GENERALIZED TRIANGULAR MATRIX RINGS AND THE FULLY INVARIANT EXTENDING PROPERTY

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ABSTRACT. A module M is called (*strongly*) *FI-extending* if every fully invariant submodule of M is essential in a (*fully invariant*) direct summand of M. A ring R with unity is called *quasi-Baer* if the right annihilator of every ideal is generated, as a right ideal, by an idempotent. For semiprime rings the FI-extending condition, strongly FI-extending condition, and quasi-Baer condition are equivalent. In this paper, we fully characterize the 2-by-2 generalized (or formal) triangular matrix rings which are either (right) FI-extending, (right) strongly FI-extending, or quasi-Baer. Examples are provided to illustrate and delimit our results.

0. INTRODUCTION

All rings are associative with unity and all modules are unital. Throughout this paper T will denote a 2-by-2 generalized (or formal) triangular matrix ring

$$\begin{pmatrix} S & M \\ 0 & R \end{pmatrix},$$

where R and S are rings and M is an (S, R)-bimodule.

Generalized triangular matrix rings have proven to be extremely useful in ring theory. They provide a good source of examples and counterexamples (e.g., see [11, pp 46-48 and 79-80] and [10]) as well as providing a framework to explore the connections between $\mathrm{End}(M_R), M$ and R when $S = \mathrm{End}(M_R)$.

Recently, several aspects of injectivity and projectivity in the context of generalized triangular matrix rings have been investigated by Haghany-Varadarajan [8, 9] and Tercan [13]. Tercan was able to obtain a characterization of the right nonsingular right extending (or CS) condition on T when ${}_SM$ is faithful (recall a module is *extending* (or *CS*) if every submodule is essential in a direct summand).

In [1], [4], and [5] the FI-extending property was introduced and investigated. A module is said to be (*strongly*) *FI-extending* if every fully invariant submodule is essential in a (fully invariant) direct summand. Observe that many distinguished submodules of a module are fully invariant (e.g., Jacobson radical, singular submodule, socle, torsion submodule, etc.). Thus, in an FI-extending module, these submodules can be "essentially split-off." From [4, Theorem 4.7] and [5, Proposition 1.5], for nonsingular modules and semiprime rings the FI-extending and strongly FI-extending properties are equivalent. A description of the

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strongly FI-extending Abelian groups was obtained in [1]. The classes of (strongly) FI-extending rings and modules, in general, exhibit better behavior with respect to various algebraic constructions than the class of extending modules. For example, the class of FI-extending modules is closed under direct sums; and the class of right strongly FI-extending rings is Morita invariant. Thus these results show, at a minimum, how much of the extending property is preserved by these constructions. For further details and examples see [4] and [5].

In the first two sections of this paper we fully characterize the generalized triangular matrix rings which are right FI-extending and right strongly FI-extending. In [13, Theorem 2.4] Tercan determines four conditions which are satisfied by a right extending generalized triangular matrix ring. However in [13, Example 3.5] he shows that these conditions are not sufficient to ensure that a generalized triangular matrix ring is right extending. Our Theorem 1.4 shows that these conditions do ensure that the generalized triangular matrix ring is right FI-extending.

Chatters and Khuri [6, Theorem 2.1] showed that a right nonsingular right extending ring is a Baer ring. In [4, Proposition 4.4 and Theorem 4.7], it was shown that a right FI-extending ring which is either semiprime or right nonsingular is quasi-Baer. Recall that a ring R is (*quasi-*) *Baer* if the right annihilator of every (ideal) nonempty subset is generated, as a right ideal, by an idempotent. In Section 3, we characterize the quasi-Baer generalized triangular matrix rings. Some examples to illustrate and delimit our results are presented in the last section.

We use ${}_SM$ or M_R to denote that M is a left S-module or a right R-module, respectively. The symbols $N_R \leq M_R$, $N_R \leq^{\text{ess}} M_R$, ${}_SN \leq {}_SM$, and ${}_SN_R \leq {}_SM_R$ are used for N is a right R-submodule, N is an essential right R-submodule, N is a left S-submodule, and N is a sub-bimodule of M, respectively. Some subscripts may be omitted if the context is clear. A submodule $N_R \leq M_R$ is called *fully invariant* in M_R, denoted $N \trianglelefteq_R M$ (or simply, $N \trianglelefteq M$), if $f(N) \subseteq N$ for all $f \in \text{End}(M_R)$. Observe that the fully invariant submodules of R_R are the ideals of R. An idempotent $e \in R$ is called *left* (*right*) *semicentral* if $Re = eRe\,(eR = eRe)$. The set of all left (right) semicentral idempotents is denoted by $\mathcal{S}_\ell(R)$ ($\mathcal{S}_r(R)$). Equivalently, $e = e^2 \in R$ is left (right) semicentral if $eR \trianglelefteq R$ ($Re \trianglelefteq R$). An idempotent e is called *semicentral reduced* if $\mathcal{S}_\ell(eRe) = \{0, e\}$. If $1 \in R$ is semicentral reduced, then R is said to be *semicentral reduced.* (See [2] or [3] for further details on semicentral idempotents). The Jacobson radical and the right singular ideal of R are denoted by $\mathbf{J}(R)$ and $Z(R_R)$, respectively. If $N_R \leq M_R$ (resp. ${}_SN \leq {}_SM$), then $\text{Ann}_R(N) = \{r \in R \mid Nr = 0\}$ (resp. $\text{Ann}_S(N) = \{s \in S \mid sN = 0\}$). If $\emptyset \neq B \subseteq S$ and M is a left S-module, then $r_M(B) = \{m \in M \mid Bm = 0\}$ and $r_S(B) = \{a \in S \mid Ba = 0\}$. The ring of n-by-n upper triangular matrices over R is denoted by $T_n(R)$.

1. THE FI-EXTENDING PROPERTY

In this section we completely characterize the FI-extending property for a generalized triangular matrix ring T. This characterization is refined under the assumptions that ${}_SM$ is faithful or $S = \text{End}(M_R)$. We include the following two lemmas for completeness since they are used repeatedly in the sequel.

Lemma 1.1. [4, Theorem 1.3] Direct sums of modules with the FI-extending property have again the FI-extending property.

Lemma 1.2. [1, Lemma 1.2] If the module $A = B \oplus C$ has the FI-extending property and B is a fully invariant summand, then both B and C have the FI-extending property.

