MODULES WITH FULLY INVARIANT SUBMODULES ESSENTIAL IN FULLY INVARIANT SUMMANDS

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ABSTRACT

A module M is called (*strongly*) *FI-extending* if every fully invariant submodule is essential in a (fully invariant) direct summand. The class of strongly FI-extending modules is properly contained in the class of FI-extending modules and includes all nonsingular FI-extending (hence nonsingular extending) modules and all semiprime FI-extending rings. In this paper we examine the behavior of the class of strongly FI-extending modules with respect to the preservation of this property in submodules, direct summands, direct sums, and endomorphism rings.

Key Words: extending, FI-extending, strongly FI-extending, fully invariant, nonsingular, non-M-singular, polyform, semicentral idempotent, quasi-Baer

0. INTRODUCTION

All rings are associative and R denotes a ring with unity. The word "ideal", used without the adjectives "left" or "right", means two-sided ideal. All modules are unitary right R-modules unless indicated otherwise. Recall a submodule K of M is called *fully invariant* if $\lambda(K) \subseteq K$ for all $\lambda \in \mathrm{End}_R(M)$. Many distinguished submodules of a module are fully invariant (e.g., the Jacobson radical, the singular submodule, the socle, torsion submodules, etc.). Furthermore the fully invariant submodules of an injective module are quasi-injective. Observe that the fully invariant submodules of R_R are exactly the ideals of R. In (1), a module M is called *FI-extending* if every fully invariant submodule is essential in a direct summand of M. The FI-extending $\mathbb{Z}$-modules (i.e., Abelian groups) were investigated in (2), where $\mathbb{Z}$ is the ring of integers. Unlike the class of extending modules (M is an *extending* or *CS* module if every submodule is essential in a direct summand), the class of FI-extending modules is closed with respect to direct sums and n-by-n full and upper triangular matrix rings. Hence the FI-extending property indicates, at a minimum, "how much" of the extending property is preserved in the formation of direct sums of extending modules or of various matrix rings over (right) extending rings.

Upon examining the FI-extending property it is natural to ask: *When does a module have the property that every fully invariant submodule is essential in a fully invariant direct summand?* At first glance, this property may seem to be an inconsequential specialization of the FI-extending property. However, this property is evident for nonsingular FI-extending modules (1, Proposition 1.10), for semiprime quasi-Baer rings (3, Lemma 2.2), and was characterized for Abelian groups in (2, Theorem 7.1). A module M is called *strongly FI-extending* if every fully invariant submodule is essential in a fully invariant direct summand of M. If R_R (resp. ${}_RR$) is strongly FI-extending then we say R is *right* (resp. *left*) *strongly FI-extending.* Other examples of strongly FI-extending modules and rings are: uniform modules, semisimple modules, prime rings, semiprime right FPF rings, and semiprime right Noetherian group algebras. Moreover, if M is an FI-extending module in which the singular submodule is not essential, then M has a nontrivial strongly FI-extending direct summand. Further examples appear in the sequel.

In this paper, we investigate the class of strongly FI-extending modules. Although the class of strongly FI-extending modules and the class of extending modules are proper subclasses of the class of FI-extending modules, we show by examples that they are incomparable. However the class of strongly FI-extending modules shares certain properties with the class of extending modules which may not hold for the class of FI-extending modules. For example both the class of strongly FI-extending modules and the class of extending modules are closed under direct summands which may not be true in the class of FI-extending modules. In contrast, the classes of right FI-extending rings and right strongly FI-extending rings are closed with respect to n-by-n full and upper triangular matrix rings (see Corollary 4.3, (1), and (4)), whereas the class of right extending rings is not closed under these matrix ring constructions.

In Section 1, we prove that for a nonsingular module M (and observe that for a semiprime ring R) the following conditions are equivalent: (i) FI-extending, (ii) strongly FI-extending, (iii) every fully invariant essentially closed submodule of M (resp. R_R) is a direct summand. If a ring R is right nonsingular, we provide a characterization of the (strongly) FI-extending property for R in terms of the idempotents of its right maximal ring of quotients. Examples are provided which distinguish the classes of FI-extending, strongly FI-extending, and extending modules, respectively.

It is presently unknown whether, in general, a direct summand of an FI-extending module is FI-extending (1), (2), and (5). In Section 2, we prove that a direct summand of a strongly FI-extending module does inherit the strongly FI-extending property. In particular, a nonsingular direct summand of an FI-extending module is (strongly) FI-extending. We show that for a left and right strongly FI-extending ring R, every ideal can be embedded in a smallest ring direct summand of R by a two-step process, such that it is essential as an ideal in that ring direct summand.

Section 3 focuses on direct sums of strongly FI-extending modules. It is shown that an arbitrary direct sum of strongly FI-extending modules is not necessarily strongly FI-extending. We prove that an arbitrary direct sum M of strongly FI-extending modules M_i is strongly FI-extending, if either the M_i's are fully invariant in M or if the M_i's are isomorphic copies of each other. As a consequence, if a ring R is right strongly FI-extending, then so is every projective right R-module.

It is well known that the right extending property is not a Morita invariant property. In Section 4 we prove that the right strongly FI-extending property is a Morita invariant property. We also show that the endomorphism ring of a free strongly FI-extending module is right strongly FI-extending. Finally, we mention that since a fully invariant submodule N of an R-module M is just a Λ-R-bimodule where $\Lambda = \mathrm{End}_R(M)$, much of this theory carries over to the general theory of bimodules.

Throughout this paper, if M is an R-module and $A \subseteq M$, then we use $A \leq M$, $A \leq^{\text{ess}} M$, $A \trianglelefteq M$, and $E(M)$ to denote that A is a submodule, essential submodule, fully invariant submodule, and the injective hull of M, respectively. If $M = R$ then $A \leq_r R$ ($A \leq_\ell R$) and $A \leq_r^{\text{ess}} R$ ($A \leq_\ell^{\text{ess}} R$) denote that A is a right (left) ideal of R and that A is right (left) essential in R, respectively. For a submodule X of $M, \langle X\rangle$, X^c and $X \leq^{\oplus} M$ denote the fully invariant closure of X in M, an essential closure of X in M, and X is a direct summand of M, respectively. For $A \leq M$ and $y \in M, y^{-1}(A) = \{r \in R \mid yr \in A\}$. The singular and second singular submodules of M are denoted by $Z(M)$ and $Z_2(M)$, respectively. Let $\emptyset \neq X \subseteq R$ then $\ell_R(X)$ and $r_R(X)$ denote the left and right annihilators of X in R, respectively (we delete the subscript if the context is clear). A ring R is called (*quasi-*)

