac{1}{\pi} \int_{-\xi/2}^{\xi/2} \left[\frac{g'(\xi + \xi') - g'(\xi)}{\xi'} \right] d\xi' | \ \le \ C_1 \ |\xi - 2i|^{-\tau - 1} \ \|(\xi - 2i)^{\tau+2} g''\|_\infty$$

The second term of (A.8) is easily seen to be bounded by

$$|\frac{2}{\pi \xi} \left(g(\frac{3}{2}\xi) + g(\frac{\xi}{2}) \right)| \ \le \ C \ |\xi - 2i|^{-\tau-1} \ \|(\xi - 2i)^\tau g\|_\infty$$

Using arguments similar to Lemma A.1, with $k = \frac{1}{2}$, the third term in (A.8) is also bounded:

$$|\frac{1}{\pi} \left(\int_{\xi/2}^{\infty} + \int_{-\infty}^{-\frac{3}{2}\xi} \right) \frac{g(\xi + \xi')}{\xi'^2} d\xi' |$$

$$\le \ C \xi^{-1-\tau} \|(\xi - 2i)^\tau g\|_\infty \ \left[\int_{1/2}^{\infty} \frac{(1 + \hat{\xi})^{-\tau}}{\hat{\xi}^2} d\hat{\xi} \ + \ \int_{3/2}^{\infty} \frac{(\hat{\xi} - 1)^{-\tau}}{\hat{\xi}^2} d\hat{\xi} \right]$$

Now, on change of variable $\xi' + \xi = \xi''$, the last term on the right of (A.8) can be bounded by

$$(A.9) \qquad |\int_{-\xi/2}^{\xi/2} \frac{g(\xi'')}{(\xi'' - \xi)^2} d\xi'' | \ = \ C \xi^{-1-\tau} \ \|(\xi - 2i)^\tau g\|_\infty \int_{-1/2}^{1/2} \frac{|\hat{\xi}|^{-\tau}}{(1 - \hat{\xi})^2} d\hat{\xi}$$

Therefore, combining bounds on each term, we get

$$(A.10) \qquad |\mathcal{H}(g')[\xi]| \ \le \ |\xi - 2i|^{-1-\tau} \left(C_2 \|(\xi - 2i)^{\tau+2} g''\|_\infty \ + \ C_0 \|(\xi - 2i)^\tau g\|_\infty \right)$$

We now consider $0 \le \xi \le 1$. In this case, it is convenient to write

(A.11)

$$\pi \mathcal{H}(g')[\xi] = \int_{-1}^{1} \frac{g'(\xi' + \xi) - g'(\xi)}{\xi'} d\xi' - [g(\xi + 1) + g(\xi - 1)] + \left(\int_1^\infty + \int_{-\infty}^{-1} \right) \frac{g(\xi + \xi')}{\xi'^2} d\xi'$$

Consider first term in (A.11):

$$(A.12) \qquad |\int_{-1}^{1} \frac{g'(\xi' + \xi) - g'(\xi)}{\xi'} d\xi' | \ \le \ C_1 \|(\xi - 2i)^{2+\tau} g''\|_\infty$$

Consider second term in (A.11):

(A.13) $$|g(\xi+1)+g(\xi-1)| \;\le\; C\;\|(\xi-2i)^\tau g\|_0$$

Consider the third term in (A.11):

(A.14)
$$|\left(\int_1^\infty + \int_{-\infty}^{-1}\right)\frac{g(\xi+\xi')}{\xi'^2}| \;\le\; C\|(\xi-2i)^\tau g\|_\infty \left[\int_1^\infty \xi'^{-2-\tau}d\xi' + \int_1^\infty \frac{(\xi'-1)^{-\tau}}{\xi'^2}d\xi'\right]$$

Combining the above inequalities it follows that (A.10) holds for $0\le\xi\le1$ as well. Also, it is to be noted that as in Lemma A.1, for $\xi < 0$, $\mathcal{H}(g)[\xi]$ can be related to the Hilbert transform of $g(-\xi)$ evaluated at $-\xi$. Thus, the same inequalities as above hold for $\xi < 0$. Therefore, (A.10) holds for all $\xi\in(-\infty,\infty)$ and the lemma follows. □