Corollary 1.3. For a ring R, let e be a left semicentral idempotent of R. Then R_R is FI-extending if and only if eR_R and $(1-e)R_R$ are FI-extending.

Proof. It follows immediately from Lemmas 1.1 and 1.2. □

Theorem 1.4. For rings S and R, assume that ${}_SM_R$ is an (S,R)-bimodule. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ be a generalized triangular matrix ring. Then the following are equivalent:

(1) T_T is FI-extending.

(2) (i) For any ${}_SN_R \leq {}_SM_R$ and any ideal I of S with $IM \subseteq N$, there is $f = f^2 \in S$ such that $I \subseteq fS, N_R \leq^{\text{ess}} fM_R$, and $(I \cap \text{Ann}_S(M))_S \leq^{\text{ess}} (fS \cap \text{Ann}_S(M))_S$; and

(ii) R_R is FI-extending.

Proof. Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$.

(1)⇒(2) Since $\begin{pmatrix} \text{Ann}_S(M) & 0 \\ 0 & 0 \end{pmatrix} \trianglelefteq T$, there exists an idempotent $c \in T$ such that $\begin{pmatrix} \text{Ann}_S(M) & 0 \\ 0 & 0 \end{pmatrix}_T \leq^{\text{ess}} cT_T = cE_{11}T = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} eS & eM \\ 0 & 0 \end{pmatrix}$, for some $e = e^2 \in S$.

If $eM \neq 0$, then choose $0 \neq em \in eM$ with $m \in M$. So we have $0 \neq \begin{pmatrix} 0 & em \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} \text{Ann}_S(M) & 0 \\ 0 & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 & em \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} \text{Ann}_S(M) & 0 \\ 0 & 0 \end{pmatrix} = 0$, a contradiction. Therefore $eM = 0$ and hence $e \in \text{Ann}_S(M)$. Thus $eS \subseteq \text{Ann}_S(M)$ and so $\text{Ann}_S(M) = eS$.

For (i), let ${}_SN_R \leq {}_SM_R$ and I be an ideal of S with $IM \subseteq N$. Then $\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}$ is a fully invariant T-submodule of $E_{11}T$. As above, there exists $f = f^2 \in S$ such that

$$\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} E_{11}T_T = \begin{pmatrix} fS & fM \\ 0 & 0 \end{pmatrix}.$$

If $fM = 0$, then $N = 0$ and so $N_R \leq^{\text{ess}} fM_R$. Suppose $fM \neq 0$. For $0 \neq fm \in fM$, we have $\begin{pmatrix} 0 & fm \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} \neq 0$ and so $fmR \cap N \neq 0$. Thus $N_R \leq^{\text{ess}} fM_R$.

Next, if $fS \cap eS = 0$, then $I \cap eS = 0$. Thus $(I \cap \text{Ann}_S(M))_S \leq^{\text{ess}} (fS \cap \text{Ann}_S(M))_S$. Assume $fS \cap eS \neq 0$. Then for $0 \neq fs \in fS \cap eS$ with $s \in S$, we have that

$$\begin{pmatrix} fs & 0 \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} fsS & fsM \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} fsS & 0 \\ 0 & 0 \end{pmatrix}.$$

So it follows that

$$0 \neq \begin{pmatrix} fs & 0 \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} fsS & 0 \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}.$$

Thus we have $0 \neq fsS \cap I = fsS \cap (I \cap eS)$. Therefore $(I \cap eS)_S \leq^{\text{ess}} (fS \cap eS)_S$. Since E_{11} is left semicentral, (ii) follows immediately from Corollary 1.3.

(2)⇒(1) Suppose (i) and (ii) hold. By (ii), $(1 - E_{11})T_T$ is FI-extending. Now to prove $E_{11}T_T$ is FI-extending, let $\mathfrak{A}$ be a fully invariant T-submodule of $E_{11}T$. Then $\mathfrak{A} = \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}$ with I an ideal of S, ${}_SN_R \leq {}_SM_R$, and $IM \subseteq N$. By (ii), there is $f = f^2 \in S$ such that $\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & M \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} fS & fM \\ 0 & 0 \end{pmatrix}$. In this case, note that $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in \text{End}(E_{11}T_T)$. So $\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & M \\ 0 & 0 \end{pmatrix}$ is a T-direct summand of $E_{11}T$. Now we claim that

$$\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & M \\ 0 & 0 \end{pmatrix}_T = \begin{pmatrix} fS & fM \\ 0 & 0 \end{pmatrix}.$$

Take $0 \neq \begin{pmatrix} fs & fm \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} fS & fM \\ 0 & 0 \end{pmatrix}$.

Case 1. $fm \neq 0$. Then since $N_R \leq^{\text{ess}} fM_R$, $N \cap fmR \neq 0$ and so

$$\begin{pmatrix} fs & fm \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} \neq 0.$$

Case 2. $fm = 0$. Then $fs \neq 0$. Thus $\begin{pmatrix} fs & fm \\ 0 & 0 \end{pmatrix} T = \begin{pmatrix} fsS & fsM \\ 0 & 0 \end{pmatrix}$. If $fsM \neq 0$, then $fsm_0 \neq 0$, for some $m_0 \in M$. So $\begin{pmatrix} 0 & fsm_0 \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} fsS & fsM \\ 0 & 0 \end{pmatrix}$ and hence $\begin{pmatrix} 0 & fsm_0R \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} fsS & fsM \\ 0 & 0 \end{pmatrix}$. But since $fsm_0R \cap N \neq 0$, we have that $\begin{pmatrix} fsS & fsM \\ 0 & 0 \end{pmatrix} \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} \neq 0$. If $fsM = 0$, then $fs \in \text{Ann}_S(M)$ and so $0 \neq fs \in fS \cap \text{Ann}_S(M)$. Thus by (ii), $fsS \cap (I \cap \text{Ann}_S(M)) \neq 0$, so

$$\begin{pmatrix} fs & 0 \\ 0 & 0 \end{pmatrix} T \cap \begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix} \neq 0.$$

¿From Cases 1 and 2, $\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} fS & fM \\ 0 & 0 \end{pmatrix}_T$, and hence $E_{11}T_T$ is FI-extending. Therefore T_T is FI-extending, by Corollary 1.3. □

Corollary 1.5. Let T_T be FI-extending. Then there exists a left semicentral idempotent $e \in S$ such that $\text{Ann}_S(M) = eS$ and eS_S is FI-extending. In particular, if $M \neq 0$ and S is semicentral reduced, then ${}_SM$ is faithful.

