MODULES IN WHICH EVERY FULLY INVARIANT SUBMODULE IS ESSENTIAL IN A DIRECT SUMMAND

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Abstract

A module M is called *extending* if every submodule of M is essential in a direct summand. We call a module *FI-extending* if every fully invariant submodule is essential in a direct summand. Initially we develop basic properties in the general module setting. For example, in contrast to extending modules, a direct sum of FI-extending modules is FI-extending. Later we largely focus on the specific case when a ring is FI-extending (considered as a module over itself). Again, unlike the extending property, the FI-extending property is shown to carry over to matrix rings. Several results on ring direct decompositions of FI-extending rings are obtained, including a proper generalization of a result of C. Faith on the splitting-off of the maximal regular ideal in a continuous ring.

Key Words: extending, FI-extending, fully invariant, quasi-Baer, semicentral idempotent, nonsingular.

INTRODUCTION

In recent years the theory of extending modules and rings has come to play an important role in the theory of rings and modules. Recall a module M is called an *extending* (also known as a CS) module if every submodule of M is essential in a direct summand. Although this generalization of injectivity is extremely useful, it does not satisfy some important properties. For example, direct sums of extending modules are not necessarily extending, and full or upper triangular matrix rings over right extending rings are not necessarily right extending. Much work has been done on finding necessary and sufficient conditions to ensure that the extending property is preserved under various extensions. (cf., [1]).

Another way to look at this problem is to ask: *How much of the extending property is preserved in direct sums of extending modules or in matrix rings over right extending rings (without adding any additional conditions)?* In this paper, we determine a generalization of the extending property which is not only preserved under various extensions including direct sums and several matrix constructions, but it is an interesting topic to investigate in its own right. We say a module is *FI-extending* if every fully invariant submodule is essential in a direct summand. One advantage of this generalization of the extending property over various other generalizations is that the underpinnings (i.e., the fully invariant submodules) form a complete modular sublattice of the lattice of submodules and are well behaved with respect to endomorphisms. The class of fully invariant submodules includes many of the most significant submodules of a module (e.g., the Jacobson radical, the socle, the singular submodule, etc.). Moreover, the lattice connection naturally follows the lattice theoretic view that was originally indicated in von Neumann's formulation of continuous geometries [2] and Utumi's formulation of continuous (regular) rings [3].

Observe that when R is considered as a right R-module, then the two-sided ideals of R are exactly the fully invariant submodules of R_R. Hence we say R is a *right (left) FI-extending* ring if every ideal is right (left) essential in an idempotent generated right (left) ideal of R. Observe that every prime ring is right and left FI-extending. This condition has been previously considered in [4], [5], [6], and [7]. Many unpublished results from [7] are included in this paper. Recently, motivated by an early version of this paper, Birkenmeier, Călugăreanu, Fuchs and Goeters [8] investigated the FI-extending property for Abelian groups.

After providing preliminaries and examples, in Section 1 we prove basic results and properties of FI-extending modules. It is shown that any direct sum of FI-extending modules is FI-extending without any additional assumptions (Theorem 1.3) and that a direct product of FI- extending modules need not be FI-extending (Example 1.6). In Section 2, we consider matrix constructions for FI-extending rings. We prove that a ring R is right FI-extending if and only if the upper triangular matrix ring is so (Proposition 2.2). Moreover it is shown that the FI-extending property of a ring R carries over to the full matrix ring $M_n(R)$, $n \geq 1$. The focus in Section 3 is on rings which are FI-extending on both sides. Ring direct decompositions and generalized triangular matrix representations are obtained for these rings. In [9], C. Faith has shown that the maximal regular ideal is a ring direct summand in a two-sided continuous ring. As an application of the theory of FI-extending modules, we obtain a proper generalization of this main result of [9] by replacing the continuity of the ring on one side by the FI-extending property (Theorem 3.6). Our theory also enables us to obtain generalizations of other results in [9]. Recall from [10] and [11], a ring R is *(quasi-) Baer* if the right annihilator of every (ideal) nonempty subset S is generated, as a right ideal, by an idempotent. In Section 4, we investigate the interconnections between the FI-extending condition and various related conditions (e.g., extending, quasi-Baer, quasicontinuous, etc.). For rings, these connections become tighter in the presence of the semiprime, right nonsingular, and/or complement bounded conditions. Some results are reminiscent of a well known result of Chatters and Khuri ([12], Theorem 2.1) which illustrates the relation between the right extending and Baer conditions in a right nonsingular ring. Finally, in Section 5, we obtain a ring decomposition for a right nonsingular right FI-extending ring using the concept of an orthogonal pair of module classes.

Throughout this paper all rings are associative and R will always denote a ring with unity. Modules are unital right R-modules unless indicated otherwise. Recall that a submodule X of M is called *fully invariant* if for every $h \in \mathrm{End}_R(M)$, $h(X) \subseteq X$. If M is an R-module and $A \subseteq M$, then we use $A \leq M$, $A \leq^e M$, $A \lhd M$, and $E(M)$ to denote that A is a submodule, essential submodule, fully invariant submodule, and the injective hull of M, respectively. If $M = R$ then $A \leq_r R$ ($A \leq_l R$) and $A \leq_r^e R$ ($A \leq_l^e R$) denote that A is a right (left) ideal of R and that A is right (left) essential in R, respectively. The singular and second singular submodules of M are denoted by $Z(M)$ and $Z_2(M)$, respectively. If $\emptyset \neq X \subseteq R$ then $l(X)$ and $r(X)$ denote the left and right annihilators of X in R, respectively. Let $e = e^2 \in R$. Then e is called a *left (right) semicentral* idempotent if $xe = exe$ ($ex = exe$), for all $x \in R$ [13]. The set of all left (right) semicentral idempotents is denoted by $\mathcal{S}_l(R)$ ($\mathcal{S}_r(R)$). Also, we use $\mathcal{B}(R)$ for the set of central idempotents. Observe $\mathcal{B}(R) = \mathcal{S}_l(R) \cap \mathcal{S}_r(R)$. Finally $\mathcal{P}(R), \mathcal{J}(R), \mathcal{M}(R),\ M_n(R)$, and $T_n(R)$, denote the prime radical of R, the Jacobson radical, the maximal regular ideal, the full ring of n-by-n matrices over R, and the ring of n-by-n upper triangular matrices, respectively.

1. FI-EXTENDING MODULES

Since fully invariant submodules are crucial to the development of our theory, we begin this section by recording some basic facts about them.

LEMMA 1.1 Let M be a module.

(i) Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).

(ii) If $X \leq Y \leq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.

(iii) If $M = \bigoplus_{i \in I} X_i$ and S is a fully invariant submodule of M, then $S = \bigoplus_{i \in I} \pi_i(S) = \oplus(X_i \cap S)$, where π_i is the i-th projection homomorphism of M.

PROPOSITION 1.2 Let M be a module and X a fully invariant submodule of M. If M is FI-extending, then X is FI-extending.