Baer if the right annihilator of every (ideal) nonempty subset is generated by an idempotent as a right ideal of R (6) and (7). Let $e = e^2 \in R$. Then e is called a *left* (*right*) *semicentral idempotent* if $xe = exe$ ($ex = exe$), for all $x \in R$ (equivalently, eR (Re) is an ideal of R) (8). The set of all left (right) semicentral idempotents is denoted by $\mathcal{S}_\ell(R)$ ($\mathcal{S}_r(R)$). Observe that $\mathbf{B}(R) = \mathcal{S}_\ell(R) \cap \mathcal{S}_r(R)$, where $\mathbf{B}(R)$ is the set of all central idempotents. A submodule N of M is *rational* (or *dense*) in M if for any $x, y \in M$ with $0 \neq x$, then $x \cdot y^{-1}(N) \neq 0$. A module M is called *polyform* (equivalently, *non-M-singular*) (9, p.78) if every essential submodule is rational in M. The class of modules which are non-M-singular properly contains the class of nonsingular modules (e.g., every singular simple module M is non-M-singular). Note if $M = R$ then non-M-singularity is equivalent to nonsingularity. We use the terms polyform or non-M-singular interchangeably, depending on which seems most appropriate. Finally $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$ denote the ring of integers, the ring of integers modulo n, and the field of rationals, respectively.

1. NONSINGULAR FI-EXTENDING MODULES

In this section we show that for nonsingular (more generally, non-M-singular) modules and for semiprime rings the FI-extending and strongly FI-extending conditions are equivalent. Moreover, we give an example of a ring R such that R_R is strongly FI-extending, $Z(R_R) \neq 0$, R is not semiprime, and R_R is not extending. Criteria are determined for $M_{\mathbb{Z}}$ to be strongly FI-extending for an FI-extending module M_R with $(M, +)$ torsionfree. For a right nonsingular ring R the FI-extending property has been characterized in terms of the idempotents of the right maximal ring of quotients $Q(R)$ of R. Examples are provided to show that the class of strongly FI-extending modules is properly contained in the class of FI-extending modules and is distinct from the class of extending modules. We begin with an example which shows that even a well behaved extending ring may not be strongly FI-extending. Recall that a ring is *strongly bounded* if every nonzero one-sided ideal contains a nonzero ideal and it is *semicentral reduced* if $\mathcal{S}_\ell(R) = \{0, 1\}$ (10).

Example 1.1. There is a finite strongly bounded QF-ring which is neither right nor left strongly FI-extending: Let $R = \mathbb{Z}_3[S_3]$, the group algebra of the symmetric group S_3 over the field $\mathbb{Z}_3$ with three elements. It can be seen that R is semicentral reduced. Let $e = 2 + \tau \in R$, where $\tau = (12) \in S_3$. Then $e^2 = e$ and eR_R is uniform. From (11), R is strongly bounded. So there exists a nonzero ideal I of R such that $I \subseteq eR$. Now if $I \leq_r^{\text{ess}} gR$ with $g \in \mathcal{S}_\ell(R)$, then $g = 1$ and so $I \leq_r^{\text{ess}} R$, a contradiction. Hence R is not right strongly FI-extending. Similarly, R is not left strongly FI-extending.

Our first three lemmas are either known or routine facts which are useful in the development of our results. We record these lemmas here for the reader's convenience.

Lemma 1.2. *Let $\Lambda = End_R(M)$ and $e = e^2 \in \Lambda$. Then*:

(i) *$e \in \mathcal{S}_\ell(\Lambda)$ if and only if $eM \trianglelefteq M$.*

(ii) *If $b, c \in \mathcal{S}_\ell(\Lambda)$, then there exists $e \in \mathcal{S}_\ell(\Lambda)$ such that $bM \cap cM = eM$.*

(iii) *If $e \in \mathcal{S}_\ell(\Lambda)$, then $Hom_R(eM, M) \subseteq Hom_R(eM, eM)$.*

Proof: (i) This part is (1, Lemma 1.9).

(ii) Set $e = bc$ then $e^2 = e \in \mathcal{S}_\ell(\Lambda)$ and $bM \cap cM = eM$.

(iii) This part follows from the definition of a semicentral idempotent. $\square$

Part (ii) of the next lemma follows from [12, 3.2].

Lemma 1.3. (i) *Let $0 \neq Y \leq M$ and $A \leq^{\text{ess}} Y$. If $y \in Y$ and $K \leq M$ such that $A \cap K = 0$, then $y^{-1}(A) = y^{-1}(A \oplus K)$.*

(ii) *If M is polyform and $X \leq M$, then X is polyform and X has a unique essential closure.*

Proof: (i) Clearly, $y^{-1}(A) \subseteq y^{-1}(A \oplus K)$. Let $s \in y^{-1}(A \oplus K)$. Then $ys = a + k$, where $a \in A$ and $k \in K$. Thus $ys - a \in Y \cap K = 0$. So $s \in y^{-1}(A)$, hence $y^{-1}(A) = y^{-1}(A \oplus K)$.

(ii) Assume to the contrary that X is not polyform. Then there exists $L \leq^{\text{ess}} X$ such that L is not rational. So there exists $0 \neq x \in X$ and $y \in X$ such that $xy^{-1}(L) = 0$. Let K be a complement of L in M. Then $L \oplus K$ is rational in M. By part (i), $xy^{-1}(L \oplus K) = 0$, a contradiction. Therefore X is polyform.

Now assume $X \leq^{\text{ess}} A$ and $X \leq^{\text{ess}} B$ such that $A \nsubseteq B$ and $B \nsubseteq A$. Let $a \in A$ and $b \in B$ such that $0 \neq a + b$. By the above argument, A and B are polyform. Hence X is rational in A and B. Therefore there exists $r \in a^{-1}(X)$ such that $0 \neq (a+b)r$ and $ar \in X$. Also there exists $s \in (br)^{-1}(X)$ such that $0 \neq (a+b)rs$ and $brs \in X$. Thus $0 \neq (a+b)rs \in X$. Hence $X \leq^{\text{ess}} A + B$. Therefore X has a unique essential closure. $\square$

Lemma 1.4. (i) *Let M be a non-M-singular module and $X \trianglelefteq M$. Then $X^c \trianglelefteq M$.*

(ii) *If M is a strongly FI-extending module and $K \trianglelefteq M$, then K is essential in a unique (fully invariant) direct summand of M.*

Proof: (i) Let $k \in X^c$ and $f \in \text{End}_R(M)$ such that $f(k) \neq 0$. Let $I = k^{-1}(X)$. Then $f(kI) = f(k)I \subseteq X$ since $X \trianglelefteq M$. Let K be a complement of X. By Lemma 1.3, $I = k^{-1}(X \oplus K)$. Since $X \oplus K$ is rational in M, $0 \neq f(k)I \subseteq X$. The uniqueness of X^c yields $f(k) \in X^c$. Therefore $X^c \trianglelefteq M$.