Lemma A.3. *Let $g\in\mathcal{C}^3(-\infty,\infty)$ such that $\|(\xi-2i)^\tau g\|_\infty$, $\|(\xi-2i)^{\tau+1}g'\|_\infty$, $\|(\xi-2i)^{\tau+3}g'''\|_\infty$ are each bounded for some $\tau\in(0,1)$. Then,*

(A.15)
$$\|(\xi-2i)^{\tau+2}\mathcal{H}(g'')\|_\infty \;\le\; C_3\|(\xi-2i)^{\tau+3}g'''\|_\infty + C_1\;\|(\xi-2i)^{\tau+1}g'\|_\infty + C_0\;\|(\xi-2i)^\tau g\|_\infty$$

Proof. For $\xi > 1$, we decompose

(A.16)
$$\pi\mathcal{H}(g'')[\xi] = \int_{-\xi/2}^{\xi/2}\frac{g''(\xi'+\xi)-g''(\xi)}{\xi'}\;d\xi' + \frac{4}{\xi^2}\left[-g(\frac{3}{2}\xi)+g(\frac{\xi}{2})\right] - \frac{2}{\xi}\left[g'(\frac{3}{2}\xi)+g'(\frac{\xi}{2})\right]$$
$$+\left(\int_{\xi/2}^\infty + \int_{-\infty}^{-\frac{3}{2}\xi}\right)\frac{2g(\xi'+\xi)}{\xi'^3}d\xi' \;+\; \int_{-\xi/2}^{\xi/2}\frac{2g(\xi'')}{(\xi''-\xi)^3}d\xi''$$

For $\xi > 1$, we then get using estimates for each term in the above using the same procedure as in previous lemma A.2, to get

(A.17)
$$||\mathcal{H}(g'')[\xi]| \le |\xi-2i|^{-2-\tau}\left[C_3\|(\xi-2i)^{3+\tau}g'''\|_\infty \;+\; C_1\|(\xi-2i)^{1+\tau}g'\|_\infty \;+\; C_0\|(\xi-2i)^\tau g\|_\infty\right]$$

For $0 \le \xi \le 1$, we decompose

$$\pi \mathcal{H}(g'')[\xi] = \int_{-1}^{1} \frac{g''(\xi'+\xi) - g''(\xi)}{\xi'} \, d\xi' \;+\; \left(\int_{1}^{\infty} + \int_{-\infty}^{-1} \right) \frac{2g(\xi+\xi')}{\xi'^3} d\xi' \\ + \left[g(\xi-1) - g(\xi+1) - g'(\xi+1) - g'(\xi-1) \right] \tag{A.18}$$

As before in previous Lemma A.2, each term can be estimated and one gets (A.17) valid again. Again, for $\xi < 0$, $\mathcal{H}(g)[\xi]$ can be related to the Hilbert transform of $g(-\xi')$ evaluated at $-\xi$; hence the inequality (A.17) is valid in that case as well. Therefore, the Lemma follows. $\square$

Lemma A.4. *If F satisfies conditions (i)-(iii) and assumption (i), then* $\sup_{\xi \in (-\infty,\infty)} |\xi + 2i|^{1+\tau} |F'| < \infty$

Proof. Define $g(\xi) = \epsilon^2 \; Im \; \ln \left[1 + \frac{F'}{H} \right]$ on the real ξ axis. From condition (i),

$$g'(\xi) = -\epsilon^2 Im \, \frac{H'}{H}(\xi) + |F' + H| Re \; F = O(\xi^{-\tau-1}) \text{ as } \xi \to \pm\infty$$

Hence, on integration, $g(\xi) = O(\xi^{-\tau})$ as $\xi \to \pm\infty$. We note that $\ln\,(1 + F'/H)$ is analytic in $\mathbf{Z}^+$ and so on the real ξ axis, $\epsilon^2 Re \; \ln\,(1 + F'/H) = \mathcal{H}(g)[\xi]$. Since conditions of previous Lemma A.1 are met by $g(\xi)$, it follows that

$$\mathcal{H}(g)[\xi] = O(\xi^{-\tau}) \text{ as } \xi \to \pm\infty$$

and therefore $\epsilon^2 \ln(1 + \frac{F'}{H}) = O(\xi^{-\tau})$ as $\xi \to \pm\infty$, which implies $F' = O(\xi^{-1-\tau})$. The lemma follows since F is continuously differentiable in $(-\infty, \infty)$. $\square$