Proof. In the proof of (1)⇒(2) of Theorem 1.4, $\text{Ann}_S(M) = eS$ for some left semicentral idempotent e of S. To show that eS_S is FI-extending, let $I_S \leq eS_S$ be a fully invariant

S-submodule of S. Since $eS \trianglelefteq S$, I is an ideal of S. Applying condition 2(i) of Theorem 1.4 with $N = 0$, we see that $fM = 0$, hence $f \in eS$. So $fS \subseteq eS$. Now $I = (I \cap eS)_S \leq^{\text{ess}} (fS \cap eS)_S = fS$ and fS is an S-direct summand of eS by the modular law. Thus eS_S is FI-extending. □

Corollary 1.6. Let ${}_SM$ be faithful. Then the following are equivalent:

(1) T_T is FI-extending.

(2) (i) For any ${}_SN_R \leq {}_SM_R$, there exists $f = f^2 \in S$ such that $N_R \leq^{\text{ess}} fM_R$; and

(ii) R_R is FI-extending.

Proof. (1)⇒(2) Assume that T_T is FI-extending. Since ${}_SM$ is faithful, $\text{Ann}_S(M) = 0$. By taking $I = 0$ in Theorem 1.4, we have (i). Then (ii) follows from Theorem 1.4.

(2)⇒(1) Assume (i) and (ii) hold. Let ${}_SN_R \leq {}_SM_R$ and I an ideal of S such that $IM \subseteq N$. By (i), there is $f = f^2 \in S$ such that $N_R \leq^{\text{ess}} fM_R$. Since $IM \subseteq N \subseteq fM$, $fn = n$ for all $n \in N$, in particular $fsm = sm$ for any $s \in I$ and $m \in M$. Thus $(s - fs)M = 0$ for any $s \in I$ and hence $s - fs = 0$, for any $s \in I$. So $I = fI \subseteq fS$. Therefore by Theorem 1.4, T_T is FI-extending. □

Since M_R is always a left S-module for $S = \text{End}(M_R)$ or $S = \mathbb{Z}$, we consider these cases in our next two results.

Corollary 1.7. Let $S = \mathbb{Z}$. Then T_T is FI-extending if and only if ${}_{\mathbb{Z}}M$ is faithful, M_R is uniform, and R_R is FI-extending.

Proof. Since $\mathbb{Z}$ is semicentral reduced, Corollaries 1.5 and 1.6 yield the result. □

Corollary 1.8. [4, Theorem 2.4] Let $S = \text{End}(M_R)$. Then T_T is FI-extending if and only if M_R and R_R are FI-extending.

Proof. It follows immediately from Corollary 1.6. □

Thus from Corollary 1.8 and [4, Proposition 1.2], if $I \trianglelefteq R$ and $S = \text{End}(I_R)$ then T_T is FI-extending if and only if R_R is FI-extending. The next corollary applies our results to the endomorphism ring of certain Abelian groups.

Corollary 1.9. Let G be an Abelian group such that $G = M \oplus C$ where M is a direct sum of finite cyclic groups and C is an infinite cyclic group. Then $\text{End}(G_{\mathbb{Z}})$ is right FI-extending.

Proof. Observe $\text{End}(G_{\mathbb{Z}}) \cong \begin{pmatrix} \text{End}(M_{\mathbb{Z}}) & M \\ 0 & \mathbb{Z} \end{pmatrix}$. Since every cyclic group is an FI-extending $\mathbb{Z}$-module, Lemma 1.1 shows that M is an FI-extending $\mathbb{Z}$-module. Now Corollary 1.8 yields the result. □

¿From our previous results, we have two classes of rings which are right FI-extending, but not left FI-extending as the following examples illustrate.

Example 1.10. Note that if $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ is left FI-extending, then by a similar method as in the proof of (1)⇒(2) of Theorem 1.4, $\text{Ann}_R(M) = Rf$ for some right semicentral idempotent f of R.

(i) Let R be a right self-injective ring with $\mathbf{J}(R) \neq 0$. Let

$$T = \begin{pmatrix} R/\mathbf{J}(R) & R/\mathbf{J}(R) \\ 0 & R \end{pmatrix}.$$

Then the ring $R/\mathbf{J}(R)$ is right self-injective. So it can be easily checked that $R/\mathbf{J}(R)$ is an FI-extending right R-module because $R/\mathbf{J}(R) \cong \mathrm{End}((R/\mathbf{J}(R))_R)$. Thus the ring T is right FI-extending by Corollary 1.8. If ${}_T T$ is FI-extending, then $\mathrm{Ann}_R((R/\mathbf{J}(R))_R) = \mathbf{J}(R) = Rf$ for some right semicentral idempotent f of R. Thus $f = 0$ and hence $\mathbf{J}(R) = 0$, a contradiction.

(ii) Let R be a prime ring with a nonzero prime ideal P. Let

$$T = \begin{pmatrix} R/P & R/P \\ 0 & R \end{pmatrix}.$$

Note that prime rings are both left and right strongly FI-extending. Therefore as in part (i) the ring T is right FI-extending, but not left FI-extending.

(iii) Let R be a left or right principal ideal domain and let I be a nonzero proper ideal of R. Then the ring R/I is QF. Thus as in part (i) the ring

$$T = \begin{pmatrix} R/I & R/I \\ 0 & R \end{pmatrix}$$

is right FI-extending, but not left FI-extending.

2. THE STRONGLY FI-EXTENDING PROPERTY

The ring T in Example 1.10(ii) is isomorphic to $\Lambda = \begin{pmatrix} \mathrm{End}((R/P)_R) & R/P \\ 0 & R \end{pmatrix}$. By Corollary 1.8, T is right FI-extending because R/P and R in the right hand column are FI-extending. Since R and R/P are prime rings, then R_R and $(R/P)_R$ are strongly FI-extending. However, in contrast to the FI-extending case, the right hand column being strongly FI-extending in each component does not ensure that Λ_Λ is strongly FI-extending. In fact Λ_Λ is not strongly FI-extending because $\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \trianglelefteq \Lambda$, but there does not exist $b \in \mathcal{S}_\ell(\Lambda)$ such that $\begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix}$ is right essential in $b\Lambda$.