Proof. Assume M is an FI-extending module. Let S be a fully invariant submodule of X. By Lemma 1.1 (ii), S is fully invariant in M. Hence there is a direct summand D of M such that $S \leq^e D$. Let $\pi : M \to D$ be the projection endomorphism. Then

$$S = \pi(S) \leq \pi(X) \cap D = \pi(X).$$

Hence $S \leq^e \pi(X)$ and $\pi(X)$ is a direct summand of X. □

THEOREM 1.3 Let $M = \underset{i \in I}{\oplus} X_i$. If each X_i is an FI-extending module, then M is an FI-extending module.

Proof. Assume each X_i is an FI-extending module, and S is a fully invariant submodule of M. Since, for each i such that $\pi_i(S) \neq 0$, $\pi_i(S)$ is a fully invariant submodule of X_i there exists D_i, a direct summand of X_i, such that $\pi_i(S) \leq^e D_i$. Using Lemma 1.1 (iii), $S = \oplus \pi_i(S) \leq^e \oplus D_i$. Since $\oplus D_i$ is a direct summand of M, we have that M is an FI-extending module. □

COROLLARY 1.4 If M is a direct sum of extending (e.g., uniform) modules, then M is FI-extending.

Observe that every module with finite uniform dimension is an essential extension of an FI-extending module and is essential in an FI-extending module (e.g., its injective hull). Applying Theorem 1.3 to Abelian groups, we obtain the next corollary.

COROLLARY 1.5 Let M be a $\mathbb{Z}$-module (i.e., an Abelian group). If M satisfies any of the following conditions, then M is an FI-extending $\mathbb{Z}$-module.

(i) M is finitely generated.

(ii) M is of bounded order (i.e., $nM = 0$, for some positive integer n).

(iii) M is divisible.

Proof. (i) Every finitely generated Abelian group is a direct sum of uniform $\mathbb{Z}$-modules.

(ii) By ([14], p.262), M is a direct sum of cyclic torsion groups. Hence M is again a direct sum of uniform $\mathbb{Z}$-modules.

(iii) Since M is divisible it is an extending $\mathbb{Z}$-module. □

EXAMPLE 1.6 There exists a $\mathbb{Z}$-module M such that M is *not* FI-extending. Let

$$M = \prod_{p \in P} \mathbb{Z}/(p) \text{ (where } p \text{ varies through all primes).}$$

The torsion subgroup, τM, is fully invariant and essentially closed. By ([14], p.244), τM is not a direct summand of M. Hence M is *not* FI-extending. This example also shows that a direct product of FI-extending modules need *not* be an extending module.

Further results on FI-extending Abelian groups appear in [8]. The next two results establish connections between an FI-extending module and its injective hull.

PROPOSITION 1.7 Let M be a module. Then M is FI-extending if and only if for each fully invariant submodule S of M there exists $e = e^2 \in End_R(E(M))$ such that $S \leq^e e(E(M))$ and $e(M) \subseteq M$.

Proof. ($\Rightarrow$) Assume M is FI-extending. There exists a direct summand X of M such that $S \leq^e X$, and $Y \leq M$ such that $M = X \oplus Y$. Hence there exist injective hulls $E(X)$ and $E(Y)$ such that $E(M) = E(X) \oplus E(Y)$. Let $e : E(M) \to E(X)$ be the projection endomorphism. Then $e(M) \leq M$ and $S \leq^e e(E(M))$.

($\Leftarrow$) Conversely, let S be a fully invariant submodule of M. Then $S \leq^e M \cap e(E(M) = e(M)$. But $e(M)$ is a direct summand of M. Hence M is FI-extending. $\square$

PROPOSITION 1.8 Let M be FI-extending and $S = M \cap I$, where I is a fully invariant direct summand of $E(M)$. Then S is a fully invariant direct summand of M.

Proof. Let $f \in End_R(M)$. There exists $\overline{f} \in End_R(E(M))$ which extends f. Let $s \in S$, then $f(s) \in M$ and $f(s) = \overline{f}(s) \in I$. So $f(s) \in S$. Hence S is fully invariant in M. Since M is FI-extending, there exists a direct summand X of M such that $S \leq^e X$. Now $E(S) = I$ and $E(X) \simeq I$. Since I is fully invariant $E(X) = I$. Hence $X \subseteq M \cap E(X) = M \cap I = S$. So $S = X$. $\square$

LEMMA 1.9 Let $H = End_R M$ and $e = e^2 \in H$.

(i) For $A \lhd M$, $(eM + A) \lhd M$ if and only if $(1-e)H(eM) \subseteq A$.

(ii) $eM \lhd M$ if and only if $e \in \mathcal{S}_l(H)$.

Proof. (i) ($\Rightarrow$) Let $t \in (1-e)He$ and $m \in M$. Then $tm = tem \in eM + A$. But $tm = (1-e)tm \in (1-e)A \leq A$.

($\Leftarrow$) Let $h \in H$. Then $h(eM) = (eh+(1-e)h)eM \subseteq eM+(1-e)HeM \subseteq eM+A$. Thus $eM + A \lhd M$.

(ii) ($\Rightarrow$) Let $h \in H$ and $m \in M$. Then there exists $k \in M$ such that $hem = ek$. Hence $ehem = e^2k = ek = hem$. So $e \in \mathcal{S}_l(H)$.

($\Leftarrow$) Let $h \in H$ and $m \in M$. Then $hem = ehem \in eM$. So $eM \lhd M$. $\square$

In the following result we consider the behavior of direct summands of M which are essential extensions of fully invariant submodules of M.

PROPOSITION 1.10 Let $H = End_R M$, $e = e^2 \in H$, and $A \lhd M$ such that $A \leq^e eM$. Then:

(i) $(1-e)H(eM) \subseteq Z(M)$.

(ii) $eM + Z(M) \lhd M$.

(iii) If $Z_2(M) = fM$ for some $f = f^2$, then $eM + Z_2(M) = (e + f - fe)M \lhd M$ and $e + f - fe \in \mathcal{S}_l(H)$.

(iv) If $Z(M) \subseteq eM$, then $eM \lhd M$. Moreover, if $A \leq^e X$, then $X \subseteq eM$. In particular, $Z_2(M) \subseteq eM$.

Proof. (i) Let $m \in M$. There exists an essential right ideal L of R such that $emL \subseteq A$. Then $(1-e)HemL \subseteq eM \cap (1-e)M = 0$. So $(1-e)H(eM) \subseteq Z(M)$.

(ii) This part is a consequence of part (i) and Lemma 1.9 (i).

(iii) Let $em + fk \in eM + Z_2(M)$, for some $m, k \in M$. Since $Z_2(M) \lhd M$, Lemma 1.9 (ii) yields $f \in \mathcal{S}_l(H)$. So $ef = fef$. Then $(e+f-fe)^2 = e+f-fe$ and $(e+f-fe)(em+fk) = em+fk$. Hence $(e+f-fe)M = eM+Z_2(M)$. From Lemma 1.9 (i) and part (i), $eM + Z_2(M) \lhd M$. By Lemma 1.9 (ii), $e + f - fe \in \mathcal{S}_l(H)$.