Lemma A.5. *If f is analytic in the upper half plane $\mathbf{Z}^+$ and continuous on $\bar{\mathbf{Z}}^+$, the closure of $\mathbf{Z}^+$, and $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau_1} |f(\xi)| = \delta < \infty$ for some $\tau_1 > 0$, then*

$$\sup_{\xi \in \mathbf{Z}^+} |\xi + 2i|^{\tau_1} |f(\xi)| = \delta \tag{A.19}$$

On the otherhand, if f were analytic in the lower half-plane $\mathbf{Z}^-$ and continuous on $\bar{\mathbf{Z}}^-$ with $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau_1} |f(\xi)| = \delta < \infty$, then

(A.20)
$$\sup_{\xi \in \mathbf{Z}^-} |\xi - 2i|^{\tau_1} |f(\xi)| \leq \delta$$

Proof. Since f is analytic in the upper half plane, $\sup_{\xi \in \bar{\mathbf{Z}}^+} |f(\xi)| \leq M_0$.
Let us define integer $n = Int\, [\frac{\tau_1}{2}] + 2$. Consider that $h_{\epsilon_1}(\xi) = \frac{1}{(1 - i\xi\epsilon_1)^{2n}}$, we have

$$|h_{\epsilon_1}(\xi)| = \frac{1}{(1 + \epsilon_1 Im\ \xi)^2 + \epsilon_1^2 (Re\ \xi)^2)^n} \leq 1$$

Consider $g(\xi) = f(\xi)(\xi + 2i)^{\tau_1} h_{\epsilon_1}(\xi)$, and domain $\mathbf{D} := \{Im\ \xi \geq 0, |\xi| \leq 2^{\tau_1/2} \frac{M_0}{\epsilon_1^2 \delta}\}$. We will assume that ϵ_1 is small enough so that $2^{\tau_1/2}\ \frac{M_0}{\epsilon_1^2 \delta}\ >\ 1$. On circular part of $\partial \mathbf{D}$,

$$|g(\xi)| \leq \frac{M_0 [(Re\ \xi)^2 + (Im\ \xi + 2)^2]^{\tau_1/2}}{(1 + \epsilon_1 Im\ \xi)^2 + (Re\ \xi)^2 \epsilon_1^2} \leq \frac{2^{\frac{\tau_1}{2}} M_0 [(Re\ \xi)^2 + (Im\ \xi)^2]^{\tau_1/2}}{\epsilon_1^2 ((Re\ \xi)^2 + (Im\ \xi)^2)^n} \leq \delta$$

On straight part of $\partial \mathbf{D}$,$|g| \leq \delta$. So $|g| \leq \delta$ inside $\mathbf{D}$, from maximum principle. Also outside $\mathbf{D}$, but for $Im\ \xi \geq 0$, it is clear that $|g| \leq \delta$. So for $Im\ \xi \geq 0$,$|g| \leq \delta$. So for any fixed ξ, as $\epsilon_1 \to 0$, $g(\xi) \to f(\xi)(\xi + 2i)^{\tau_1}$. So $|f(\xi)||\xi + 2i|^{\tau_1} \leq \delta$ for all $\xi \in \bar{\mathbf{Z}}^+$. The proof of the second part is very similar. □

Proof of Lemma 1.5

(a) and (b) follow from Lemma A.5 on using Lemma A.4.

Since $g(\xi) = \epsilon^2 Im\ \ln(1 + F'/H)$ satisfies

$$g' = -\epsilon^2 Im\ \frac{H'}{H} + |F' + H| Re\ F,$$

It is clear that $g' = O(\xi^{-1-\tau})$ as $\xi \to \pm\infty$.