In this section we determine necessary and sufficient conditions to ensure that a 2-by-2 generalized triangular matrix ring is right strongly FI-extending.

Lemma 2.1. Let X be a right ideal of R such that $X_R \leq^{\mathrm{ess}} bR_R$, for some $b \in \mathcal{S}_\ell(R)$. If $X_R \leq^{\mathrm{ess}} eR_R$, where $e = e^2$, then $bR = eR$ and $e \in \mathcal{S}_\ell(R)$.

Proof. Observe that $X_R \leq^{\mathrm{ess}} (eR \cap bR)_R$. Then $eR \cap bR = ebR$, where $eb = (eb)^2$. Hence $eR = ebR = bR$. Since $eR \trianglelefteq R$, $e \in \mathcal{S}_\ell(R)$.

Definition 2.2. Let $N_R \leq M_R$. We say N_R has a *direct summand cover* $\mathcal{D}(N_R)$ if there exists $e = e^2 \in \mathrm{End}(M_R)$ such that $N_R \leq^{\mathrm{ess}} eM_R = \mathcal{D}(N_R)$. In general a submodule may have several direct summand covers, however Lemma 2.1 yields that if M_R is a strongly FI-extending module then every fully invariant submodule has a *unique* direct summand cover.

Let M be an (S, R)-bimodule and ${}_SN_R \leq {}_SM_R$. If there is $e = e^2 \in \mathcal{S}_\ell(S)$ such that $N_R \leq^{\text{ess}} eM_R$, then we write $\mathcal{D}_S(N_R) = eM$.

For $N_R \leq M_R$, let $(N_R : M_R) = \{a \in R \mid Ma \subseteq N\}$. Then $\mathcal{D}((N_R : M_R)_R)$ denotes a direct summand cover of the right ideal $(N_R : M_R)$ in R_R.

Lemma 2.3. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix} \in T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where $e_1 = e_1^2$ and $e_2 = e_2^2$.

(1) $e \in \mathcal{S}_\ell(T)$ if and only if
  (i) $e_1 \in \mathcal{S}_\ell(S)$;
  (ii) $e_2 \in \mathcal{S}_\ell(R)$;
  (iii) $e_1k = k$; and
  (iv) $e_1me_2 = me_2$, for all $m \in M$.

(2) $e_1k = k$ if and only if $eT \subseteq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T$.

(3) If $e_1me_2 = me_2$, for all $m \in M$, then $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T \subseteq eT$.

(4) If $e \in \mathcal{S}_\ell(T)$, then $eT = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T$.

Proof. Observe $e = e^2$ if and only if $e_1 = e_1^2$, $e_2 = e_2^2$, and $e_1k + ke_2 = k$. Let $t = \begin{pmatrix} s & m \\ 0 & r \end{pmatrix} \in T$. Then $te = \begin{pmatrix} se_1 & sk + me_2 \\ 0 & re_2 \end{pmatrix}$ and $ete = \begin{pmatrix} e_1se_1 & e_1sk + e_1me_2 + kre_2 \\ 0 & e_2re_2 \end{pmatrix}$.

(1) Assume $e \in \mathcal{S}_\ell(T)$. Then $te = ete$. Hence conditions (i) and (ii) are satisfied. Letting $s = 1, m = 0$, and $r = 0$ yields $k = e_1k$. So condition (iii) is satisfied. Also $k = e_1k + ke_2$ implies $ke_2 = 0$. Since $sk = se_1k = e_1se_1k = e_1sk$ and $kre_2 = ke_2re_2 = 0$, then $e_1me_2 = me_2$. Hence condition (iv) is satisfied. The converse is routine.

(2) This proof is straightforward.

(3) Observe $e\begin{pmatrix} e_1 & -ke_2 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. Thus $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T \subseteq eT$.

(4) This is a consequence of the previous parts. □

The next result gives a characterization for the strongly FI-extending condition for a 2-by-2 generalized triangular matrix ring.

Theorem 2.4. Assume M is an (S, R)-bimodule, and let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent:

(1) T_T is strongly FI-extending.

(2) (i) For any ${}_SN_R \leq {}_SM_R$ and any ideal I of S with $IM \subseteq N$, there exists $e \in \mathcal{S}_\ell(S)$ such that $I \subseteq eS$, $N_R \leq^{\text{ess}} eM_R$ and $(I \cap \text{Ann}_S(M))_S \leq^{\text{ess}} (eS \cap \text{Ann}_S(M))_S$;
  (ii) R_R is strongly FI-extending;
  (iii) For any ${}_SN_R \leq {}_SM_R$, $\mathcal{D}_S(N_R)\mathcal{D}((N_R : M_R)_R) = M\mathcal{D}((N_R : M_R)_R)$.

Proof. (1)⇒(2). Assume T_T is strongly FI-extending. Then by [5, Theorem 2.4]

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_T$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_T$ are strongly FI-extending. So as in Theorem 1.4, we can show that (i) and (ii) hold. For (iii), let ${}_SN_R \leq {}_SM_R$ and put $A = (N_R : M_R)$. By (i) and (ii), there are $e \in \mathcal{S}_\ell(S)$ and $f \in \mathcal{S}_\ell(R)$ such that $\mathcal{D}_S(N_R) = eM$ and $\mathcal{D}(A_R) = fR$. Since $MA \subseteq N$, it follows that $\begin{pmatrix} 0 & N \\ 0 & A \end{pmatrix} \trianglelefteq T$. So there exists $\theta^2 = \theta \in \mathcal{S}_\ell(T)$ such that $\begin{pmatrix} 0 & N \\ 0 & A \end{pmatrix}_T \leq^{\text{ess}} \theta T_T$. By Lemma 2.3, there exist $e_0 \in \mathcal{S}_\ell(S)$ and $f_0 \in \mathcal{S}_\ell(R)$ such that $\theta T = \begin{pmatrix} e_0 & 0 \\ 0 & f_0 \end{pmatrix} T$ and $\begin{pmatrix} e_0 & 0 \\ 0 & f_0 \end{pmatrix} \in \mathcal{S}_\ell(T)$. Hence $N_R \leq^{\text{ess}} e_0M_R$ and $A_R \leq^{\text{ess}} f_0R_R$. So $\mathcal{D}_S(N_R) = eM = e_0M$ and $\mathcal{D}(A_R) = fR = f_0R$. Thus, from the fact that $e_0Mf_0 = Mf_0$, it follows that $eMf = Mf$. Therefore $\mathcal{D}_S(N_R)\mathcal{D}((N_R : M_R)_R) = M\mathcal{D}((N_R : M_R)_R)$.