(iv) By part (ii), $eM \lhd M$. Let $x \in X$. Then there exists an essential right ideal L of R such that $xL \subseteq A$. Then $(1-e)x \in Z(M) \subseteq eM$. But $x = ex + (1-e)x \in eM$. Thus $X \leq eM$. □

The next result shows that every right FI-extending ring has a maximal nonsingular FI-extending direct summand which is also a right nonsingular right FI-extending ring.

PROPOSITION 1.11 A ring R is right FI-extending if and only if $R = S \oplus Z_2(R_R)$ (right ideal decomposition) with S and $Z_2(R_R)$ FI-extending R-modules.

Proof. Assume R is right FI-extending. From ([5],Theorem 2.2), $R = S \oplus Z_2(R_R)$ (right ideal decomposition), where S is a right FI-extending ring. Since $S = eR$, where $e \in \mathcal{S}_r(R)$, every S-endomorphism is a R-homomorphism on S. Hence every fully invariant R-submodule of S is a fully invariant S-submodule of S. Thus S is an FI-extending R-module. By Proposition 1.2, $Z_2(R_R)$ is an FI-extending R-module. The converse follows from Theorem 1.3. □

2. FI-EXTENDING MATRIX RINGS

LEMMA 2.1 For an idempotent e of R, the following conditions are equivalent:

(i) $e \in \mathcal{S}_l(R)$;

(ii) $1 - e \in \mathcal{S}_r(R)$;

(iii) $Re = eRe$;

(iv) $(1-e)Re = 0$;

(v) eR is an ideal of R;

(vi) $eR(1-e)$ is an ideal of R and $eR = eR(1-e) \oplus Re$, as a direct sum of left ideals.

Proof. This proof is routine. □

LEMMA 2.2 Let X be a right ideal of R and $e = e^2 \in R$ such that $X \leq^e eR$. Then $M_n(X) \leq^e dM_n(R)$, where d is the diagonal n-by-n matrix with e in all the diagonal positions.

Proof. The proof involves a case-by-case verification as is illustrated in the following proof for $n = 2$. Let a, b, c, $d \in R$ such that $0 \neq \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix} \in dM_2(R)$.

Case 1. Assume $ea \neq 0$. Then there exists $s \in R$ such that $0 \neq eas \in X$. If $ecs = 0$, then $0 \neq \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \in M_2(X)$. If $ecs \neq 0$, there exists $\alpha \in R$ such that $0 \neq ecs\alpha \in X$. Hence $0 \neq \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix} \begin{bmatrix} s\alpha & 0 \\ 0 & 0 \end{bmatrix} \in M_2(X)$.

Case 2. Assume $ea = 0$ but $eb \neq 0$. Then there exists $u \in R$ such that $0 \neq ebu \in X$. If $edu = 0$, then $0 \neq \begin{bmatrix} 0 & eb \\ ec & ed \end{bmatrix} \begin{bmatrix} 0 & 0 \\ u & 0 \end{bmatrix} \in M_2(X)$. If $edu \neq 0$, there exists $\beta \in R$ such that $0 \neq edu\beta \in X$. Hence $0 \neq \begin{bmatrix} 0 & eb \\ ec & ed \end{bmatrix} \begin{bmatrix} 0 & 0 \\ u\beta & 0 \end{bmatrix} \in M_2(X)$.

Case 3. Assume $ea = eb = 0$, but $ec \neq 0$. Then there exists $t \in R$ such that $0 \neq ect \in X$. Hence $0 \neq \begin{bmatrix} 0 & 0 \\ ec & ed \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} \in M_2(X)$.

Case 4. Assume $ea = eb = ec = 0$, but $ed \neq 0$. Then there exists $v \in R$ such that $0 \neq edv \in X$. Hence $0 \neq \begin{bmatrix} 0 & 0 \\ 0 & ed \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \in M_2(X)$.

The above cases show that $M_2(X) \leq^e dM_2(R)$. The proof for $n > 2$ follows a similar pattern. □

PROPOSITION 2.3 If R is right FI-extending, then $M_n(R)$ is right FI-extending, for all positive integers n.

Proof. Let $I \lhd M_n(R)$. There exists $X \lhd R$ such that $I = M_n(X)$. Also there exists $e = e^2 \in R$ such that $X \leq^e eR$. Now the result is an immediate consequence of Lemma 2.2. □

Example 4.5 shows that a full matrix ring of a right extending ring is not necessarily right extending. At this time, the authors do not know if the right FI-extending property is a Morita invariant property and so this question is open.

THEOREM 2.4 Let M be an H-R-bimodule where $H = End_R(M)$. Then R_R and M_R are FI-extending if and only if $T = \begin{bmatrix} H & M \\ 0 & R \end{bmatrix}$ is right FI-extending.

Proof. ($\Rightarrow$) Assume R_R and M_R are FI-extending and $0 \neq I \lhd T$. Then there exists $X \lhd H$, $Y \lhd R$, and K a H-R submodule of M such that $MY \subseteq K$, $XM \subseteq K$, and $I = \begin{bmatrix} X & K \\ 0 & Y \end{bmatrix}$. Since K is a H-R submodule of M, then K is fully invariant and there exists $e = e^2 \in H$ such that $K \leq^e eM$. Also there exists $c = c^2 \in R$ such that $Y \leq^e cR$.

We claim that $I \leq^e \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T$. Let $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \in \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix} T$. Consider the following cases:

Case 1. Assume $em \neq 0$. There exists $b \in R$ such that $0 \neq emb \in K$. If $crb = 0$, then $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & emb \\ 0 & 0 \end{bmatrix} \in I$.

If $crb \neq 0$, then there exists $d \in R$ such that $0 \neq crbd \in Y$. Hence $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & bd \end{bmatrix} = \begin{bmatrix} 0 & embd \\ 0 & crbd \end{bmatrix} \in I$.

Case 2. Assume $eh \neq 0$. Then there exists $n \in M$ such that $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ehn \\ 0 & 0 \end{bmatrix}$. Now Case 1 can be used to show $0 \neq \left(\begin{bmatrix} 0 & ehn \\ 0 & 0 \end{bmatrix} T \right) \cap I$.

Case 3. Assume $cr \neq 0$, but $em = 0$. There exists $d \in R$ such that $0 \neq crd \in Y$. Then $0 \neq \begin{bmatrix} eh & em \\ 0 & cr \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & crd \end{bmatrix} \in I$. Since $\begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix}^2 = \begin{bmatrix} e & 0 \\ 0 & c \end{bmatrix}$, T is right FI-extending.

