Improved estimates for the approximation numbers of Hardy-type operators

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Abstract

We consider a Hardy-type integral operator $T : L^p(a,b) \to L^p(a,b)$, $-\infty \leq a < b \leq \infty$, which is defined by

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)dt.$$

In papers [EEH1],[EEH2] and [EHL2], upper and lower estimates and asymptotic results were obtained for the approximation numbers $a_n(T)$ of T. In case $p = 2$ for "nice" u and v these results were improved in [EKL]. In this paper we extend these results for $1 < p < \infty$ by using a new technique from [EHL2]. We will show that under suitable conditions on u and v,

$$\limsup_{n\to\infty} n^{1/2} \left| \alpha_p \int_a^b |u(t)v(t)|dt - na_n(T) \right|$$
$$\leq c(\|u'\|_{p'/(p'+1)} + \|v'\|_{p/(p+1)})(\|u\|_{p'} + \|v\|_p) + 3\alpha_p \|uv\|_1,$$

where $\|w\|_p = (\int_a^b |w(t)|^p dt)^{1/p}$ and $\alpha_p = A((0,1),1,1)$.

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1 Introduction.

In [EEH1] and [EEH2] the approximation numbers $a_n(T)$ of

$$(Tf)(x) = v(x) \int_0^x u(t)f(t)dt, \tag{1. 1}$$

as an operator from $L^p(\mathbf{R}^+)$ to itself were studied. Here $\mathbf{R}^+ = (0, \infty)$, $1 < p < \infty$, and u, v are real–valued functions with $u \in L^{p'}_{loc}(\mathbf{R}^+)$, and $v \in L^p(\mathbf{R}^+)$; as usual, $p' = p/(p-1)$.

In [EEH1] it was shown that if T is bounded from $L^2(\mathbf{R}^+)$ to itself, then to each $\varepsilon > 0$ there corresponds $N(\varepsilon) \in \mathbf{N}$ such that

$$a_{N(\varepsilon)+2}(T) \le \frac{\varepsilon}{\sqrt{2}} \le a_{N(\varepsilon)}(T). \tag{1. 2}$$

The estimate (1. 2) was improved in [EEH2], in which it was shown that

$$\lim_{n\to\infty} n a_n(T) = \frac{1}{\pi} \int_0^\infty |u(t)v(t)|dt, \tag{1. 3}$$

given certain restrictions on u and v. (For related results see also [NS].)

In [EHL2] it was showed that (1. 3) is true also for the Hardy-type operator on trees and for $1 < p < \infty$. For cases $p = 1$ and $p = \infty$ was found a similar formula like (1. 3), see [EHL1] and [EHL2].

Further extensions were given in [LL] and [LMN] to deal with the cases in which T is viewed as a map from L^p to L^q, for any $p, q \in [1, \infty]$.

In paper [EKL] an estimate (1. 3) was improved in the case $p = 2$ (L^2 is the Hilbert space and then it is simple to find the closes element from any closed subspace). It was shown that under some conditions on u and v we have

$$\limsup_{n\to\infty} n^{1/2} \left| n a_n(T) - \frac{1}{\pi} \int_a^b |uv| \right|$$

$$\le 3\sqrt{2}(\|u'\|_{2/3,I} + \|v'\|_{2/3,I})(\|u\|_{2,I} + \|v\|_{2,I}) + \frac{3}{\pi}\|uv\|\|_{1,I},$$

I being an arbitrary interval in $\mathbf{R}$.

In the present paper we will extend this result to $1 < p < \infty$. Under further conditions on u and v we get for the approximation numbers of the map $T : L^p(I) \to L^p(I)$ the following estimates:

$$\limsup_{n\to\infty} n^{1/2} \left| n a_n(T) - \alpha_p \int_a^b |uv| \right|$$

$$\leq 3c(p,p')(\|u'\|_{p'/(p'+1),I} + \|v'\|_{p/(p+1),I})(\|u\|_{p',I} + \|v\|_{p,I})$$

$$+3\alpha_p \|uv\|_{1,I}.$$

Thus,

$$a_n(T) = \frac{1}{\pi n} \int_I |u(t)v(t)|dt + O(n^{-3/2});$$

and under the conditions which we impose, the exponent $-3/2$ cannot be much improved. This is the first theorem of this kind which is covering the case $p \neq 2$ and it is surprising that there is the same power $n^{1/2}$ for any $1 < p < \infty$. We do not know at the moment whether or not it is possible to show the existence of a genuine second term in the expansion of $a_n(T)$. Our results follow from the systematic use of the function A introduced in [EHL1] together with techniques based on those in [EEH2] and [EKL].