(2)⇒(1). Let $\begin{pmatrix} I & N \\ 0 & A \end{pmatrix} \trianglelefteq T$. Then ${}_SN_R \leq {}_SM_R$, $I \trianglelefteq S$, and $IM \subseteq N$. So, by (i), there exists $e \in \mathcal{S}_\ell(S)$ such that $I \subseteq eS$ and $\mathcal{D}_S(N_R) = eM$. Since $A \trianglelefteq R$, by (ii), there exists $f \in \mathcal{S}_\ell(R)$ such that $\mathcal{D}(A_R) = fR$. Also, by (ii), $\mathcal{D}((N_R : M_R)_R) = f_0R$ for some $f_0 \in \mathcal{S}_\ell(R)$. Since $\begin{pmatrix} I & N \\ 0 & A \end{pmatrix} \trianglelefteq T$, we have $MA \subseteq N$ and so $A \subseteq (N_R : M_R)$. Thus $A_R \leq^{\text{ess}} (fR \cap f_0R)_R = f_0fR$ with $f_0f \in \mathcal{S}_\ell(R)$. So $\mathcal{D}(A_R) = f_0fR$. By Lemma 2.1, $fR = f_0fR$ and hence $f_0f = f$. Since $\mathcal{D}_S(N_R)\mathcal{D}((N_R : M_R)_R) = M\mathcal{D}((N_R : M_R)_R)$, part (iii) yields $eMf_0R = Mf_0R$. So $eMf_0 = Mf_0$. Thus $eMf_0f = Mf_0f$, so $eMf = Mf$. Since $I \subseteq eS$, we have $\begin{pmatrix} I & N \\ 0 & A \end{pmatrix}_T \leq \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} T_T$. By (i), $\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} T_T$. Because $A_R \leq^{\text{ess}} fR_R$, we have $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} T_T$. So $\begin{pmatrix} I & N \\ 0 & A \end{pmatrix}_T \leq^{\text{ess}} \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} T_T$. Since $eMf = Mf$, Lemma 2.3 yields $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in \mathcal{S}_\ell(T)$. Therefore T_T is strongly FI-extending. □

Corollary 2.5. Let ${}_SM$ be faithful. Then the following are equivalent:

(1) T_T is strongly FI-extending.

(2) (i) For any ${}_SN_R \leq {}_SM_R$, there exists $e^2 = e \in \mathcal{S}_\ell(S)$ such that $N_R \leq^{\text{ess}} eM_R$.

(ii) R_R is strongly FI-extending.

(iii) For any ${}_SN_R \leq {}_SM_R$, $\mathcal{D}_S(N_R)\mathcal{D}((N_R : M_R)_R) = M\mathcal{D}((N_R : M_R)_R)$.

Proof. The proof is similar to that of Corollary 1.6 □

Corollary 2.6. Let $S = \mathbb{Z}$. Then T_T is strongly FI-extending if and only if ${}_{\mathbb{Z}}M$ is faithful, M_R is uniform, and R_R is strongly FI-extending.

Proof. Since $\mathbb{Z}$ is semicentral reduced, Corollaries 1.5 and 2.5 yield the result. □

Observe in Theorem 2.4 that for $S = \text{End}(M_R)$ and T_T strongly FI-extending if $A \trianglelefteq R$ and $MA = 0$, then $M\mathcal{D}(A_R) = 0$.

Corollary 2.7. For a right R-module M, let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ with $S = \text{End}(M_R)$. Then the following are equivalent:

(1) T_T is strongly FI-extending.

(2) (i) M_R is strongly FI-extending.

(ii) R_R is strongly FI-extending.

(iii) For any $N \trianglelefteq_R M$, $\mathcal{D}(N_R)\mathcal{D}((N_R : M_R)_R) = M\mathcal{D}((N_R : M_R)_R)$.

Proof. The proof is similar to that of Corollary 1.8. □

For a ring R and a positive integer n, let $T_n(R)$ be the n-by-n upper triangular matrix ring over R.

Theorem 2.8. Assume R is a ring. Then the following are equivalent:

(1) R is right strongly FI-extending.

(2) $T_n(R)$ is right strongly FI-extending for every positive integer n.

(3) $T_n(R)$ is right strongly FI-extending for some positive integer $n > 1$.

Proof. (1)⇒(2). Assume that R is right strongly FI-extending. We proceed by induction.

Step 1. Assume $n = 2$. Then $T_2(R) = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$. Take $M = R$, then ${}_RM$ is faithful. Let ${}_RN_R \leq {}_RM_R$. Since R_R is strongly FI-extending, there exists $e = e^2 \in \mathcal{S}_\ell(R)$ such that $N_R \leq^{\text{ess}} eM_R$. Now note that $(N_R : M_R) = N_R \leq^{\text{ess}} eR_R = eM_R$. So we have that $\mathcal{D}_R(N_R)\mathcal{D}((N_R : M_R)_R) = eReR = ReR = M\mathcal{D}((N_R : M_R)_R)$. Therefore $T_2(R)$ is a right strongly FI-extending ring by Corollary 2.5.

Step 2. Assume that $T_n(R)$ is right strongly FI-extending. Then we need to show that $T_{n+1}(R)$ is right strongly FI-extending. Note that $T_{n+1}(R) = \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$, where $M = (R, R, \ldots, R)$ (n-tuple). Let ${}_RN_{T_n(R)} \leq {}_RM_{T_n(R)}$ with $N = (N_1, N_2, \ldots, N_n)$. Then $N_i \trianglelefteq R$ for each i and $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n$. Since R_R is strongly FI-extending, there exists $e \in \mathcal{S}_\ell(R)$ such that ${N_n}_R \leq^{\text{ess}} eR_R$. It can be easily checked that $N = (N_1, N_2, \ldots, N_n)_{T_n(R)} \leq^{\text{ess}} e(R, R, \ldots, R)_{T_n(R)} = eM$.