($\Leftarrow$) Conversely assume T is right FI-extending. Observe $R \simeq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{S}_r(T)$. By [5, Lemma 2.1], R is right FI-extending. Now let $0 \neq K$ be a fully invariant submodule of M. Then $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \lhd T$. Hence there exists $e = e^2 \in H$ and $n \in M$ such that $en = n$ and $\begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix} \leq^e \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} T$. Let $0 \neq x \in e(M)$. Then $0 \neq \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} e & n \\ 0 & 0 \end{bmatrix} T$. Hence there exists $\begin{bmatrix} h & m \\ 0 & r \end{bmatrix} \in T$ such that $0 \neq \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h & m \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & xr \\ 0 & 0 \end{bmatrix} \in K$. Therefore $K \leq^e e(M)$. So M_R is FI-extending. □

COROLLARY 2.5 R is right FI-extending if and only if $T_n(R)$ is right FI-extending, for any $n > 1$.

Proof. For $n = 2$, let $M = R$ in Theorem 2.4. Then $H = R$. For $n > 2$ a tedious but straightforward proof can be made by induction on the proof of Theorem 2.4. □

EXAMPLE 2.6 Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. By Corollary 2.5, R is right FI-extending. We claim that R is neither right nor left extending. Assume to the contrary that R is either right or left extending. Since R is both right and left nonsingular, ([12],Theorem 2.1) yields that R is a Baer ring. This is a contradiction ([11], p.16, Exercise 2).

3. TWO-SIDED FI-EXTENDING RINGS

In this section we consider ring decompositions and matrix representations of rings which are FI-extending on both sides. For any given ideal A of such a ring, we provide ring decompositions, such that the ideal is essential (on both sides) in one of the ring direct summands under additional conditions. Applications include several specific cases when the ideal is itself a ring direct summand. An upper triangular matrix representation for a two-sided FI-extending ring is provided.

LEMMA 3.1 Let A be an ideal of R and $e \in \mathcal{S}_l(R)$ such that $A \leq_r^e eR$. If $l(A) \cap A = 0$, then $l(A) = R(1-e)$.

Proof. Clearly $R(1-e) \subseteq l(A)$. By Lemma 2.1 (v), $eR \lhd R$ so $l(A) \cap eR = 0$. From Lemma 2.1 (vi), $eR = Re \oplus eR(1-e)$. Hence $l(A) \cap Re = 0$. Thus $l(A) = R(1-e)$. □

THEOREM 3.2 Let A be an ideal of R.

(i) If R is right FI-extending and $A \cap l(A) = 0$, then there exists $e \in \mathcal{S}_r(R)$ such that $A \leq_r^e eR$.

(ii) If R is left FI-extending and $A \cap r(A) = 0$, then there exists $f \in \mathcal{S}_l(R)$ such that $A \leq_l^e Rf$.

(iii) If R is left and right FI-extending, $A \cap l(A) = 0$, and $A \cap r(A) = 0$ then there exists $c \in \mathcal{B}(R)$ such that $A \leq_{r,l}^e cR$ and $l(A) = r(A) = (1-c)R$.

Proof. (i) There exists $e = e^2$ such that $A \leq_r^e eR$. Let $0 \neq y \in eR(1-e)$. There exists $s \in R$ such that $0 \neq ys \in A$. But $ysA \subseteq eR(1-e)A = 0$. Hence $ys \in A \cap l(A) = 0$, a contradiction. Thus $eR(1-e) = 0$, and so $e \in \mathcal{S}_r(R)$, by Lemma 2.1.

(ii) The proof of this part is similar to part (i).

(iii) From (i) and (ii) $A \leq_r^e eR$ and $A \leq_l^e Rf$ where $e \in \mathcal{S}_r(R)$ and $f \in \mathcal{S}_l(R)$, respectively. Since $e \in \mathcal{S}_r(R)$, $(1-e)R \subseteq r(A)$. Also $e(r(A)) \subseteq eR \cap r(A) = 0$. Hence $r(A) = (1-e)R$. Similarly, $l(A) = R(1-f)$. Since $(1-e) \in l(A)$ and $(1-f) \in r(A)$, then $l(A) = r(A)$. So $(1-e)R = R(1-f)$. Therefore $e = f \in \mathcal{S}_l(R) \cap \mathcal{S}_r(R) = \mathcal{B}(R)$. Let $e = c$. The remainder of the proof is a consequence of Lemma 3.1. □

COROLLARY 3.3 Let R be left and right FI-extending, and let A be an ideal of R. Then A is a ring direct summand of R if and only if $A \cap l(A) = 0 = A \cap r(A)$ and $A = lr(A)$.

Proof. Clearly if A is a ring direct summand then $A \cap l(A) = 0 = A \cap r(A)$ and $A = lr(A)$. The converse is an immediate consequence of Theorem 3.2(iii). □

COROLLARY 3.4 Let R be a left and right FI-extending. Then every ideal, which is semiprime (as a ring), is left and right essential in a ring direct summand. In particular, an ideal which is maximal among semiprime ideals (considered as rings) is a ring direct summand.

Proof. Let A be a semiprime ideal (considered as a ring). By Theorem 3.2 (iii), A is left and right essential in a ring direct summand. Moreover by a Zorn's Lemma argument, A is contained in an ideal M which is maximal among semiprime ideals (considered as rings). Since a ring direct summand which is either a right or left essential extension of M must also be a semiprime ring, the maximality of M yields that M is a ring direct summand. □

The following result generalizes a result of Jeremy on quasi-continuous rings ([15], Proposition 5.10).

COROLLARY 3.5 A left and right FI-extending ring is a direct sum of a reduced ring and a ring in which every nonzero ideal contains a nonzero nilpotent element of R.

Proof. The proof is similar to that of Corollary 3.4. □

Let $\#$ denote a map on the class of all rings (including those not necessarily having unity) such that $\#(R)$ is an ideal of R. We say R is *$\#$-regular* if $R/\#R$ is von Neumann regular and $\#(R) \cap \mathcal{M}(R) = 0$ (recall that $\mathcal{M}(R)$ is the maximal regular ideal of R). For example every semilocal ring or semiregular ring [16] is $\mathcal{J}$-regular, every left continuous ring is $\mathcal{J}$-regular and $Z({}_RR)$-regular, every commutative zero dimensional ring is $\mathcal{P}$-regular, etc. Observe that Faith's [9] "top regular" rings are just the $\mathcal{J}$-regular rings. Our next result generalizes the main results of ([9], Theorems 6, 7, 8, and 9). Note that if S is a nonempty subset of a ring T then we say S is *ideal essential* in T if S has nonzero and nonempty intersection with every nonzero ideal of T.

THEOREM 3.6 Let R be a right FI-extending ring.

(i) If R is left FI-extending, then $\mathcal{M}(R)$ is left and right essential in a ring direct summand of R.

(ii) If $\mathcal{M}(R)$ is a left annihilator ideal, then $\mathcal{M}(R)$ is a ring direct summand of R.