2 Preliminaries.

Throughout the paper we shall assume that $-\infty \leq a < b \leq \infty$ and that

$$u \in L^{p'}(a,b), \quad v \in L^p(a,b) \quad \text{and } u, v > 0 \text{ on } (a,b). \tag{2. 1}$$

Under these restrictions on u and v it is well known (see, for example, [EEH1], Theorem 1) that the norm $\|T\|$ of the operator $T : L^p(a,b) \to L^p(a,b)$ in (1. 1) satisfies

$$\|T\| \sim \sup_{x\in(a,b)} \|u\chi_{(a,x)}\|_{p',(a,b)} \|v\chi_{(x,b)}\|_{p,(a,b)}. \tag{2. 2}$$

Here χ_S denotes the characteristic function of the set S and

$$\|f\|_{p,I} = \left(\int_I |f(t)|^p dt\right)^{1/p}, \quad 1 < p < \infty, \quad I \subset (a,b).$$

Moreover, by $F_1 \sim F_2$ we mean that $C^{-1}F_1 \le F_2 \le CF_1$ for some positive constant $C \ge 1$ independent of any variables in $F_1, F_2 \ge 0$.

Given any interval $I = (c,d) \subset (a,b)$, define

$$J(I) = \sup_{x \in I} \|u\chi_{(c,x)}\|_{p',I}\|v\chi_{(x,d)}\|_{p,I}.$$

A straightforward modification of Lemma 2.1 of [EHL1] shows that for any $d \in (a,b)$, the function $J((.,d))$ is continuous and non-increasing on (a,d). Now, for any $x \in I = (c,d) \subset (a,b)$, set

$$(T_I f)(x) = v(x)\chi_I(x)\int_a^x u(t)\chi_I(t)f(t)dt.$$

Then the norm of the operator $T_I : L^p(I) \to L^p(I)$ satisfies

$$\|T_I\| \sim J(I).$$

We next introduce a function A which will play a key role in the paper. Given $I = (c,d) \subset (a,b)$, set

$$A(I) := \sup_{\|f\|_{p,I}=1} \inf_{\alpha \in \mathbf{R}} \|Tf - \alpha v\|_{p,I}.$$

From (2. 1) it follows that T is a compact operator from L^p into L^p (see [EGP] or [OK]) and then from [EHL2, Theorem 3.8] we have that

$$A(I) = \inf_{x \in I} \|T_{x,I}|L^p(I) \to L^p(I)\|,$$

where

$$T_{x,I}f(.) := v(.)\chi_I(.)\int_x^{.} v(t)\chi_I(t)dt.$$

Lemma 2.1 *Let $I = (c,d) \subset (a,b)$ and $1 \le p \le \infty$, then $\|T_{x,I}|L^p(I) \to L^p(I)\|$ is continuous in x.*

Proof. See Lemma 3.4 in [EHL2] and put $\Gamma = (a,b)$ and $K = I$. $\square$

Lemma 2.2 *Suppose that u and v satisfy (2. 1) and $a \le c < d \le b$. Then:*

1. *The function $A(.,d)$ is non-increasing and continuous on (a,d).*
2. *The function $A(c,.)$ is non-decreasing and continuous on (c,b).*
3. $\lim_{y \to a_+} A(a,y) = \lim_{y \to b_-} A(y,b) = 0$.

Proof. The proof of 1 illustrates the techniques necessary to prove 2 and 3 also. That $A(.,d)$ is non-increasing is easy to see. To get the continuity, fix $y \in (a,d)$. Then, there exists $h_0 > 0$ such that for $0 < h < h_0$