Note that

$$(N_{T_n(R)} : M_{T_n(R)}) = \begin{pmatrix} N_1 & N_2 & \cdots & N_n \\ 0 & N_2 & \cdots & N_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_n \end{pmatrix}_{T_n(R)} \leq^{\text{ess}} (eI_n)T_n(R)_{T_n(R)}$$

where I_n is the identity matrix in $T_n(R)$. Hence $\mathcal{D}_R(N_{T_n(R)})\mathcal{D}((N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}) = e(R, R, \ldots, R)(eI_n)T_n(R)$ and so we have $M\mathcal{D}((N_{T_n(R)} : M_{T_n(R)})_{T_n(R)}) = M(eI_n)T_n(R) = eM(eI_n)T_n(R) = \mathcal{D}_R(N_{T_n(R)})\mathcal{D}((N_{T_n(R)} : M_{T_n(R)})_{T_n(R)})$ because $e \in \mathcal{S}_\ell(R)$.

Next, by the induction hypothesis, $T_n(R)$ is a right strongly FI-extending ring. Therefore from Corollary 2.5, $T_{n+1}(R)$ is a right strongly FI-extending ring.

(2)⇒(3) is obvious. (3)⇒(1) is a consequence of Theorem 2.4. □

Corollary 2.9. [4, Corollary 2.5] A ring R is right FI-extending if and only if $T_n(R)$ is right FI-extending for every positive integer n if and only if $T_n(R)$ is right FI-extending for some positive integer $n > 1$.

Proof. The proof follows by using Theorem 1.4 and an argument similar to that used in the proof of Theorem 2.8. □

3. QUASI-BAER RINGS

As indicated in the introduction, for rings, the FI-extending property and the quasi-Baer property are closely linked. In fact for semiprime rings, R_R is FI-extending if and only if R_R is strongly FI-extending if and only if R is quasi-Baer [4, Theorem 4.7]. In this section, we characterize the quasi-Baer property for 2-by-2 generalized triangular matrix rings.

Lemma 3.1. Let $\begin{pmatrix} I & N \\ 0 & L \end{pmatrix} \trianglelefteq T$. Then $r_T\left(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}\right) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \mathrm{Ann}_R(N) \end{pmatrix}$.

Proof. Clearly $\begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \mathrm{Ann}_R(N) \end{pmatrix} \subseteq r_T\left(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}\right)$. Let $\begin{pmatrix} s & m \\ 0 & r \end{pmatrix} \in r_T\left(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}\right)$. Then $Is = 0$, $Lr = 0$, and $Im + Nr = 0$. Hence $s \in r_S(I)$, $r \in r_R(L) \cap \mathrm{Ann}_R(N)$, and $m \in r_M(I)$. So $r_T\left(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}\right) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \mathrm{Ann}_R(N) \end{pmatrix}$.
□

Theorem 3.2. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent:

(1) T is quasi-Baer.
(2) (i) R and S are quasi-Baer;
(ii) $r_M(I) = (r_S(I))M$ for all $I \trianglelefteq S$; and
(iii) if N is any ${}_SN_R \leq {}_SM_R$ then $\mathrm{Ann}_R(N) = aR$ for some $a = a^2 \in R$.

Proof. (1)⇒(2). By [13, p.128] R and S are quasi-Baer. Let $I \trianglelefteq S$. Then $\begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix} \trianglelefteq T$. Hence $r_T\left(\begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix}\right) = eT$, where $e \in \mathcal{S}_\ell(T)$. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$, so $eT = \begin{pmatrix} e_1S & e_1M + kR \\ 0 & e_2R \end{pmatrix}$. By Lemma 2.3, $kR = e_1kR$. Thus $e_1M = e_1M + kR$. By Lemma 3.1, $e_1S = r_S(I)$ and $r_M(I) = e_1M = e_1SM = (r_S(I))M$.

Now let ${}_SN_R \leq {}_SM_R$. Then $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \trianglelefteq T$. So $r_T\left(\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}\right) = cT$, where $c \in \mathcal{S}_\ell(T)$. Let $c = \begin{pmatrix} c_1 & h \\ 0 & c_2 \end{pmatrix}$. By Lemma 3.1, $\mathrm{Ann}_R(N) = r_R(0) \cap \mathrm{Ann}_R(N) = c_2R$. Therefore conditions (i), (ii), and (iii) are satisfied.

(2)⇒(1). Let $\begin{pmatrix} I & N \\ 0 & L \end{pmatrix} \trianglelefteq T$. Since $I \trianglelefteq S$, $L \trianglelefteq R$, and ${}_SN_R \leq {}_SM_R$, there exist $e_1 \in \mathcal{S}_\ell(S)$, $f \in \mathcal{S}_\ell(R)$, and $a = a^2 \in R$ such that $r_S(I) = e_1S$, $r_R(L) = fR$, and $\mathrm{Ann}_R(N) = aR$. Observe that since $\mathrm{Ann}_R(N) \trianglelefteq R$, then $a \in \mathcal{S}_\ell(R)$. Let $e_2 = af$. Then $af \in \mathcal{S}_\ell(R)$ and $afR = r_R(L) \cap \mathrm{Ann}_R(N)$. Let $e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$. Then $eT =$

$\begin{pmatrix} e_1S & e_1M \\ 0 & e_2R \end{pmatrix} = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \mathrm{Ann}_R(N) \end{pmatrix}$. From Lemma 3.1, $eT = r_T\left(\begin{pmatrix} I & N \\ 0 & L \end{pmatrix}\right)$. Therefore T is a quasi-Baer ring. □

Theorem 3.2 easily yields that if $R = S$ and $M \trianglelefteq R$, then T is quasi-Baer if and only if R is quasi-Baer. Observe that [4, Example 4.11] provides a 2-by-2 generalized triangular matrix ring T which is quasi-Baer, left and right nonsingular, but neither right nor left FI-extending.

Corollary 3.3. Let $S = \mathbb{Z}$. Then T is quasi-Baer if and only if
(i) R is quasi-Baer,
(ii) ${}_{\mathbb{Z}}M$ is torsion-free, and
(iii) if $N_R \leq M_R$, then $\mathrm{Ann}_R(N) = aR$ for some $a = a^2 \in R$.

One can construct examples illustrating Corollary 3.3 by taking R to be a direct sum of simple rings and M any R-module whose additive group is torsion-free.

Corollary 3.4. Let $S = \mathrm{End}(M_R)$. Then the following are equivalent:
(1) T is quasi-Baer.
(2) (i) R is quasi-Baer;
(ii) $r_M(I)$ is a direct summand of M for all $I \trianglelefteq S$; and
(iii) if ${}_SN_R \leq {}_SM_R$ then $\mathrm{Ann}_R(N) = aR$ for some $a = a^2 \in R$.