(ii) By Lemma 1.2(i), there exists $e \in \mathcal{S}_\ell(\Lambda)$, where $\Lambda = \text{End}_R(M)$ such that $K \leq^{\text{ess}} eM$. Assume that $c = c^2 \in \Lambda$ and $K \leq^{\text{ess}} cM$. Then $(ce)^2 = cece = ce$ and $ceK = K$. Hence $K \leq^{\text{ess}} ceM \leq cM$, so $ceM = cM$. By (13, Lemma 3.1) $ce\Lambda \leq e\Lambda$ implies that $ceM \leq eM$, therefore $cM \leq eM$. But as $K \leq^{\text{ess}} eM$, $cM = eM$. $\square$

Proposition 1.5. *Let M be a non-M-singular module, then the following conditions are equivalent:*

(i) *M is FI-extending;*

(ii) *M is strongly FI-extending;*

(iii) *every fully invariant essentially closed submodule of M is a direct summand.*

Proof: This result follows from Lemma 1.4 and the non-M-singularity of M. □

We remark that, in general, (i)⇒(iii) in Proposition 1.5 always holds true without the non-M-singular condition. Hence (i)⇒(iii) generalizes (14, Corollary 6.6). On the other hand, if M is not nonsingular then (iii)⇒(ii) may not hold true even when M is FI-extending, in general, as the next example illustrates.

Proposition 1.5 naturally motivates one to ask: *If R_R is nonsingular and M is an FI-extending R-module then is M strongly FI-extending?* Again the next example provides a negative answer to this question. It also shows that the class of strongly FI-extending $\mathbb{Z}$-modules is properly contained in the class of FI-extending $\mathbb{Z}$-modules.

Example 1.6. The module $M = \mathbb{Z} \oplus \mathbb{Z}_p$ is an FI-extending $\mathbb{Z}$-module (i.e., Abelian group) for any prime p by (1, Theorem 1.3), however, M is not strongly FI-extending by (2, Theorem 7.1).

The following example illustrates Proposition 1.5, and with Example 1.1, shows that the classes of extending modules and strongly FI-extending modules are distinct.

Example 1.7. (1, Example 2.6) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then R_R and ${}_RR$ are FI-extending but neither one is extending. Since $Z(R_R) = 0 = Z({}_RR)$, R_R and ${}_RR$ are strongly FI-extending.

Observe that from Proposition 1.5, if M is a nonsingular module which is strongly FI-extending then so is its injective hull $E(M)$. However for the ring R in Example 1.1, even though $\mathrm{Soc}(R_R)$ is strongly FI-extending (in fact, every semisimple module is strongly FI-extending), its injective hull $E(\mathrm{Soc}(R_R)) = R_R$ is not strongly FI-extending.

A monoid G is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq G$ there exists an element $x \in G$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see (15) and (16)). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid is cancellative. Especially, in (16) group algebras of a u.p.-group are extensively studied in the investigation of the zero divisor problem.

From (1, Theorem 4.7), we see that for a semiprime ring the conditions (i), (ii), and (iii) of Proposition 1.5 are equivalent. The next result provides a large class of strongly FI-extending rings.

Proposition 1.8. (i) *Let $F[G]$ be a semiprime group algebra of a group G over a field F. Then $F[G]$ is right strongly FI-extending if and only if each annihilator ideal is finitely generated. In particular, if $F[G]$ is right Noetherian, then $F[G]$ is right strongly FI-extending. If, in addition, $F[G]$ is commutative, then $F[G]$ is a finite direct sum of domains (hence $F[G]$ is an extending ring).*

(ii) *Let $R[G]$ be the monoid ring of a u.p.-monoid G over a ring R. Then R is a semiprime (right) strongly FI-extending if and only if $R[G]$ is a semiprime (right) strongly FI-extending ring.*

Proof: (i) This result is a consequence of (1, Theorem 4.7) and (17, Proposition 1.7 and Corollary 1.9).

(ii) Assume R is a semiprime (right) strongly FI-extending ring. Then R is quasi-Baer by (1, Theorem 4.7). Thus by (17, Theorem 1.2) $R[G]$ is quasi-Baer. To show that $R[G]$ is semiprime, let $\alpha = a_1g_1 + a_2g_2 + \cdots + a_ng_n \in R[G]$ with $a_i \in R$ and $g_i \in G$ such that $\alpha R[G]\alpha = 0$. Then by adopting the technique for the proof of (17, Theorem 1.2) we have $a_iRa_i = 0$, for all i. Hence $a_i = 0$ and thus $\alpha = 0$. So $R[G]$ is a semiprime ring. Therefore by (1, Theorem 4.7) $R[G]$ is (right) strongly FI-extending.

Conversely, if $R[G]$ is semiprime (right) strongly FI-extending, then R is semiprime. Thus by (1, Theorem 4.7) and (17, Theorem 1.2) R is (right) strongly FI-extending. $\square$

From (1, Theorem 4.7), the class of strongly FI-extending rings includes all semiprime right FPF rings (18, p.168). Note that Example 1.1 provides an example of a non-semiprime finite QF group algebra which is not strongly FI-extending. Also there is a semiprime Noetherian integral group ring which is not strongly FI-extending (17, Example 1.10). Observe that, in general, left and right strongly FI-extending rings need not be quasi-Baer (e.g., $\mathbb{Z}_4$). Although a right FI-extending ring which is either right nonsingular or semiprime is right strongly FI-extending, the following example provides a finite right strongly FI-extending ring, which is neither right nonsingular, nor semiprime, nor right extending, nor quasi-Baer.

Example 1.9. Let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. By (1, Corollary 2.5), R_R and ${}_RR$ are FI-extending. By considering the ideals of R, it can be seen that R_R and ${}_RR$ are strongly FI-extending. However R is not semiprime, $Z(R_R) \neq 0$, and $Z({}_RR) \neq 0$. Moreover by a routine calculation, R is not quasi-Baer and neither R_R nor ${}_RR$ are extending.

Let R be a right nonsingular ring and let $Q(R)$ denote its right maximal ring of quotients. It is well known (13, Theorem 2.8) that R is right quasi-continuous if and only if every idempotent of $Q(R)$ lies in R. We obtain the following analogue.

Theorem 1.10. *Let R be a right nonsingular ring and let $Q = Q(R)$ be the right maximal ring of quotients of R. Then R is right FI-extending if and only if for every idempotent $e = e^2 \in Q$ such that $Re = eRe$, there exists $f \in \mathcal{S}_\ell(R)$ such that $eQ = fQ$.*

Proof: Assume R is right FI-extending. Let $e = e^2 \in Q$ such that $Re = eRe$. Then $R \cap eQ$ is an ideal of R. By Lemma 1.2 and Proposition 1.5 there exists $f \in \mathcal{S}_\ell(R)$ such that $R \cap eQ \leq^{\text{ess}} fR$. Observe that $R \cap eQ$ is essentially closed in R, hence $R \cap eQ = fR$. So $fR \leq^{\text{ess}} eQ$ and therefore, $eQ = fQ$ holds.