Also

$$g'' = -\epsilon^2 Im\ (\frac{H'}{H})' + |F' + H| Re\ F' + |F' + H| Re\ \left[\frac{F'' + H'}{F' + H}\right] Re\ F,$$

Since $\mathcal{H}(g')[\xi]$ is *a priori* $O(\xi^{-1})$ as $\xi \to \pm\infty$, it follows that

$$Re \left[\epsilon^2 \frac{d}{d\xi} \ \ln \ (F' + H) - \epsilon^2 \frac{H'}{H} \right] \ = \ \mathcal{H}(g')[\xi] \ = \ O(\xi^{-1}) \text{ at least}$$

Therefore, $Re \, \frac{F''+H'}{F'+H} \ = \ O(\xi^{-1})$ for large $|\xi|$. Also, using large $|\xi|$ behavior: $Im \left(\frac{H'}{H} \right)' \ = \ O(\xi^{-3})$, $|H + F'| \ = \ O(\xi^{-1})$, and using $Re \ F \ = \ O(\xi^{-\tau})$ and part (b) result: $Re \ F' \ = \ O(\xi^{-1-\tau})$, it follows that $g'' \ = \ O(\xi^{-2-\tau})$. From using Lemma A.2, it follows that $\mathcal{H}(g')[\xi] \ = \ O(\xi^{-1-\tau})$. So,

$$\frac{F'' + H'}{F' + H} - \frac{H'}{H} \ = \ g' + i\mathcal{H}(g)[\xi] \ = \ O(\xi^{-1-\tau})$$

Therefore, $F'' = O(\xi^{-2-\tau})$ as $\xi \to \pm\infty$ and hence

$$\sup_{\xi \in (-\infty, \infty)} |\xi - 2i|^{\tau+2} |F''(\xi)| \equiv \delta_2 \ < \ \infty$$

Using previous Lemma A.5, with f replaced F'', the proof of Lemma 1.5 is complete.

Appendix B. Properties of function $P(\xi)$

In this section,we discuss properties of the following function:

$$P(\xi) = \int_{-i\gamma}^{\xi} iL^{1/2}(t)dt = i \int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4} (\gamma + it)^{1/4}}{(1 + t^2)} dt \tag{B.1}$$

we choose branch cut $\{\xi : \xi = \rho i, \rho > \gamma\}, -\pi \leq \arg(\gamma + i\xi) \leq \pi$ for the function $(\gamma + i\xi)^{1/4}$ and branch cut $\{\xi : \xi = -\rho i, \rho > \gamma\}, -\pi \leq \arg(\gamma - i\xi) \leq \pi$ for the function $(\gamma - i\xi)^{3/4}$.

Proof of Property 1:

(1.) First consider, $\xi \in (-\infty, 0)$,

$$\text{Re } P(\xi) = \int_{\xi}^{0} \frac{(\gamma^2 + t^2)^{1/2}}{(1 + t^2)} \sin\{\frac{1}{2} \arg(\gamma - it)\} dt + \text{Re } P(0);$$

Clearly,Re $P(-\infty) = \infty$ since $\arg(\gamma - it) \to \frac{\pi}{2}$ as $t \to -\infty$, and Re $P(\xi)$ decreases as ξ increases since $arg\ (\gamma - it) \in \left(0, \frac{\pi}{2}\right)$.

(2.) For $-b < \rho < 0$,

$$P(\rho i) = -\int_0^\rho \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(1 - t^2)} dt + P(0),$$

so

$$ReP(\rho i) = -\int_0^\rho \frac{(\gamma + t)^{3/4}(\gamma - t)^{1/4}}{(1 - t^2)} dt + ReP(0),$$

On inspection as ρ increases in the interval $(-b, 0)$, Re $P(i\rho)$ decreases.

Proof of Property 2:

$$P'(t) = \frac{e^{\frac{\pi}{4}i}(t + i\gamma)^{3/4}(t - i\gamma)^{1/4}}{(t + i)(t - i)}$$

It is to be noted that $|t - 2i||P'(t)|$ is nonzero upper and lower bounds in the domain $\mathcal{R}$. Further, on a ray $t(s) = \xi - se^{i\varphi}, 0 \le s < \infty$ where $0 \le \varphi < \pi/2$, as $s \to \infty$, it is clear from the behavior of $P'(t)$ for large t that since $\arg P'(t(s)) \sim -\frac{5\pi}{4} - \varphi$,

$$\frac{d}{ds}\, Re\, P(t(s)) \;=\; Re\, \left[P'(t(s))e^{i\varphi + i\pi}\right] \;=\; |P'(t(s))|\, \cos\, [arg\, P'(t(s) + \pi + \varphi)] \;>\; \frac{C}{|t(s) - 2i|}$$

satisfies property 2.