Proof. The proof follows from Theorem 3.2 and a routine argument which shows that the condition "$r_M(I)$ is a direct summand of M" is equivalent to "S is quasi-Baer and condition (ii) of Theorem 3.2." □

Corollary 3.5. Let M_R be a nonsingular FI-extending module and $S = \mathrm{End}(M_R)$. Then the following are equivalent:
(1) T is quasi-Baer.
(2) (i) R is quasi-Baer; and
(ii) for $N \trianglelefteq M, \mathrm{Ann}_R(N) = aR$ for some $a = a^2 \in R$.

Proof. (1)⇒(2). This implication follows from Theorem 3.2.
(2)⇒(1). By [5, Proposition 4.8], S is quasi-Baer. Since M_R is FI-extending and $r_M(I) \trianglelefteq M$, there exists $e = e^2 \in \mathrm{End}(M_R)$ such that $r_M(I)_R \leq^{\mathrm{ess}} eM_R$. Let $em \in eM$. There exists $L_R \leq^{\mathrm{ess}} R_R$ such that $IemL = 0$. Hence $Iem = 0$. Thus $r_M(I) = eM$. By Corollary 3.4, T is quasi-Baer. □

Examples illustrating Corollary 3.5 can be constructed by taking R to be a finite direct sum of simple rings and M any nonsingular FI-extending R-module (e.g., any fully invariant submodule of a projective R-module). For another illustrating example, take R to be a right primitive ring and M a faithful irreducible R-module. By Corollary 1.8, the above examples are at least right (and in some cases strongly) FI-extending.

4. EXAMPLES AND CONSTRUCTIONS

In this section, we provide some examples and constructions illustrating and delimiting our results in previous sections.

¿From [4, Theorem 4.7], if R is semiprime and either quasi-Baer or FI-extending, then R is strongly FI-extending. By [5, Proposition 1.5], if R_R is nonsingular and FI-extending, then R_R is strongly FI-extending. Hence one may wonder if there are any right strongly FI-extending rings R that are neither semiprime, quasi-Baer, nor right nonsingular. Our first example provides a class of such rings.

Example 4.1. Let A be a commutative principal ideal domain which is not a field. Let p be a nonzero prime in A. For $n \geq 2$, let $R = T_2(A/p^nA)$. Then: (1) R is not semiprime; (2) R is not right nonsingular; (3) R is not right extending; (4) R is not quasi-Baer; but (5) R_R is strongly FI-extending.

Clearly R is neither semiprime nor right nonsingular. Consider the right ideal

$$X = \begin{pmatrix} 0 & 1 \\ 0 & p \end{pmatrix} R.$$

Assume R is right extending. Then there exists $e = e^2 \in R$ such that $X_R \leq^{\text{ess}} eR_R$. But the only possible such e is the unity. So $X_R \leq^{\text{ess}} R_R$. But $X \cap \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} R = 0$, which is a contradiction. So R is not right extending. Since A/p^nA is commutative QF and not reduced, A/p^nA is strongly FI-extending but not quasi-Baer. By Theorems 2.8 and 3.2, the ring R is right strongly FI-extending.

By [4, Proposition 1.2], fully invariant submodules of an FI-extending submodule are FI-extending. However this does not hold for the case of strongly FI-extending modules as indicated in our next example.

Example 4.2. Let R be as in Example 4.1. Then R_R is strongly FI-extending, but R contains a nonzero ideal I such that I_R is not strongly FI-extending. Let

$$I = \begin{pmatrix} 0 & A/p^nA \\ 0 & p^{n-1}A/p^nA \end{pmatrix}.$$

Then $I \trianglelefteq R$. First we show that $\text{End}(I_R) \cong \begin{pmatrix} A/p^nA & A/p^nA \\ p^{n-1}A/p^nA & A/p^nA \end{pmatrix}$. Let $g \in \text{End}(I_R)$. Then g is completely determined by $g\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$ and $g\left[\begin{pmatrix} 0 & 0 \\ 0 & p^{n-1} \end{pmatrix}\right]$. Let $g\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & a \\ 0 & p^{n-1}b \end{pmatrix}$ and $g\left[\begin{pmatrix} 0 & 0 \\ 0 & p^{n-1} \end{pmatrix}\right] = \begin{pmatrix} 0 & p^{n-1}c \\ 0 & p^{n-1}d \end{pmatrix}$. Then it can be checked that $g(\alpha) = \begin{pmatrix} a & c \\ p^{n-1}b & d \end{pmatrix} \cdot \alpha$, for $\alpha \in I$. So $\text{End}(I_R) \cong \begin{pmatrix} A/p^nA & A/p^nA \\ p^{n-1}A/p^nA & A/p^nA \end{pmatrix}$.

Now let $J = \begin{pmatrix} 0 & p^{n-1}A/p^nA \\ 0 & 0 \end{pmatrix}$. Then $J \trianglelefteq R$ and $J \subseteq I$. It is easy to see that J is a fully invariant submodule of I_R. We show that I_R is not strongly FI-extending. Assume to the contrary that I_R is strongly FI-extending. Then since J_R is a fully invariant submodule of I_R, there exists a fully invariant R-direct summand K of I_R such that $J_R \leq^{\text{ess}} K_R$. Since K_R is a fully invariant submodule of I_R, K_R is a fully invariant submodule of R_R by [4, Proposition 1.2]. Hence $K \trianglelefteq R$. So candidates for K are of the form $\begin{pmatrix} 0 & C \\ 0 & D \end{pmatrix} I$ with $C \subseteq A/p^nA, D \subseteq p^{n-1}A/p^nA$, and $D \subseteq C$. Since $D \subseteq p^{n-1}A/p^nA$, we have the following two cases.

Case 1. $D = 0$. Then $K = \begin{pmatrix} 0 & p^kA/p^nA \\ 0 & 0 \end{pmatrix}$ where $0 \leq k \leq n$.