$$\begin{aligned}
& A^p(y,d) \leq A^p(y-h,d) \\
= & \sup_{\|f\|_{p,(y-h,d)}=1} \inf_{|\alpha|\leq \|u\|_{p',(y-h_0,d)}} \|v\chi_{(y-h,d)} \left[\int_a^{\cdot} u(t)f(t)\chi_{(y-h,d)}(t)dt - \alpha\right] \|^p_{p,(y-h,d)} \\
= & \sup_{\|f\|_{p,(y-h,d)}=1} \inf_{|\alpha|\leq \|u\|_{p',(y-h_0,d)}} \Bigg[\|v \left[\int_{y-h}^{\cdot} uf\chi_{(y-h,y)}dt - \alpha\right] \|^p_{p,(y-h,y)} \\
& + \|v \left[\int_y^{\cdot} uf\chi_{(y,d)}dt - \alpha + \int_{y-h}^{y} uf dt\right] \|^p_{p,(y,d)}\Bigg] \\
\leq & \sup_{\|f\|_{p,(y-h,d)}=1} \inf_{|\alpha|\leq \|u\|_{p',(y-h_0,d)}} \Bigg[2^p\|v \int_{y-h}^{\cdot} uf\chi_{(y-h,y)}dt\|^p_{p,(y-h,y)} \\
& + 2^p\alpha^p\|v\|^p_{p,(y-h,y)} + \|v \left[\int_y^{\cdot} uf\chi_{(y,d)}dt - \alpha + \int_{y-h}^{y} uf dt\right] \|^p_{p,(y,d)}\Bigg] \\
\leq & 2^p\|v\|^p_{p,(y-h,y)}\|u\|^p_{p',(y-h,y)} + 2^p\|u\|^p_{p',(y-h_0,d)}\|v\|^p_{p,(y-h,y)} + A^p(y,d).
\end{aligned}$$

It follows that

$$\lim_{h\to 0_+} A(y-h,d) = A(y,d).$$

In the same way we see that

$$\lim_{h\to 0_+} A(y+h,d) = A(y,d),$$

and now the proof is complete. □

Lemma 2.3 *Suppose that* $T : L^p(a,b) \to L^p(a,b)$ *is compact. Let* $I = (c,d)$ *and* $J = (c',d')$ *be subintervals of* (a,b), *with* $J \subset I$, $|J| > 0$, $|I - J| > 0$, $\int_a^b v^p(x)dx < \infty$ *and* $u, v > 0$ *on* I. *Then*

$$A(I) > A(J) > 0. \tag{2. 3}$$

Proof. Let $0 \leq f \in L^p(J)$, $0 < \|f\|_{p,J} \leq \|f\|_{p,I} \leq 1$ with $\operatorname{supp} f \subset J$. Let $y \in J$ then

$$\|T_{(c',y)}\|_{p,J} > 0 \qquad \text{and} \qquad \|T_{(y,d')}\|_{p,J} > 0$$

and then from [EHL2, Lemma 3.5] we have

$$\min\{\|T_{(c',y)}\|_{p,J}, \|T_{(y,d')}\|_{p,J}\} \leq \min_{x\in J} \|T_{x,J}\|_{p,J}$$

which means $A(J) > 0$.

Next, let us suppose that $c = c' < d' < d$. By [EHL2, Theorem 3.8], there exist $x_0 \in J$ and $x_1 \in I$ such that $A(J) = \|T_{x_0,J}\|_{p,J}$ and $A(I) = \|T_{x_1,I}\|_{p,I}$. Since $u, v > 0$ on I, it is then quite easy to see that $x_0 \in J^o$ and $x_1 \in I^o$.

If $x_0 = x_1$, then, since $u, v > 0$ on I, we get

$$A(I) = \|T_{x_1,I}\|_{p,I} > \|T_{x_1,I}\|_{p,J} = \|T_{x_1,J}\|_{p,J} = A(J).$$

If $x_0 \neq x_1$, then

$$A(I) = \|T_{x_1,I}\|_{p,I} \geq \|T_{x_1,I}\|_{p,J} \geq \|T_{x_1,J}\|_{p,J} > \|T_{x_0,J}\|_{p,J} = A(J).$$

The case $c < c' < d' = d$ could be proved similarly and the case $c < c' < d' < d$ follows from previous cases and the monotonicity of $A(I)$. □

Remark 2.4 *It follows from the continuity of A that for sufficiently small $\varepsilon > 0$ there is an a_1, $a < a_1 < b$, for which $A(a_1, b) = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $b = a_0 > a_1 > \ldots > a_{N(\varepsilon)} = a$ with $A(a_i, a_{i-1}) = \varepsilon, i = 1, 2, \ldots, N(\varepsilon) - 1$ and $A(a, a_{N(\varepsilon)-1}) \leq \varepsilon$.*