Conversely, let A be an ideal of R. Then $A_R \leq^{\text{ess}} eQ_R$, for some $e = e^2 \in Q$. Hence there exists an essential right ideal I of R with $eI \subseteq A$. This yields $ReI \subseteq A$, and hence $(1-e)ReI = 0$. Therefore $(1-e)Re = 0$, as $Z(R_R) = 0$. Thus $Re = eRe$ holds, and so $eQ = fQ$ for some $f \in \mathcal{S}_\ell(R)$ by hypothesis. Hence $A_R \leq^{\text{ess}} fQ_R$ and so $A_R \leq^{\text{ess}} fR_R$. Therefore R_R is FI-extending. □

Proposition 1.11. *Let M be an FI-extending R-module and R a ring such that if L is an essential right ideal of R then $L \cap \mathbb{Z} \cdot 1 \neq 0$. If $(M, +)$ is torsionfree then $(M, +)$ is a strongly FI-extending Abelian group.*

Proof: Let X be a fully invariant subgroup of $(M, +)$. Then X is a fully invariant submodule of M. Hence there exists a direct summand A of M such that $X \leq^{\text{ess}} A$. Let $0 \neq a \in A$, then there is an essential right ideal I such that $Ia \subseteq X$. There exists $0 \neq n \in I \cap \mathbb{Z} \cdot 1_R$ since $(M, +)$ is torsionfree, $0 \neq na \in X$. Hence X is essential in $(A, +)$. Thus $(M, +)$ is strongly FI-extending. □

2. SUBMODULES AND DIRECT SUMMANDS

In this section we consider the question: *When is the strongly FI-extending condition inherited by fully invariant submodules or by direct summands?* From (1, Proposition 1.2) (resp. (19, Corollary 1.2)), we know that a fully invariant submodule of an FI-extending (resp. extending) module is an FI-extending (resp. extending) module. However this is not, in general, the case for strongly FI-extending modules as our next example demonstrates. This example provides a right strongly FI-extending ring R with an ideal I which is not a strongly FI-extending submodule of R_R but is a strongly FI-extending submodule of ${}_RR$.

Example 2.1. Let $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. As indicated in Example 1.9, R_R is strongly FI-extending. Consider the ideal $I = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$ of R. It can be seen that if $g \in \mathrm{Hom}(I_R, I_R) \cong \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$ then $g(\alpha) = \begin{pmatrix} a & c \\ 2b & d \end{pmatrix} \alpha$, for any $\alpha \in I$ and for some $a, b, c, d \in \mathbb{Z}_4$. Using this fact we can show that $J = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix} \trianglelefteq R$ is also a fully invariant R-submodule of I. Now if I_R were strongly FI-extending, then there is a fully invariant summand K_R of I_R such that $J_R \leq^{\text{ess}} K_R$. The only possible right R-direct summand K_R of I_R, essential over J_R, is $\begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$. However, $\begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$ is not fully invariant (as $g\begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & ax \\ 0 & 2bx \end{pmatrix} \mid x \in \mathbb{Z}_4 \right\} \not\subseteq \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$, for $b = 1 \in \mathbb{Z}_4$). Therefore, $I = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$ is not a strongly FI-extending R-module. By calculation, I is a strongly FI-extending submodule of ${}_RR$.

In view of Example 2.1, it is natural to ask: *What conditions ensure that fully invariant submodules of a strongly FI-extending are also strongly FI-extending?* The following result and corollary provide an answer.

Proposition 2.2. *Let M be a strongly FI-extending module and $X \trianglelefteq M$. If any of the following conditions is satisfied, then X is strongly FI-extending.*

(i) *X is indecomposable.*

(ii) *X is non-M-singular.*

(iii) *For any $e, g \in End_R(X)$ with $e = e^2$, there exists $\overline{g} \in End_R(M)$ such that $ge(X) = \overline{g}e(X)$.*

Proof: By (1, Proposition 1.2), X is FI-extending. Clearly, any indecomposable FI-extending module is strongly FI-extending. Condition (ii) follows from Proposition 1.5. So assume condition (iii). Let $S \trianglelefteq X$. Since $X \trianglelefteq M$, then $S \trianglelefteq M$. Hence there exists $D \trianglelefteq M$ such that $M = B \oplus D$ and $S \leq^{\text{ess}} D$. Moreover $X = (X \cap B) \oplus (X \cap D)$. There exists $e = e^2 \in \mathrm{End}_R(X)$ such that $e(X) = X \cap D$. Let $g \in \mathrm{End}_R(X)$. Then we have $\overline{g} \in \mathrm{End}_R(M)$ such that $ge(X) = \overline{g}e(X)$. Since $e(X) \trianglelefteq M$, $\overline{g}e(X) \subseteq e(X)$. Thus $X \cap D = e(X) \trianglelefteq X$, so X is strongly FI-extending. □

Corollary 2.3. *Let M be a strongly FI-extending module. If M is either non-M-singular or quasi-injective, then every fully invariant submodule of M is strongly FI-extending.*

Proof: This result is an immediate consequence of Proposition 2.2. □

At this time, the following problem is open: *Determine if a direct summand of an FI-extending module is FI-extending* (1), (2), and (5). Note that in (2, Theorem 3.2) an affirmative solution was obtained for Abelian groups. Our next results provide an affirmative answer for various cases including strongly FI-extending modules.

Theorem 2.4. *Every direct summand of a strongly FI-extending module is strongly FI-extending.*

Proof: Let M be a strongly FI-extending module and let B be a direct summand of M. Let $\Lambda = \mathrm{End}_R(M)$. Hence there exists $e^2 = e \in \Lambda$ such that $B = eM$. Let X be fully invariant submodule of $B = eM$. Then $\Lambda X \trianglelefteq M$. Since M is strongly FI-extending there exists $f^2 = f \in \mathcal{S}_\ell(\Lambda)$ such that $\Lambda X \leq^{\text{ess}} fM$. Obviously $X \subseteq \Lambda X \cap eM$. Since $X \trianglelefteq eM$, it can be seen that $\Lambda X \cap eM \subseteq X$. Hence $X = \Lambda X \cap eM \leq^{\text{ess}} fM \cap eM$. Therefore $eX \subseteq efM$, where $(ef)^2 = (ef)(ef) = e(fef) = e(ef) = ef$ and $efM \subseteq eM \cap fM$ as $f \in \mathcal{S}_\ell(\Lambda)$. Let $x \in eM \cap fM$, then there exist $m, m' \in M$ such that $x = em = fm'$. This implies that $ex = em = efm' = fefm' = fe\cdot em = fem = fx$. Thus $efM = eM \cap fM$ holds, where $(ef)^2 = ef \in H$. Hence $X = eX \leq^{\text{ess}} efM$, and efM is a direct summand of eM. Next we show that $efM \trianglelefteq eM$. It is easy to see that $\mathrm{End}_R(eM) = e\Lambda e$, where $\Lambda = \mathrm{End}_R(M)$. Now $e\Lambda e\cdot efM \subseteq e(\Lambda e(fM)) \subseteq e(fM)$ since $fM \trianglelefteq M$. Therefore efM is a fully invariant direct summand of $B = eM$, essential over X. □

Proposition 2.5. *Let M be an extending* (*resp. non-M-singular FI-extending*) *module such that every essential submodule is fully invariant. Then every submodule is extending* (*resp. strongly FI-extending*).