Proof of Property 3:

$P' \sim i\gamma$ as $\xi \to 0$. So $P(\xi) = P(0) + i\gamma\xi + O(\nu^2)$.

So, on $\xi = -\nu + se^{-i\pi/4}, 0 \le s \le \sqrt{2}\nu$,

$$P(\xi) \sim P(0) + i\gamma(-\nu + se^{-i\pi/4}) + O(\nu^2) \sim P(0) - i\gamma\nu + \gamma se^{i\pi/4} + O(\nu^2),$$

$$\frac{d}{ds} Re\ P(\xi(s)) \sim \gamma \cos \pi/4 + O(\nu) > C > 0$$

with C independent of ν and ϵ for sufficiently small ν.

Proof of Property 4:

(1) For $0 < \gamma < 1$

$$P'(\xi) = i(\gamma + i\xi)^{1/4}(\gamma - i\xi)^{3/4}\frac{1}{(\xi^2+1)}$$

On $l^- = \left\{\xi : \xi = -ib - e^{i\pi/4}s\right\}$, it suffices to consider $\arg(-e^{i\pi/4}P')$ and ensure it is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, modulo an additive multiple of 2π. This will ensure property 4, since $|P'||\xi - 2i|$ has a lower bound in the region $\mathcal{D}$

Consider

$$\xi^2 + 1 = 1 - b^2 + is^2 + 2e^{3i\pi/4}bs = (1 - b^2 - \sqrt{2}bs) + i(\sqrt{2}bs + s^2),$$

$$\arg(\xi^2+1) = \pi - \arctan(\frac{\sqrt{2}bs + s^2}{\sqrt{2}bs - (1-b^2)}).$$

put $s = \sqrt{2}b\rho$ to get

$$\arg(\xi^2+1) = \pi - \arctan\left[\frac{2b^2(\rho^2+\rho)}{2b^2(\rho - \frac{1-b^2}{2b^2})}\right] = \pi - \arctan\left[\frac{\rho^2+\rho}{\rho - q}\right]$$

where $q = \frac{1-b^2}{2b^2}$. In the range $\rho > q$, the minimum of function $\frac{\rho^2+\rho}{\rho-q}$ is $(\sqrt{1+q}+\sqrt{q})^2$. Since $q = \frac{1-b^2}{2b^2} \geq \frac{1-\gamma^2}{2\gamma^2} := q_{min}$, define $\theta_{min} = \tan^{-1}(\sqrt{1+q_{min}} + \sqrt{q_{min}})^2 > \frac{\pi}{4}$, then

$$-\pi + \theta_{min} \leq -\arg(\xi^2+1) \leq 0 \tag{B.2}$$

Consider

$$\arg i(\gamma + i\xi)^{1/4} = \arg\left[e^{i\pi/2}e^{i\pi/8}(\xi - i\gamma)^{1/4}\right] = \frac{5\pi}{8} + \frac{1}{4}(-\frac{3\pi}{4}, -\frac{\pi}{2}] = (\frac{7\pi}{16}, \frac{\pi}{2}], \tag{B.3}$$

Let $\epsilon_2 = \frac{1}{2}(\theta_{min} - \frac{\pi}{4})$. Near $\xi = -i\gamma$:

$$P'(\xi) = \frac{i(2\gamma)^{1/4}}{(1-\gamma^2)}(\gamma - i\xi)^{3/4}\{1 + O(\gamma - i\xi)\} \tag{B.4}$$