Case 2. $D = p^{n-1}A/p^nA$. Then $K = \begin{pmatrix} 0 & p^kA/p^nA \\ 0 & p^{n-1}A/p^nA \end{pmatrix}$, where $0 \leq k \leq n$. Since $J_R \leq^{\text{ess}} K_R$, Case 2 and the case when $K = 0$, cannot hold. Also note that $\begin{pmatrix} 0 & p^kA/p^nA \\ 0 & 0 \end{pmatrix}$, with $1 \leq k \leq n-1$, cannot be an R-direct summand of I_R. So the only possible candidate for K is $\begin{pmatrix} 0 & A/p^nA \\ 0 & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 & A/p^nA \\ 0 & 0 \end{pmatrix}$ is not a fully invariant submodule of I_R. In fact, take $g \in \text{End}(I_R)$ such that g is represented as right multiplication by $\begin{pmatrix} a & c \\ p^{n-1}b & d \end{pmatrix}$. Then $g\left[\begin{pmatrix} 0 & A/p^nA \\ 0 & 0 \end{pmatrix}\right] = \{\begin{pmatrix} 0 & ax \\ 0 & p^{n-1}bx \end{pmatrix} \mid x \in A/p^nA\}$ which may not be contained in $\begin{pmatrix} 0 & A/p^nA \\ 0 & 0 \end{pmatrix}$ by choosing $b = 1$. Therefore the fully invariant submodule I_R of the strongly FI-extending module R_R is not a strongly FI-extending module.

As in Example 4.2, let $I = \begin{pmatrix} 0 & A/p^nA \\ 0 & p^{n-1}A/p^nA \end{pmatrix}$. Then it can be seen that $\text{End}({}_RI) \cong A/p^nA$, so every left R-module homomorphism of ${}_RI$ can be represented as a right multiplication by an element in A/p^nA. Thus all fully invariant submodules of ${}_RI$ are all ideals of R contained in I. Also it can be verified that all these nonzero ideals are essential submodules of ${}_RI$. Thus ${}_RI$ is strongly FI-extending.

We also can apply our characterizations of strongly FI-extending generalized matrix rings to construct a right strongly FI-extending ring which is not left FI-extending. Thereby showing that the strongly FI-extending property is not left-right symmetric.

Example 4.3. Assume that R is a right strongly FI-extending ring (e.g., a prime ring). Let $M = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$. Then M can be considered as a left R- right $T_2(R)$-bimodule. Now we show that the generalized triangular matrix ring

$$T = \begin{pmatrix} R & M \\ 0 & T_2(R) \end{pmatrix}$$

is right strongly FI-extending, but it is not left FI-extending (hence not left strongly FI-extending). Note that ${}_RM$ is faithful. For any ${}_RN_{T_2(R)} \leq {}_RM_{T_2(R)}$, let $N = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. Then $I \trianglelefteq R$. Since R_R is strongly FI-extending, there is $e \in \mathcal{S}_\ell(R)$ such that $I_R \leq^{\text{ess}} eR_R$. Therefore we have that $N = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}_{T_2(R)} \leq^{\text{ess}} e\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}_{T_2(R)}$. Since R is right strongly FI-extending, $T_2(R)$ is also right strongly FI-extending by Theorem 2.8.

Finally, let ${}_RN_{T_2(R)} \leq {}_RM_{T_2(R)}$. Then, as before, $I = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \trianglelefteq R$ and $I_R \leq^{\text{ess}} eR_R$. Now $\mathcal{D}_R(N_{T_2(R)}) = \begin{pmatrix} 0 & eR \\ 0 & 0 \end{pmatrix} = e\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} = eM$. Also $(N_{T_2(R)} : M_{T_2(R)}) = \begin{pmatrix} R & R \\ 0 & I \end{pmatrix}_{T_2(R)} \leq^{\text{ess}} \begin{pmatrix} R & R \\ 0 & eR \end{pmatrix}_{T_2(R)}$. Observe that $\begin{pmatrix} R & R \\ 0 & eR \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} T_2(R)$ and $\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \in \mathcal{S}_\ell(T_2(R))$. So $\mathcal{D}((N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}) = \begin{pmatrix} R & R \\ 0 & eR \end{pmatrix}$. Therefore we have that

$$\mathcal{D}_R(N_{T_2(R)})\mathcal{D}((N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}) = \begin{pmatrix} 0 & eR \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R & R \\ 0 & eR \end{pmatrix} = \begin{pmatrix} 0 & eReR \\ 0 & 0 \end{pmatrix}$$

and

$$M\mathcal{D}((N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}) = \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R & R \\ 0 & eR \end{pmatrix} = \begin{pmatrix} 0 & ReR \\ 0 & 0 \end{pmatrix}.$$

Since $e \in \mathcal{S}_\ell(R), ReR = eReR$ and so it follows that $\mathcal{D}_R(N_{T_2(R)})\mathcal{D}((N_{T_2(R)} : M_{T_2(R)})_{T_2(R)}) = M\mathcal{D}((N_{T_2(R)} : M_{T_2(R)})_{T_2(R)})$. Therefore T_T is strongly FI-extending by Corollary 2.5. But note that $\text{Ann}_{T_2(R)}(M)$ is not generated, as a left ideal, by a right semicentral idempotent in $T_2(R)$. Thus ${}_TT$ is not FI-extending.

Since the quasi-Baer condition is left-right symmetric and is related to the strongly FI-extending condition, one may conjecture that a quasi-Baer right strongly FI-extending ring is left FI-extending. In Example 4.3 by taking R to be a prime ring and using Theorem 3.2, it can be seen that T is quasi-Baer and right strongly FI-extending but not left FI-extending.

In the following example, which appears in [7], there is a right self-injective and right strongly bounded (i.e., every nonzero right ideal contains a nonzero ideal) ring which is not strongly FI-extending on either side, and is not quasi-Baer.

Example 4.4. [7, Example 5.2] Let $R = \begin{pmatrix} D & S \\ 0 & Q \end{pmatrix}$, where Q is non-semisimple commutative injective regular ring, M is a maximal essential ideal of Q, $S = Q/M$ and $D = \text{End}(S_Q)$. Then R is right self-injective, right strongly bounded, $Z(R_R) \neq 0$ but $Z({}_RR) = 0$. Take $\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \trianglelefteq R$. Then $\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}_R \leq^{\text{ess}} \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}_R$ but $\begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ is not an ideal of R. So R_R is not strongly FI-extending.

On the other hand, $\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$ is not essential as a left R-submodule of R. Also it is not essential as a left R-submodule of $\begin{pmatrix} 0 & S \\ 0 & Q \end{pmatrix}$. Thus R is not left strongly FI-extending. From Corollary 3.4, R is not quasi-Baer.

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