Proof: Let $X \leq M$. There exists $C \leq M$ such that $X \oplus C \leq^{\text{ess}} M$. Hence $X \oplus C \trianglelefteq M$. If M is extending, then (19, Corollary 1.2(ii)) yields that $X \oplus C$ is an extending module. Hence X is an extending module. If M is non-M-singular FI-extending, then by (1, Proposition 1.2), Proposition 1.5 and Theorem 2.4, X is strongly FI-extending. □

Recall from (10) that an ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in R is called a set of *left triangulating idempotents* of R if all of the following hold:

(i) $1 = b_1 + \cdots + b_n$;

(ii) $b_1 \in \mathcal{S}_\ell(R)$; and

(iii) $b_{k+1} \in \mathcal{S}_\ell(c_k R c_k)$, where $c_k = 1 - (b_1 + \cdots + b_k)$ for $1 \leq k \leq n-1$.

From part (iii) it can be shown that a set of left triangulating idempotents is a set of pairwise orthogonal idempotents. We call an element of R a *triangulating idempotent* if it is a member of some set of left triangulating idempotents of R.

As can be seen in (2), the result (2, Lemma 2.2) effectively splits the study of the FI-extending property for Abelian groups into the torsion and torsion-free cases. Using the above definition and Lemma 1.2(i), (2, Lemma 1.2) can be generalized as follows.

Proposition 2.6. *Let M be an FI-extending module and $e = e^2 \in \Lambda = End_R(M)$. If e is a triangulating idempotent of Λ, then eM and $(1-e)M$ are FI-extending.*

Proof: This result follows from an induction argument using Lemma 1.2(i), the above definitions, and (2, Lemma 1.2). □

Proposition 2.7. *If R_R is FI-extending (resp. extending), then every nonsingular cyclic module is strongly FI-extending (resp. extending).*

Proof: Assume R_R is FI-extending (the extending case is similar). Let M be a nonsingular cyclic module. Then there exists $X \leq_r R$ such that $M \cong R/X$. Let $I/X \trianglelefteq R/X$. Then $I \trianglelefteq R$, so there exists $e = e^2 \in R$ such that $I \leq^{\text{ess}} eR$. Let $0 \neq K/X \leq R/X$ such that $K/X \leq eR/X$. Let $k \in K$ such that $k \notin X$. There exists $L \leq_r^{\text{ess}} R$ such that $kL \subseteq I$. Since R/X is nonsingular, $kL \nsubseteq X$. Hence there exists $y \in L$ such that $0 \neq ky + X \in I/X \cap K/X$. So $I/X \leq^{\text{ess}} eR/X$. Therefore, by Proposition 1.5, M is strongly FI-extending. □

The next result shows that if N is a nonsingular direct summand of an FI-extending module, then N is strongly FI-extending.

Proposition 2.8. *Let M be an FI-extending module such that either $K \leq^{\oplus} M$ or $K \trianglelefteq M$. If M/K is nonsingular, then $M = Z_2(M) \oplus X \oplus Y$, where*
(i) *$Z_2(M)$ is FI-extending;*
(ii) *$K = Z_2(M) \oplus X$ is FI-extending;*
(iii) *$X \oplus Y$ is strongly FI-extending.*

Proof: Since $Z_2(M)$ is a fully invariant closed submodule of M, by Lemma 1.2(i), there exists $e \in \mathcal{S}_\ell(\text{End}_R(M))$ such that $Z_2(M) = eM$. First assume $M = K \oplus N$ for some N. Since M/K is nonsingular, $Z_2(M) \subseteq K$. Thus $K = Z_2(M) \oplus X$, where $X = (1-e)M \cap K$. By Propositions 1.5 and 2.6, $Z_2(M)$ is FI-extending and $X \oplus N$ is strongly FI-extending. So let $Y = N$.

Now assume $K \trianglelefteq M$. Then there exists a direct summand D such that $K \leq^{\text{ess}} D$. Then D/K is a singular module. Hence $D = K$. So $M = K \oplus Y$, where $M/K \cong Y$. The remainder of the proof is as above. □

Observe, from Theorem 2.4, that if M is strongly FI-extending in Proposition 2.8, then $Z_2(M)$ and K are strongly FI-extending. Our next result, which is the module version of (2, Proposition 3.1), shows that if M is an FI-extending module in which $Z(M)$ is not essential, then M has a nontrivial strongly FI-extending direct summand.

Corollary 2.9. *Let M be a module. Then M is FI-extending if and only if $M = S \oplus Z_2(M)$ where S is strongly FI-extending and $Z_2(M)$ is FI-extending.*

Proof: This result follows from Proposition 2.8 with $K = Z_2(M)$ and (1, Theorem 1.3) □

If $M = R$ in Corollary 2.9, further details are available on the decomposition in (20, Theorem 2.2).

A ring has the *IFP* (*Insertion of Factors Property*) if $ab = 0$ implies $aRb = 0$ (equivalently, $r(a) \trianglelefteq R$ for all $a \in R$) (21). Note that every reduced ring or every left or right duo ring has the IFP.

Corollary 2.10. *If R is a right FI-extending ring with IFP, then R is right strongly FI-extending and every nonsingular cyclic submodule of a projective module is projective and strongly FI-extending.*

Proof: Since every idempotent is central in a ring with IFP, R is right strongly FI-extending. Let P be a projective R-module. We can take $P \leq^{\oplus} \bigoplus_{i \in I} R_i$, where each $R_i \cong R$. Let xR be a nonsingular cyclic submodule of P. Then $xR \cong R/r(x)$. Now $r(x) = \cap_{i \in I} r(x_i)$, where the x_i are the components of x in R. Since each $r(x_i) \trianglelefteq R$, then $r(x) \trianglelefteq R$. Now Proposition 2.8 yields the result. □

In [22] Vanaja defines the *second M-singular submodule*, $Z_M^2(N)$, *of* $N \in \sigma[M]$ to be the submodule of N where $Z_M(M/Z_M(N)) = Z_M^2(N)/Z_M(N)$. It can be seen that $Z_M^2(N) \trianglelefteq N$ and $Z_M^2(N) \subseteq Z_2(N)$. From [22, Lemma 3.2], $Z_M^2(N)$ is closed in N. It now follows that Proposition 2.8, Corollary 2.9, Corollary 2.10, and the comment between Proposition 2.8 and Corollary 2.9 remain true if nonsingular, singular, $Z(M)$, and $Z_2(M)$ are replaced by non-M-singular, M-singular, $Z_M(M)$, and $Z_M^2(M)$, respectively. One advantage of this replacement can be seen in Corollary 2.9 in that it provides a "larger" strongly FI-extending direct summand S.