Clearly, there exists R_0 large enough (depending on b) so that for $\xi \in l^-$,

$$\frac{3}{4} arg\ (\gamma - i\xi) \ \in \ [\frac{9}{16}\pi - \epsilon_2, \frac{9}{16}\pi) \text{ for } |\xi + i\gamma| \ \geq \ R_0 \tag{B.5}$$

and

$$\frac{3}{4} arg\ (\gamma - i\xi) \ \in \ [0, \frac{9}{16}\pi - \epsilon_2) \text{ for } |\xi + i\gamma| \ < \ R_0$$

From geometric consideration, it is clear that $R_0(b) \ \rightarrow \ 0$ as $b \ \rightarrow \ \gamma^-$. We choose b so close to γ so that the approximation in (B.4) is good enough to ensure that on l^-,

$$\arg(P'(\xi) e^{-3\pi i/4}) \in (-\frac{5\pi}{16}, \frac{7\pi}{16}), \text{ for } |\xi + i\gamma| \ < \ R_0(b)$$

On the otherhand, on l^- for $|\xi + i\gamma| \ \geq \ R_0(b)$, using (B.5), along with (B.2) and (B.3), it follows that

$$\arg(e^{i\frac{-3\pi}{4}} P'(\xi)) \in (-\frac{3\pi}{4} + \frac{\pi}{4} + (\theta_{min} - \frac{\pi}{4}) - \epsilon_2, \frac{5\pi}{16}) \subset (-\frac{\pi}{2}, \frac{\pi}{2}) \tag{B.6}$$

(2) Now consider $\gamma > 1$ ($i.e.$ $\lambda \ > \ \frac{1}{2}$ but restrict to $\gamma - 1$ small enough so that we can choose b so that $10(\gamma - 1) \leq |b - 1|$. We want to show that on ray $l^- \equiv \{\xi = -bi - se^{i\pi/3}, 0 \leq s < \infty\}$.

$$\frac{d}{ds} Re P(\xi(s)) = Re\{P'(\xi) e^{-i2\pi/3}\} > \frac{C}{|\xi(s) - 2i|} \ > \ 0. \tag{B.7}$$

We note that since $|\xi - 2i||P'$ is bounded above and below by nonzero constants, it suffices to show that

$$arg\ \left(P'(\xi(s)) e^{-i2\pi/3}\right) \ \in \ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ modulo } 2\pi$$

Note:

$$P'(\xi) = i \frac{(\xi - i\gamma)}{(\xi + i)} \left(\frac{\gamma + i\xi}{\gamma - i\xi}\right)^{1/4} \frac{1}{(1 + i\xi)}; \tag{B.8}$$

Let $B(s)$ be the positive angle between $\xi(s)+\gamma i$ and $\xi(s)+i$,then by geometry:

$$\arg\left(e^{-i2\pi/3}P'(\xi)\right)\in(-B-\frac{5\pi}{12},0);$$

we can see that $B<\frac{\pi}{12}$ implies (B.7).

Let $d_1=|b-1|+|\gamma-1|$, by geometry:

$$\cos B=\frac{(s^2-2s|b-1|\sin\frac{\pi}{3}+|b-1|^2)+(s^2-2sd_1\sin\frac{\pi}{3}+d_1^2)-|\gamma-1|^2}{2\sqrt{(s^2-2s|b-1|\sin\frac{\pi}{3}+|b-1|^2)}\sqrt{(s^2-2sd_1\sin\frac{\pi}{3}+d_1^2)}}$$

Let $t=\frac{|\gamma-1|^2s}{|b-1|\sin\frac{\pi}{3}}, d=\frac{|\gamma-1|}{|b-1|}$,

$$\cos B=\frac{(t-1-\frac{d}{2})^2+(1+d)\cot^2\frac{\pi}{3}-\frac{1}{4}d^2}{\sqrt{(t-1)^2+\cot^2\frac{\pi}{3}}\sqrt{(t-1-d)^2+(1+d)^2\cot^2\frac{\pi}{3}}}$$

The min of the above function over $0<t<\infty, d\le 0.1$ is .9688749307,but $\cos\frac{\pi}{12}=0.9330127$,so $B<\frac{\pi}{12}$.