(iii) If R is $\#$-regular and either $\mathcal{M}(R)$ is a maximal regular right ideal or R is left FI-extending, then $R = \mathcal{M}(R) \oplus B$ (ring direct sum) and B is an ideal essential extension of $\#(R) + Z(R_R)$.

(iv) If R is left continuous, then $R = \mathcal{M}(R) \oplus B$(ring direct sum), where B is an ideal essential extension of $\mathcal{J}(R)$.

Proof. (i) This part is a direct consequence of Theorem 3.2(iii).

(ii) There exists $X \subseteq R$ such that $\mathcal{M}(R) = l(X)$. Hence $\mathcal{M}(R) = l(X) = lrl(X) = lr(\mathcal{M}(R))$. Therefore, by Corollary 3.3, $\mathcal{M}(R)$ is a ring direct summand.

(iii) By Theorem 3.2(i), there exists $e \in \mathcal{S}_r(R)$ such that $\mathcal{M}(R) \leq_r^e eR$. Then $(1-e)R$ is an ideal of R and $\#(R) \subseteq (1-e)R$. Hence $R/(1-e)R \simeq eR$ is a regular ring. If $\mathcal{M}(R)$ is a maximal regular right ideal then $\mathcal{M}(R) = eR$. If R is left FI-extending then, by Theorem 3.2(iii), e is central. Hence eR is a regular ideal of R. So $\mathcal{M}(R) = eR$. Therefore in either case $\mathcal{M}(R)$ is a ring direct summand. So there exists an ideal B of R such that $R = \mathcal{M}(R) \oplus B$. Since B is right FI-extending there exists $f = f^2 \in B$ such that $\#(R) + Z(R_R) \leq_r^e fB$ (note $Z(R_R) = Z(B_B)$). By Proposition 1.10, $fB \lhd B$. Let $I \lhd B$ such that $f(B) \cap I = 0$. Let $b = b^2 \in B$ such that $bR = B$ (i.e., b is the unity of B). Let $c = b - f$. Then $c \in \mathcal{S}_r(B)$ and cR is a regular ring with $I \subseteq cR$. Hence I is a regular ideal of R. So $I \subseteq \mathcal{M}(R) \cap B = 0$. Therefore B is an ideal essential extension of $\#(R) + Z(R_R)$.

(iv) Since R is left continuous, $R/\mathcal{J}(R)$ is regular. Hence R is $\mathcal{J}$-regular. Using left-right symmetry and the fact that $\mathcal{J}(R) = Z({}_RR)$, this part is a consequence of part (iii). □

Observe in Theorem 3.6(i) that the ring direct summand containing $\mathcal{M}(R)$ is a semiprime quasi-Baer ring in which each nonzero one-sided ideal contains a nonzero idempotent. The following examples illustrate and delimit Theorem 3.6.

EXAMPLE 3.7 There exist left and right FI-extending rings R in which $\mathcal{M}(R) \leq_{l,r}^e R$, but R is neither left (nor right) extending nor regular. Let D be a commutative integral domain which is not Prüfer and let F be its field of fractions. Let T be the subring of $\prod_{i=1}^{\infty} F_i$ (each $F_i \simeq F$) consisting of sequences whose components are eventually from D. Then T is a commutative reduced extending ring with $\mathcal{M}(T) \leq^e T$, but T is not regular. Let $R = M_2(T)$. By Proposition 2.3, R is left and right FI-extending. Clearly $\mathcal{M}(R) \leq_{l,r}^e R$, but R is not regular. By an argument similar to that used in ([11], p.17, Exercise 3), R is not a Baer ring. Since R is left and right nonsingular, then R is neither right nor left extending ([12], Theorem 2.1).

EXAMPLE 3.8 Since every prime ring is left and right FI-extending, every prime ring R with nonzero socle has $\mathcal{M}(R) \leq_{l,r}^e R$. See ([17], p.158) for such examples which are neither regular nor extending.

EXAMPLE 3.9 There are rings R which are right FI-extending, left continuous but not right continuous. Let S be a left continuous regular ring which is not right continuous (e.g., the endomorphism ring of an infinite dimensional vector space). Then S is a semiprime Baer ring. By ([4], Lemma 2.2) S is left and right FI-extending. Let T be a continuous ring. Take $R = S \oplus T$. Observe that T can be chosen so that R is not regular.

If R is left and right FI-extending then, by Proposition 1.11, $Z_2(R_R)$ and $Z_2({}_RR)$ are direct summands. It is easy to see that if $Z_2(R_R) = Z_2({}_RR)$, then $Z_2(R_R)$ is a ring direct summand ([6], Lemma 3.5). However it is not known whether the two second singular ideals coincide even for right and left self-injective rings [18].

PROPOSITION 3.10 Let R be left and right FI-extending and A be an ideal of R which is right closed. If $\mathcal{P}(R) \subseteq A$, then $A \leq_l^e cR$ where $c \in \mathcal{B}(R)$.

Proof. Since R is right FI-extending, there exists $e \in \mathcal{S}_l(R)$ such that $A = eR$. By Lemma 2.1 (vi), $r(A) = r(Re) \cap r(eR(1-e)) = (1-e)R \cap r(eR(1-e)) \subseteq (1-e)R$. Hence $A \cap r(A) = 0$. If $L \leq_l R$ such that $r(A) \leq_l^e L$, then $A \cap L = 0$. Hence $AL = 0$. So $r(A)$ is a left complement for A. There exists $f \in \mathcal{S}_r(R)$ such that $r(A) = Rf$. From Lemma 2.1, $Rf = fR \oplus (1-f)Rf$ and $(1-f)Rf \lhd R$. Since $(1-f)Rf \subseteq \mathcal{P}(R) \subseteq A$ and $(1-f)Rf \subseteq r(A)$,we have $(1-f)Rf = 0$. So $f \in \mathcal{B}(R)$. Let $c = 1-f$. Then $A \leq_l^e cR$. □

PROPOSITION 3.11 Let R be left and right FI-extending. Then
$R = T \simeq \begin{bmatrix} R_1 & R_{12} & R_{13} \\ 0 & R_2 & R_{23} \\ 0 & 0 & R_3 \end{bmatrix}$, where
(i) R_1 is a left nonsingular left FI-extending ring;
(ii) R_3 is a right nonsingular right FI-extending ring;
(iii) $\overline{T} = \begin{bmatrix} R_2 & R_{23} \\ 0 & R_3 \end{bmatrix}$ is a right FI-extending ring;
(iv) each R_{ij} is a $R_i - R_j$ bimodule;
(v) $Z_2({}_TT) = \begin{bmatrix} 0 & R_{12} & R_{13} \\ 0 & R_2 & R_{23} \\ 0 & 0 & R_3 \end{bmatrix}$;
(vi) $Z_2(\overline{T}_{\overline{T}}) = \begin{bmatrix} R_2 & R_{23} \\ 0 & 0 \end{bmatrix}$;
(vii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & R_2 & R_{23} \\ 0 & 0 & 0 \end{bmatrix} \subseteq Z_2({}_TT) \cap Z_2(T_T)$.