We say a submodule N of M is *invariantly essential* in M, denoted $N \trianglelefteq^{\text{ess}} M$, if $K \cap N \neq 0$ for all $0 \neq K \trianglelefteq M$.

Theorem 2.11. *Let M be a strongly FI-extending module such that $\Lambda = End_R(M)$ a left strongly FI-extending ring. If $K \trianglelefteq M$, then there exists $b \in \mathcal{S}_\ell(\Lambda)$ and $c \in \mathbf{B}(\Lambda)$ such that $b\Lambda \leq_\ell^{\text{ess}} c\Lambda$ and $K \leq^{\text{ess}} bM \trianglelefteq^{\text{ess}} cM$.*

Proof: By Lemma 1.2(i), there exists $b \in \mathcal{S}_\ell(\Lambda)$ such that $K \leq^{\text{ess}} bM$. Since $b\Lambda \trianglelefteq \Lambda$, there exists $c \in \mathcal{S}_r(\Lambda)$ such that $b\Lambda \leq_\ell^{\text{ess}} \Lambda c$. Then $\Lambda c \trianglelefteq \Lambda$, $(1-c)\Lambda c \trianglelefteq \Lambda$, and $\Lambda c = c\Lambda \oplus (1-c)\Lambda c$ (right ideal direct sum). So $bc = b = bcb$. But $b \in \mathcal{S}_\ell(\Lambda)$, hence $bc = b = cb$. Then $b\Lambda \subseteq c\Lambda$. So $(1-c)\Lambda c = 0$. Thus $c \in \mathbf{B}(\Lambda)$ and ${}_\Lambda b\Lambda \leq^{\text{ess}} {}_\Lambda c\Lambda$. Let $0 \neq X \trianglelefteq M$ such that $X \leq cM$ and $bM \cap X = 0$. There exists $d \in \mathcal{S}_\ell(\Lambda)$ such that $X \leq^{\text{ess}} dM$. Since $d\Lambda \trianglelefteq \Lambda$ and $d\Lambda \leq c\Lambda$, then $b\Lambda \cap d\Lambda \neq 0$. Consequently, $bM \cap dM = b\Lambda M \cap d\Lambda M \neq 0$, a contradiction. □

The following corollary generalizes a result in (23) for nonsingular FI-extending rings and shows that for a left and right strongly FI-extending ring any ideal can be "essentially" embedded in a ring direct summand. In particular this can be done for various radical ideals, the socles and the singular ideals.

Corollary 2.12. *Let R be left and right strongly FI-extending. If $I \trianglelefteq R$ then there exists $b \in \mathcal{S}_\ell(R)$, $e \in \mathcal{S}_r(R)$ and $c \in \mathbf{B}(R)$ such that $I \leq_r^{\text{ess}} bR \leq_\ell^{\text{ess}} cR, I \leq_\ell^{\text{ess}} Re \leq_r^{\text{ess}} cR$, and $I \trianglelefteq^{\text{ess}} cR$.*

Proof: The result is a consequence of Theorem 2.11. □

The following example shows that both the left and right strongly FI-extending conditions are needed in Corollary 2.12 to ensure that the two-step process essentially embeds I in a ring direct summand. Moreover it provides an example of a ring which is right strongly FI-extending but not left strongly FI-extending.

Example 2.13. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where $S = \mathbb{Z}_4$, $M = \begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. The ideals of T have the form $\begin{pmatrix} P & N \\ 0 & L \end{pmatrix}$ where $P \trianglelefteq S$, $L \trianglelefteq R$, and N is an S-R bimodule of M, satisfying $ML \subseteq N$ and $PM \subseteq N$. By a straightforward case by case calculation, we can show that every ideal of T is essential in an ideal direct summand. Hence T is a right strongly FI-extending ring. Next, let $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Then $I \leq_r^{\text{ess}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T$. Assume there exists $c \in \mathbf{B}(T)$ such that $c = \begin{pmatrix} e & h \\ 0 & g \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T \leq_\ell^{\text{ess}} cT$. Hence $e = 0$ or $1 \in \mathbb{Z}_4$

Case 1. $e = 0$. Then $c = \begin{pmatrix} 0 & h \\ 0 & g \end{pmatrix}$. Since $c \in \mathbf{B}(T)$, $\begin{pmatrix} 0 & h \\ 0 & g \end{pmatrix} \begin{pmatrix} f & m \\ 0 & r \end{pmatrix} = \begin{pmatrix} 0 & hr \\ 0 & gr \end{pmatrix}$ and $\begin{pmatrix} f & m \\ 0 & r \end{pmatrix} \begin{pmatrix} 0 & h \\ 0 & g \end{pmatrix} = \begin{pmatrix} 0 & fh + mg \\ 0 & rg \end{pmatrix}$, it follows that $g \in \mathbf{B}(R)$ and $hr = fh + mg$ for all f, m, r. In particular, if $m = 0$ and $r = 0$, then $fh = 0$ for all $f \in \mathbb{Z}_4$. So $h = 0$. Thus $mg = 0$ for all $m \in M$. Since $g \in \mathbf{B}(R)$, $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or 0. But since $Mg = 0$, $g = 0$. Therefore $c = 0$, a contradiction.

Case 2. $e = 1$. Then $c = \begin{pmatrix} 1 & h \\ 0 & g \end{pmatrix} \in \mathbf{B}(T)$. So from $\begin{pmatrix} 1 & h \\ 0 & g \end{pmatrix} \begin{pmatrix} f & m \\ 0 & r \end{pmatrix} = \begin{pmatrix} f & m + hr \\ 0 & gr \end{pmatrix}$ and $\begin{pmatrix} f & m \\ 0 & r \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & g \end{pmatrix} = \begin{pmatrix} f & fh + mg \\ 0 & rg \end{pmatrix}$, we have $g \in \mathbf{B}(R)$ and $m + hr = fh + mg$ for all f, m, r. In particular, if $m = 0$ and $r = 0$ then $fh = 0$ for all $f \in \mathbb{Z}_4$. Thus $h = 0$. So $c = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ with $g \in \mathbf{B}(R)$ and $m = mg$ for all $m \in M$. Therefore $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $g = 0$.

Hence $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and so $c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $A = \operatorname{ann}_R(M_R) = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \trianglelefteq T$ and $\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \cap \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T = 0$. So $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T$ is not left essential in T and, from Corollary 2.12, T cannot be left strongly FI-extending (in fact, T is not even left FI-extending).