Proof of Property 5 and 6:

Recall that in showing Property 4. we showed that there exists ϕ_0 and b with $\frac{\pi}{4}\le\phi_0<\frac{\pi}{2}, 0<b<\min\{1,\gamma\}$ so that on $\xi=-ib-e^{i\phi_0}s$

$$\arg\left[P'e^{-i(\pi-\phi_0)}\right]\in(-\theta_1,\theta_2)\subset(-\pi/2,\pi/2),\ \text{where } 0<\theta_1,\theta_2<\pi/2,$$

without loss of generality, it will be assumed that $\pi/4\le\theta_1,\theta_2<\pi/2$. Then it is clear that on $\xi=-ib-e^{i\phi_0}s$,

$$\arg P'\in(\pi-\phi_0-\theta_1,(\pi-\phi_0)+\theta_2)$$

Note $[\frac{\pi}{2},\frac{3\pi}{4})\subset(\pi-\phi_0-\theta_1,(\pi-\phi_0)+\theta_2)$. On the real axis, $\arg P'=\arg i+\frac{1}{2}\arg(\gamma-i\xi)\in\frac{\pi}{2}+(0,\pi/4)=(\frac{\pi}{2},\frac{3\pi}{4})$. On the imaginary axis between O and ib, $\arg P'=\frac{\pi}{2}$. In all cases, on the boundary of the domain $\mathcal{R}^-$, bounded by negative real axis, imaginary axis between 0 and ib and line $\xi = -bi - e^{i\phi_0} s$, we have $\arg P' \in (\pi - \phi_0 - \theta_1, \pi - \phi_0 + \theta_2)$. On $\xi = -bi - s e^{i\phi}$ for $\phi < \phi_0$, as $s \to \infty$, we have

$$\arg(\xi + i\gamma) \to (\pi + \phi)\,,\ \arg(\xi - i\gamma) \to (-\pi + \phi)\,,\ \arg(\xi + i) \to (\pi + \phi)\,,\ \arg(\xi - i) \to (-\pi + \phi).$$

So, as $s \to \infty$, $\arg P' \to \frac{3\pi}{4} - \phi \in (\frac{3\pi}{4} - \phi_0, \frac{3\pi}{4})$. So as $\xi \to \infty$ and $\xi \in \mathcal{R}^-$, $\arg P' \in (\frac{3\pi}{4} - \phi_0, \frac{3\pi}{4}) \subset (\frac{\pi}{2}, \frac{3\pi}{4})$. Using Maximum Principle , $\arg P' \in (\pi - \phi_0 - \theta_1, \pi - \phi_0 + \theta_2)$ everywhere inside the domain $\mathcal{R}^-$.

Now if we choose $\mathcal{P}(\xi, -\infty) = \{t : t = \xi - e^{i\phi_0} s, 0 < s < \infty\}$, it is clear on $\mathcal{P}$, we have

$$\frac{d}{ds}(Re\ P) = |P'| \cos\left[\arg(P' e^{-i(\pi - \phi_0)})\right] > \frac{C}{|\xi - 2i|} > 0,$$

where C can be made independent of γ for γ in a compact subset of $(0,\ \infty)$. Hence property 6 follows.

Now to find $\mathcal{P}(\xi, -\nu)$ so that $Re\ P$ decreases monotonically from ξ to $-\nu$, we use line $\mathcal{P}_0 = \{t = \xi + e^{i\phi_0} s, s > 0\}$ where $\frac{d}{ds} Re\ P = |P'| \cos[\arg P' + \phi_0] \leq -\frac{C}{|\xi - 2i|} < 0$. This line intersects $\partial\mathcal{D}$ at some point $\xi_1 \in \partial\mathcal{D}$. Now, clearly ξ_1 can be connected to $\xi = -\nu$ by $\mathcal{P}_1(\xi_1, -\nu)$ on a path coinciding with $\partial\mathcal{D}$ so that $Re\ P$ decreases monotonically from ξ_1 to $-\nu$ such that $-\frac{d}{ds}(Re\ P) > \frac{C}{|\xi - 2i|} > 0$. Then $\mathcal{P}(\xi, -\nu) = \mathcal{P}_0(\xi, \xi_1) + \mathcal{P}_1(\xi_1, -\nu)$. Reversing this path, leads to the desired path $\mathcal{P}(-\nu, \xi)$ having property 5.

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