Proof. Since R is left FI-extending, there exists $e \in \mathcal{S}_l(R)$ such that $R = Re \oplus R(1-e)$ where $R(1-e) = Z_2({}_RR)$. By Lemma 2.1 and the left-hand version of Proposition 1.11, $Re = eRe$ is a left nonsingular left FI-extending ring. Now $R \simeq \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix}$. Since $(1-e) \in \mathcal{S}_r(R)$, $(1-e)R(1-e)$ is right FI-extending by [5, Lemma 2.1]. Let $\overline{R} = (1-e)R(1-e)$. Then there exists $c \in \mathcal{S}_l(\overline{R})$ such that $\overline{R} = c\overline{R} \oplus (\overline{1}-c)\overline{R}$, where $c\overline{R} = Z_2(\overline{R}_{\overline{R}})$ and $\overline{1} = 1-e$. Hence $\overline{1}-c \in \mathcal{S}_r(\overline{R})$.

By Lemma 2.1 and Proposition 1.11, $(\overline{1}-c)\overline{R} = (\overline{1}-c)\overline{R}(\overline{1}-c)$ is a right nonsingular right FI-extending ring. Now let $e = e_1$, $c = e_2$, and $1-e-c = e_3$. Let $R_1 = e_1Re_1$, $R_2 = e_2Re_2$, $R_3 = e_3Re_3$, $R_{12} = e_1Re_2$, $R_{13} = e_1Re_3$, and

$R_{23} = e_2Re_3$. Then $R \simeq T = \begin{bmatrix} R_1 & R_{12} & R_{13} \\ 0 & R_2 & R_{23} \\ 0 & 0 & R_3 \end{bmatrix}$. Parts (i)-(vi), follow from the above discussion, and part (vii) is a consequence of parts (v) and (vi). $\Box$

Note that if R is left and right extending, then in parts (i)-(iii) we may replace "FI-extending" with "extending".

4. FI-EXTENDING AND RELATED CONDITIONS

In this section we compare and contrast the FI-extending condition with various related conditions. The motivation for some of these results is provided by a theorem of Chatters and Khuri ([12], Theorem 2.1): Let R be any ring. Then R is a right nonsingular right CS ring if and only if R is a right cononsingular Baer ring. Moreover, a condition which links the extending and FI-extending conditions is introduced. We say a module M is *quasi-extending* if for any $X \leq M$ there exists a direct summand D of M such that: (i) $X \leq D$; (ii) if $K \lhd M$ with $K \cap D \neq 0$,then $K \cap X \neq 0$ (equivalently, if $0 \neq Y \leq D$ then $< Y > \cap X \neq 0$, where $< Y >$ denotes the fully invariant submodule of M generated by Y). We begin with the following two results which illustrate the connection between the quasi-Baer and FI-enxtending conditions.

PROPOSITION 4.1 [19] A ring R is quasi-Baer if and only if whenever I is an ideal of R there exists $e \in \mathcal{S}_l(R)$ such that $I \subseteq eR$, and $l(I) \cap eR = eR(1-e)$.

COROLLARY 4.2 [19] If R is a quasi-Baer ring and I is an ideal of R, then there exists $e \in \mathcal{S}_l(R)$ such that $I \subseteq eR$ and $I + eR(1-e)$ is right essential in eR and $eR(1-e)$ is an ideal of R. In particular, if I contains the prime radical of R (e.g., R is semiprime) or e is central, then I is right essential in eR. Moreover if I is not right essential in eR, then there exists a closed right ideal $0 \neq X = eX(1-e)$ such that $I \cap X = 0$ and $I \oplus X \leq_r^e eR$.

PROPOSITION 4.3 (i) If M is a quasi-extending module, then M is FI-extending.

(ii) If every ideal of R is right essential in an ideal generated by a central idempotent, then R_R is quasi-extending.

(i) Let $A \lhd M$. If A is essential in M, then we are finished. So assume X is a nonzero relative complement of A in M. Then there exists $e = e^2 \in End_R(M)$ such that $X \subseteq eM$ and $A \cap eM = 0$. Hence $A \subseteq (1-e)M$. If there exists $Y \leq M$ such that $Y \subseteq (1-e)M$ and $A \cap Y = 0$, then $(X + Y) \cap A = 0$. Hence $Y \subseteq X$, so $Y = 0$. Therefore $A \leq^e (1-e)M$.

(ii) Let $X \leq R_R$. If $X \cap K \neq 0$ for all $0 \neq K \lhd R$, then we are finished. So assume $0 \neq C$ is maximal among ideals of R such that $X \cap C = 0$. There exists $c \in \mathcal{B}(R)$ such that $C \leq_r^e cR$. Then $X \cap cR = 0$. So $C = cR$ and $X \subseteq eR$ where

$e = 1-c$. Assume $K \lhd R$ such that $K \cap eR \neq 0$. By the maximality of C, $K \cap X \neq 0$. Therefore R_R is quasi-extending. $\square$

Thus if R is an Abelian ring (i.e., every idempotent is central), Corollary 4.2 shows that the quasi-Baer condition implies the FI-extending condition, while Proposition 4.3 yields the equivalence of the FI-extending and the quasi-extending conditions. However there are commutative self-injective rings which are not Baer(e.g., $\mathbb{Z}_4$). Thus Corollary 4.2 has no converse.

PROPOSITION 4.4 Let R be right nonsingular. Then R is right FI-extending if and only if R is quasi-Baer and $A_R \leq^e rl(A)$, for all $A \lhd R$.

Proof. Assume R is right FI-extending and let $A \lhd R$. Then there exists $e = e^2$ such that $A_R \leq^e eR$. Since R is right nonsingular, $l(A) = l(eR) = R(1-e)$. Hence R is quasi-Baer. Moreover $A_R \leq^e eR = rl(eR) = rl(A)$. The converse is obvious. $\square$

PROPOSITION 4.5 Let R be right nonsingular and right FI-extending. The following conditions are equivalent:

(i) R is semiprime;

(ii) $\mathcal{S}_l(R) = \mathcal{B}(R)$;

(iii) the right essential closure equals the left essential closure for each ideal of R.

Proof. By Proposition 4.4, R is quasi-Baer. Hence the equivalence, (i)$\Leftrightarrow$(ii), follows from ([19], Proposition 1.4 (i)). For (ii)$\Rightarrow$(iii), let $A \lhd R$. By Proposition 4.4, there exists $e \in \mathcal{S}_l(R)$ such that $A \leq^e_r eR$. From (ii), $e \in \mathcal{B}(R)$. Since R is right nonsingular $l(A) = l(eR) = (1-e)R$. Since $1-e \in \mathcal{B}(R)$, $l(A) \subseteq r(A)$. Again since R is quasi-Baer there exists $f \in \mathcal{S}_l(R)$ such that $r(A) = fR$. By (ii), $f \in \mathcal{B}(R)$. Hence $l(A) = r(A) = (1-e)R$. Now there exists $L \leq_l R$ such that $A \oplus L \leq^e_l eR$. Then $AL \subseteq A \cap L = 0$. Hence $L \subseteq eR \cap (1-e)R = 0$. Therefore the right essential closure of A equals its left essential closure.