3. DIRECT SUMS

Although a direct sum of FI-extending modules is FI-extending, our next example shows that a similar result fails for strongly FI-extending modules. In fact, the following example exhibits a right FI-extending ring R such that every proper right direct summand is strongly FI-extending, but the ring itself is neither right nor left strongly FI-extending. In spite of this example, we determine criteria to ensure some direct sums of strongly FI-extending modules are strongly FI-extending.

Example 3.1. Let $R = \left(\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}; \mathbb{Z} \right)$ denote the Dorroh extension of $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ by $\mathbb{Z}$. This ring is isomorphic to the ring in (24, Example 9). From (1), (3) and (24), this example has $Z(R_R) \neq 0$, $Z({}_RR) = 0$, R is strongly right bounded, R is right FI-extending, but R is neither right extending, nor quasi-Baer, nor left FI-extending. Through calculation it can be shown that every proper direct summand is strongly FI-extending, but R is not right strongly FI-extending. Let $e = e^2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right)$; then $Z(eR_R) = Z(R_R) \cap eR = 0$ because $eR = \left\{ \left(\begin{pmatrix} \bar{n} & 0 \\ 0 & 0 \end{pmatrix}, n \right) \mid n \in \mathbb{Z} \right\}$. Hence this example also illustrates Proposition 2.8.

Also observe in Example 1.1, R is QF; thus R is a right and left FI-extending ring. Moreover R is a direct sum of uniserial (hence strongly FI-extending) modules, but R is neither right nor left strongly FI-extending.

Theorem 3.2. *Let $M = \bigoplus_{i \in I} N_i$ and let $N_i \trianglelefteq M$, for all i. Then M is strongly FI-extending if and only if N_i is strongly FI-extending, for all $i \in I$.*

Proof: If M is strongly FI-extending then each N_i is so, by Theorem 2.4.

Conversely, let each N_i be strongly FI-extending. Write $N_i = e_i M$, where $e_i^2 = e_i \in \Lambda = \operatorname{End}_R(M)$. Now if $X \trianglelefteq M$, then $X = \bigoplus_{i \in I} e_i X$. It can be seen that $e_i X \trianglelefteq e_i M = N_i$, for all $i \in I$, as $(e_i \Lambda e_i)(e_i X) = e_i(\Lambda e_i X) \subseteq e_i X$, since $\Lambda e_i X \subseteq X$ as $X \trianglelefteq M$. Therefore there exists $S_i \leq^{\oplus} N_i = e_i M$, where $S_i \trianglelefteq N_i$ such that $e_i X \leq^{\text{ess}} S_i$. Thus $X = \bigoplus_{i \in I} e_i X \leq^{\text{ess}} \bigoplus_{i \in I} S_i \leq^{\oplus} M$, and $\bigoplus_{i \in I} S_i \trianglelefteq M$. $\square$

In Example 3.1, $R = Z_2(R_R) \oplus X$ where R_R is FI-extending, and $Z_2(R_R)$ and X are strongly FI-extending. Since R_R is not strongly FI-extending but $Z_2(R_R) \trianglelefteq R$, this example shows that the condition requiring that each $N_i \trianglelefteq M$ is not superfluous in Theorem 3.2.

Theorem 3.3. *Let $M = \bigoplus_{i\in I} M_i$, where $M_i \cong M_j$, and M_i is strongly FI-extending for all $i, j \in I$. Then M is strongly FI-extending.*

Proof: Let $N \trianglelefteq M$. Then $N = \bigoplus_{i\in I}(N \cap M_i)$, where $N \cap M_i \trianglelefteq M_i$. Since M_i is strongly FI-extending, we can write $M_i = e_iM_i \oplus (1 - e_i)M_i$, where $N \cap M_i \leq^{\text{ess}} e_iM_i$ and $e_i \in \mathcal{S}_\ell(\text{End}_R(M_i))$. Set σ_{ij} to be the isomorphism from M_i to M_j. Since $N \leq^{\text{ess}} \bigoplus_{i\in I} e_iM_i \leq^{\oplus} M$, to complete the proof it suffices to show that $\bigoplus_{i\in I} e_iM_i \trianglelefteq M$. Let $h \in \text{End}_R(M)$, and let $x \in \bigoplus_{i\in I} e_iM_i$. Without loss of generality we assume $x = e_im_i$ for some $i \in I$. Therefore $h(x) = h(e_im_i) = \sum_{j\in J} m'_j$, for some $J \subseteq I$, $|J| < \infty$. To show that $h(e_im_i) \in \bigoplus_{i\in I} e_iM_i$ we consider without loss of generality $\pi_jh(e_im_i) = m'_j$, and then show that $m'_j \in e_jM_j$, where $\pi_k : M \to M_k$, $k \in I$ are the natural projections. Note that $\pi_jh(e_im_i) = \pi_jh\pi_i(e_im_i)$, hence $\sigma_{ij}^{-1}(\pi_jh\pi_i)(e_im_i) = \sigma_{ij}^{-1}(m'_j)$. The map $(\sigma_{ij}^{-1}\pi_jh\pi_i)|_{M_i} \in \text{End}_R(M_i)$ and $e_i \in \mathcal{S}_\ell(\text{End}_R(M_i))$. Hence $(e_i\sigma_{ij}^{-1}\pi_jh\pi_i)(e_im_i) = \sigma_{ij}^{-1}(m'_j)$, where $e_i(\sigma_{ij}^{-1}\pi_jh\pi_i)(e_im_i) \in e_iM_i$. Thus $\sigma_{ij}(e_i\sigma_{ij}^{-1}\pi_jh\pi_i)(e_im_i) = m'_j$. Now, as $\sigma_{ij} : M_i \to M_j$ is an isomorphism, with $N \cap M_i \cong N \cap M_j$ under σ_{ij} and since each e_iM_i is a unique direct summand essential over $N \cap M_i$ (Lemma 1.4(ii)) it follows that $e_iM_i \cong e_jM_j$ under σ_{ij} also. Hence $\sigma_{ij}(e_i\sigma_{ij}^{-1}\pi_jh\pi_i)(e_im_i) = m'_j \in e_jM_j$. As $e_i \in \mathcal{S}_\ell(\text{End}_R(M_i))$, we obtain $(\sigma_{ij}\sigma_{ij}^{-1}\pi_jh\pi_i)(e_im_i) = (\pi_jh\pi_i)(e_im_i) = m'_j \in e_jM_j$. This shows that $\bigoplus_{i\in I} e_iM_i$ is a fully invariant direct summand of M. $\square$

Corollary 3.4. *Assume R_R is strongly FI-extending. Then every projective right R-module is strongly FI-extending.*

Proof: This result follows from Theorems 2.4 and 3.3. $\square$

In (25, Example 12.20) an example of a commutative continuous (hence extending and strongly FI-extending) regular ring is provided having a finitely generated free module F which is not extending. However, by Corollary 3.4, F is strongly FI-extending.

Corollary 3.5. *Let R_R be FI-extending. If R is (semi-) hereditary, then every (finitely generated) submodule of a projective module is strongly FI-extending.*

4. MORITA INVARIANCE AND ENDOMORPHISM RINGS

In this section, we investigate the endomorphism rings of strongly FI-extending modules. We are able to show that the strongly FI-extending property is a Morita invariant, unlike the case for the extending property. In addition, we are able to show that the endomorphism ring of a nonsingular FI-extending module is quasi-Baer. This is somewhat reminiscent of the fact that the endomorphism ring of a nonsingular quasi-injective module is a right selfinjective regular ring (hence a Baer ring).

Lemma 4.1. *Let M_R be a generator in the category Mod-R of right R-modules. Let $\Lambda = End_R(M)$ and let A, B be right ideals of Λ. Then $(A \cap B)M = AM \cap BM$.*

Proof: This proof follows from the proof of (26, Theorem 1.3(2)). □

Theorem 4.2. *Let R be a right strongly FI-extending ring. Then for any projective generator P in Mod-R, $End_R(P)$ is a right strongly FI-extending ring.*

Proof: By Corollary 3.4, P is strongly FI-extending. Let $\Gamma = \mathrm{End}_R(P)$ and I be an ideal of Γ. Then IP is a fully invariant R-submodule of P. Since P is strongly FI-extending, Lemma 1.2(i) yields $e \in \mathcal{S}_\ell(\Gamma)$ such that $IP \leq^{\text{ess}} eP = e\Gamma P$. We show that $I \leq^{\text{ess}} e\Gamma$. For $0 \neq e\gamma \in e\Gamma$ with $\gamma \in \Gamma$, assume to the contrary that $I \cap e\gamma\Gamma = 0$. Then $0 = (I \cap e\gamma\Gamma)P = IP \cap e\gamma\Gamma P = IP \cap e\gamma P$ by Lemma 4.1. But since $0 \neq e\gamma P \subseteq eP$ and $IP \leq^{\text{ess}} eP$, we have a contradiction. Thus $I \leq^{\text{ess}} e\Gamma$ and $e\Gamma \trianglelefteq \Gamma$. Therefore Γ is a right strongly FI-extending ring. □

Corollary 4.3. *The right strongly FI-extending property is a Morita invariant property.*

The next lemma from (27, Lemma 3.4) is a consequence of (26, Theorem 1.3).

Lemma 4.4. *Let F be a free right R-module and set $\Lambda = End_R(F)$. Let U and V be right ideals of Λ such that $U \subseteq V$. Then U is an essential right Λ-submodule of V if and only if UF is an essential R-submodule of VF. Also, Λ is a right nonsingular ring if F is a nonsingular R-module.*

Lemma 4.5. *Let M be an R-module, and $\Lambda = End_R(M)$. Then for any ideal I of Λ, there exists a fully invariant submodule $IM \trianglelefteq M$. Conversely, for every fully invariant submodule K of M, there exists an ideal $L = \{s \in \Lambda \mid sM \subseteq K\} \trianglelefteq \Lambda$.*

Proof: Let $I \trianglelefteq \Lambda$. Then $IM \subseteq M$ and $IM \trianglelefteq M$ since for any $h \in \Lambda$, $h(IM) = (hI)(M) \subseteq IM$. Conversely, let $K \trianglelefteq M$. Set $L = \{s \in \Lambda \mid sM \subseteq K\}$. It can be seen that $Lt \subseteq L$ and $tL \subseteq L$ for any $t \in \Lambda$. Therefore $L \trianglelefteq \Lambda$. □

Using Lemma 4.5 we can show that the endomorphism ring of an (strongly) FI-extending free module is (strongly) FI-extending.

Theorem 4.6. *Let M be a free R-module and $\Lambda = End_R(M)$. If M is an FI-extending or a strongly FI-extending module then so is Λ_Λ.*

Proof: Let M be an FI-extending module and let $I \trianglelefteq \Lambda$. Then by Lemma 4.5 $IM \trianglelefteq M$. There exists $e = e^2 \in \Lambda$ such that $IM \leq^{\text{ess}} eM = e\Lambda M$. Using Lemma 4.4, it follows that $I \leq^{\text{ess}} e\Lambda$. This shows that Λ is a right FI-extending ring. The case for M strongly FI-extending follows from Theorems 2.4 and 4.2. □

Corollary 4.7. *If R_R is (strongly) FI-extending then so is Λ_Λ, where $\Lambda = End_R(F)$ and F is a free R-module.*

Proof: This result is a consequence of (1, Theorem 1.3), Corollary 3.4, and Theorem 4.6. □

We note that from (28, Exercise 8, p.220), the endomorphism ring of a torsionfree Abelian group of rank $\mathfrak{m}$ can be represented by column-finite $\mathfrak{m}$-by-$\mathfrak{m}$ matrices over $\mathbb{Q}$. Hence, by Corollary 4.7, these endomorphism rings are right strongly FI-extending.

Proposition 4.8. *Let M be a non-M-singular FI-extending module. Then $End_R(M)$ is a quasi-Baer ring.*

Proof: Let $\Lambda = \text{End}_R(M)$ and $I \trianglelefteq \Lambda$. Since $IM \trianglelefteq M$, there exists $e \in \mathcal{S}_\ell(\Lambda)$ such that $IM \leq^{\text{ess}} eM$ and $I \subseteq e\Lambda$. Clearly $\Lambda(1-e) \subseteq \ell_\Lambda(I)$. Let $\alpha \in \ell_\Lambda(I)$ and $em \in eM$. Then $\alpha em(em)^{-1}(IM) = 0$. Since M is non-M-singular, $\alpha em = 0$. So $\alpha eM = 0$. Hence $\alpha \in \ell_\Lambda(e) = \Lambda(1-e)$. Therefore $\ell_\Lambda(I) = \Lambda(1-e)$. □

Open Problems. (i) Characterize the classes of rings satisfying the condition that every (cyclic, finitely generated, projective, etc.) module is FI-extending.

(ii) Characterize the classes of rings satisfying the condition that every (cyclic, finitely generated, projective, etc.) module is strongly FI-extending.

(iii) Characterize the classes of rings such that every (finite) direct sum of strongly FI-extending modules is strongly FI-extending.

(iv) For various classes of rings describe all (cyclic, finitely generated, etc.) modules which are FI-extending or strongly FI-extending (see (2, Theorem 7.1) for a description of the strongly FI-extending Abelian groups).

(v) Characterize the classes of rings for which every (principal, finitely generated, etc.) right ideal is FI-extending or every (principal, finitely generated, etc.) right ideal is strongly FI-extending.

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