For (iii)$\Rightarrow$(ii), let $e \in \mathcal{S}_l(R)$. From Lemma 2.1, $eR(1-e) \lhd R$. From ([20], p.20, Exercise 11), $l(eR(1-e)) \leq^e_r R$. Since R is quasi-Baer, $l(eR(1-e))$ equals its left essential closure. By (iii), $l(eR(1-e)) = R$. Therefore $eR(1-e) = 0$, so $e \in \mathcal{B}(R)$. $\square$

Motivated by [21] and [22], we say a module M is *complement bounded* if every nonzero complement(i.e., essentially closed) submodule contains a nonzero fully invariant submodule of M.

PROPOSITION 4.6 Let M be a complement bounded module. Then M is quasi-extending if and only if M is extending.

Proof. Assume M is quasi-extending and $X \leq M$. There exists a direct summand D such that $X \leq D$ and if $K \lhd M$ with $K \cap D \neq 0$ then $K \cap X \neq 0$. If X is not essential in D, let C be a complement of X in D. Since D is a complement submodule of M, then C is a complement submodule of M. Hence there exists a nonzero $H \lhd M$ such that $H \subseteq C \subseteq D$. Hence $X \cap H \neq 0$, a contradiction. Therefore M is extending. The converse is obvious.

THEOREM 4.7 Consider the following conditions on a ring R.

a) R_R is FI-extending;

b) ${}_RR$ is FI-extending;

c) R is quasi-Baer;

d) every ideal is right(left) essential in a ring direct summand;

e) every ideal which is right (left) essentially closed is a direct summand;

f) R_R is quasi-extending;

g) R_R is extending;

h) R is Baer.

The following statements hold true for R:

(i) If R is semiprime, then a) through f) are equivalent.

(ii) If R is semiprime and R_R is complement bounded, the a) through g) are equivalent.

(iii) If R_R is nonsingular and complement bounded, then a) through h) are equivalent.

Proof. (i) From Lemma 2.1, $S_l(R) = S_r(R) = \mathcal{B}(R)$, using Theorem 3.2, it is evident that a), b), and d) are equivalent. From([4], Lemma 2.2), c), d), and e) are equivalent. Proposition 4.3 part (i) shows that f) implies a), and part (ii) shows that d) implies f).

(ii) This is a consequence of part (i) and Proposition 4.6.

(iii) From([21], Theorem 10), R is a reduced ring. Hence part(ii) yields the equivalence of a) through g). Since R is reduced, c) and h) are equivalent ([23], Lemma 1). □

Observe that although in Theorem 4.7(iii) the nonsingular and complement bounded conditions imply R is reduced, these conditions cannot be replaced by just the reduced condition. For example, if R is a domain which is not right Ore, then R satisfies a) through f) and h) but does not satisfy g). Moreover, there are prime right(and left) uniform rings R for which $Z(R_R) \neq 0$ ([24]). These rings are complement bounded, so Theorem 4.7(ii) holds. However, since $Z(R_R) \neq 0$, such rings are not Baer.

COROLLARY 4.8 Let R be right FPF. If R is semiprime (equivalently, right nonsingular), then R is right and left FI-extending.

Proof. In ([25], p.168), Faith shows that every semiprime right FPF ring is quasi-Baer. Now Theorem 4.7 yields the result. □

PROPOSITION 4.9 If R is right nonsingular and right quasi-continuous, then R is semiprime.

Proof. Let $A \lhd R$ with $A^2 = 0$. Then $A_R \leq^e eR$ and by Proposition 1.10 $e^2 = e \in S_l(R)$.

Claim: eR and $(1-e)R$ have no nonzero isomorphic submodules. Assume to the contrary, let $0 \neq X \leq eR$, $0 \neq Y \leq (1-e)R$, and let $\Phi : X \to Y$ be an isomorphism. Then $\Phi(X) \cap X = 0$ implies that Φ can be extended to an endomorphism of R, since R is right quasi-continuous. Thus Φ can be given as a left multiplication by an element s of R. Hence $\Phi(X) = sX = Y \subseteq eR \cap (1-e)R = 0$. Thus $\Phi = s = 0$ so eR and $(1-e)R$ are orthogonal.

Let $t \in R$. Consider the map $\Psi : (1-e)R \to eR$, defined by $\Psi((1-e)x) = et(1-e)x \in eR$. Then $Ker\,\Psi \leq^e eR$. Since if not, then $Ker\,\Psi \cap X = 0$ yields $\Psi|_X : X \to \Psi(X)$ is a monomorphism hence $X \simeq \Psi(X)$, a contradiction. Hence $\Psi((1-e)R) = et(1-e)R$ is singular, thus equals to zero as R is right nonsingular. Therefore $et(1-e)R = 0$. Thus $et(1-e) = 0$ for all $t \in R$, consequently $eR(1-e) = 0$. So $e^2 = e$ is central. Therefore, $R = R_1 \times R_2$ where $R_1 = eR,\ A \leq^e eR$ where $e^2 = e$ is central.

Next, since $A^2 = 0$, then $aA = 0$ for all $a \in A$. Thus $a \in Z(R_{1R_1}) = 0$ for all $a \in A$. We therefore conclude that $A = 0$ and R is semiprime. $\square$

Observe that if R is the 2-by-2 ring of upper triangular matrices over a field, then R is right and left extending and right and left nonsingular, but R is not semiprime. Therefore the hypothesis in Proposition 4.9 requires the quasi-continuity (condition C_3 of [26], p.18) of the ring R and the result does not hold true even if R is FI-extending on both sides.

The next three examples illustrate the contrast between the FI-extending condition and various related conditions.

EXAMPLE 4.10 Let D be a commutative domain and $R = M_n(D)$. Then R is a nonsingular prime ring. The primeness of R yields that R is quasi-Baer. So, by Theorem 4.7(i), R_R is quasi-extending (hence FI-extending). However, if $n > 1$ and D is not a Prüfer domain, then R is neither right nor left extending, nor Baer ([11], p.17) and ([12], Theorem 2.1).

EXAMPLE 4.11 Let D be a simple domain which is not a division ring. Take $R = \begin{bmatrix} D & D \oplus D \\ 0 & D \end{bmatrix}$. Then R is a quasi-Baer left and right nonsingular ring. However R is neither Baer, nor right FI-extending, nor left FI-extending. The quasi-Baer condition becomes apparent after computing the ideals of R.

To see that R is not Baer, let x be a nonzero noninvertible element of D. Let $0 \neq \begin{bmatrix} a & (b,d) \\ 0 & c \end{bmatrix} \in r \begin{bmatrix} x & (1,0) \\ 0 & 0 \end{bmatrix}$. Then $a = 0$ and $c = x(-b)$.

Assume (to obtain a contradiction) that R is a Baer ring. Since $r \begin{bmatrix} x & (1,0) \\ 0 & 0 \end{bmatrix}$ is generated by an idempotent, we can take $c = 1$. Hence x is invertible, a contradiction. Thus R is not a Baer ring.

Since $I = \begin{bmatrix} 0 & 0 \oplus D \\ 0 & 0 \end{bmatrix} \lhd R$ and the nonzero idempotents of R are of the form

$\begin{bmatrix} 1 & (b,d) \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & (b,d) \\ 0 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, I is essential in neither a right ideal direct summand nor a left ideal direct summand. Thus R is neither right nor left FI-extending.

EXAMPLE 4.12 ([27], Example 9). There exists a ring R such that $Z(R_R) \neq 0$, $Z({}_RR) = 0$, R is strongly right bounded (i.e., every nonzero right ideal contains a nonzero ideal), R is right FI -extending, but R is neither right extending, nor quasi-extending, nor quasi-Baer, nor left FI-extending. Let $T = \mathbb{Z}_2A$ be the semigroup ring with $\mathbb{Z}_2$, the integers mod 2, and A, the semigroup on the set $\{a, b\}$ satisfying the relation $xy = y$ for $x, y \in A$. Let R be the Dorroh extension of T (i.e., the ring with unity formed from $T \times \mathbb{Z}$ with componentwise addition and with multiplication given by $(x, k)(y, n) = (xy + nx + ky, kn)$). Observe that $R = (a, 1)R \oplus (T, 0)$ as right ideals. To see that R is not right extending observe that the right ideal $\{(k(a+b), 2k) | k \in \mathbb{Z}\}$ is not right essential in an idempotent generated right ideal. As R_R is complement bounded, Proposition 4.6 shows that R_R is not quasi-extending. Since $r((T, 0)) = \{(0, 2k) | k \in \mathbb{Z}\}$ is not generated by an idempotent, then R is not a quasi-Baer ring. By direct calculation R is right FI-extending. However, since $Z({}_RR) = 0$ and R is not quasi-Baer, Proposition 4.4 indicates that R is not left FI-extending.

Observe that Examples 4.10 and 4.12 show that the quasi-extending condition lies strictly between the extending and FI-extending conditions.

5. NONSINGULAR FI-EXTENDING RING DECOMPOSITIONS

In this section we obtain a ring decomposition for a right nonsingular right FI-extending ring using the concept of an orthogonal pair of module classes. Recall, from [26], two modules are called *orthogonal* if they have no nonzero isomorphic submodules. For a class $\mathfrak{A}$ of modules, $\mathfrak{A}^\perp$ denotes the class of modules orthogonal to all members of $\mathfrak{A}$. Classes $\mathfrak{A}$ and $\mathfrak{B}$ form an orthogonal pair if $\mathfrak{A}^\perp = \mathfrak{B}$ and $\mathfrak{B}^\perp = \mathfrak{A}$. We use $Q = Q(R)$ to denote the maximal right ring of quotients of R.

LEMMA 5.1 Let $Z(R_R) = 0$. Given any orthogonal pair $\mathfrak{A}$ and $\mathfrak{B}$ of classes of R-modules, there exist ideals A and B of R, such that A and B are maximal among the right ideals of R contained in $\mathfrak{A}$ and $\mathfrak{B}$, respectively, and $A \oplus B \leq^e R_R$.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be an orthogonal pair. We can use Zorn's lemma to obtain a right ideal A which is maximal among the right ideals of R in $\mathfrak{A}$ and a right ideal B which is maximal among the right ideals of R in $\mathfrak{B}$. It is clear that $A \cap B = 0$. We claim that $A \oplus B \leq^e R_R$: if not, then there exists an $0 \neq X \leq R_R$ with $(A \oplus B) \cap X = 0$. Now if X has a nonzero submodule Y in $\mathfrak{A}$, then $(A \oplus B) \cap Y = 0$. Thus $A \oplus Y \in \mathfrak{A}$ which contradicts the maximality of A. If X does not have any nonzero submodule Y in $\mathfrak{A}$, then $X \in \mathfrak{B}$. Then $B \oplus X \in \mathfrak{B}$, contradicting the maximality of B. Thus $A \oplus B \leq^e R_R$ and $Q = Q(R) = E(A) \oplus E(B)$. Thus $E(A) = eQ$, where $e^2 = e$ is a central idempotent of Q (by using $Z(R_R) = 0$). Next, $A \leq^e eQ \cap R \in \mathfrak{A}$ then maximality of A yields $A = eQ \cap R$. Similarly $B = (1-e)Q \cap R$ holds as e is a central idempotent. Therefore A and B are ideals of R. $\square$

THEOREM 5.2 Let R be a right FI-extending ring with $Z(R_R) = 0$. Then any orthogonal pair $\mathfrak{A}$ and $\mathfrak{B}$ of classes of R-modules yields a ring direct decomposition $R = R_1 \times R_2$, where R_1 and R_2 are maximal among the right ideals of R in $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

Proof: By Lemma 5.1, we obtain $R_1 \oplus R_2 \leq^e R$ where $R_1 = eQ \cap R$, and $R_2 = (1-e)Q \cap R$, for some central idempotent $e^2 = e$ in Q. Since R is right FI-extending, $R_1 \leq^e fR$, where $f^2 = f$ is an idempotent of R.

Now $E(R_1) = E(fR) = eQ$ yields $e = f \in R$ and hence $R_1 = eR$, $R_2 = (1-e)R$, and $R = R_1 \oplus R_2$ holds. $\square$

The next example shows that Theorem 5.2 fails if R is not right FI -extending in the hypothesis (the example also illustrates Lemma 5.1).

EXAMPLE 5.3 Let F be any (finite) field and let $F_i = F$, $i \in I$, where I is infinite. Define $R = \underset{i \in I}{\oplus} F_i + 1 \cdot F$, and let $I = I_1 \cup I_2$ be a nontrival disjoint union. Then R is a regular ring and $A = \underset{i \in I_1}{\oplus} F_i$, and $B = \underset{i \in I_2}{\oplus} F_i$ are ideals of R.

Consider classes $\mathfrak{A} = A^{\perp\perp}$ and $\mathfrak{B} = A^{\perp}$, then $A \oplus B \lneq^e R_R$. Note that A and B are maximal among ideals of R in $\mathfrak{A}$ and $\mathfrak{B}$, respectively, since they are closed. $\square$

From Theorem 1.3, we know that a direct sum of FI-extending modules is FI -extending. On the other hand, it is not known in general if the FI -extending property is inherited by direct summands of FI-extending modules. The answer is affirmative in the case of FI-extending Abelian groups [8].

OPEN PROBLEM. Show that a direct summand of an FI-extending module is FI-extending or provide a counter-example. In the latter case, find necessary and sufficient conditions for a direct summand of an FI-extending module to be FI-extending.

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