Lemma 2.5 *The number $N(\varepsilon)$ is a non-increasing function of ε which takes on every sufficiently large an integer value.*

Proof. Fix $c, a < c < b$. Then, (2. 3) ensures $A(c, b) = \varepsilon_0 > 0$. Moreover, as observed in Remark 2.4, there is a positive integer $N(\varepsilon_0)$ and a partition $b = a_0 > a_1 > \ldots > a_{N(\varepsilon_0)} = a$ such that $A(a_i, a_{i-1}) = \varepsilon_0,\ i = 1, 2, \ldots, N(\varepsilon_0) - 1$ and $A(a, a_{N(\varepsilon_0)-1}) \leq \varepsilon_0$. Let $d \in (a_1, b)$. According to Lemma 2.3, $A(d, b) = \varepsilon_0' < \varepsilon_0$ and the procedure outlined above applied to ε_0' gives $\infty > N(\varepsilon_0') \geq N(\varepsilon_0) + 1$. If $N(\varepsilon_0') = N(\varepsilon_0) + 1$, we stop.

Otherwise, define

$$\varepsilon_1 = \sup\{\varepsilon : 0 < \varepsilon < \varepsilon_0 \text{ and } N(\varepsilon) \geq N(\varepsilon_0) + 1\}.$$

We claim $N(\varepsilon_1) = N(\varepsilon_0) + 1$. Indeed, suppose $N(\varepsilon_1) \geq N(\varepsilon_0) + 2$ and the partition $b = a_0 > a_1 > \ldots > a_{N(\varepsilon_1)} = a$ satisfies $A(a_i, a_{i-1}) = \varepsilon_1, i = 1, 2, \ldots, N(\varepsilon_1) - 1$ and $A(a, a_{N(\varepsilon_1)-1}) \leq \varepsilon_1$. Decrease $a_{N(\varepsilon_1)-1}$ slightly to $a'_{N(\varepsilon_1)}$ so that both $A(a, a'_{N(\varepsilon_1)}) < \varepsilon_1$ and $A(a'_{N(\varepsilon_1)}, a_{N(\varepsilon_1)-1}) > \varepsilon_1$, continuing the process to get a partition of (a, b) having $N(\varepsilon_1)$ intervals such that $A(a, a'_{N(\varepsilon_1)}) < \varepsilon_1$ and $A(a'_i, a'_{i-1}) > \varepsilon_1, i = 1, 2, \ldots, N(\varepsilon_1) - 1, a'_0 = b$. Taking $\varepsilon_2 \leq \min_{2 \leq i \leq N(\varepsilon_1)-1} A(a'_i, a'_{i-1})$ we obtain $\varepsilon_2 > \varepsilon_1$ and $N(\varepsilon_2) \geq N(\varepsilon_0) + 2$, a contradiction. This establishes the claim.

An inductive argument completes the proof. □

The quantity $N(\varepsilon)$ is useful in the derivation of upper and lower estimates for the approximation numbers of T.

Lemma 2.6 *For all* $\varepsilon \in (0, \|T\|)$,

$$a_{N(\varepsilon)+2}(T) \leq \varepsilon \leq a_{N(\varepsilon)+1}(T).$$

Proof. This follows from [EHL2], Lemma 3.19 (put $K = (a, b)$).□

A version of this result, with a slightly different $N(\varepsilon)$, was first proved in [EEH1] and was then extended in [EHL1]. For general u and v it is impossible to find a simple relation between ε and $N(\varepsilon)$, but by using the properties of A the behavior of $\varepsilon N(\varepsilon)$ when $\varepsilon \to 0_+$ can be determined.

Lemma 2.7 *Given* $v \in L^p(a, b)$, $u \in L^{p'}(a, b)$ *we have*

$$\lim_{\varepsilon \to 0_+} \varepsilon N(\varepsilon) = \alpha_p \int_a^b |u(t)v(t)| dt.$$

This result follows from an adaptation of the argument of [EHL2]; see, in particular, Theorem 6.4 of that paper. Together with Lemma 2.6 this shows, again using the techniques of [EHL2], that the following theorem holds.

Theorem 2.8 *Given $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ the operator T defined in (1. 1) satisfies*

$$\lim_{n\to\infty} n a_n(T) = \alpha_p \int_a^b |u(t)v(t)|dt,$$

where $\alpha_p = A((0,1),1,1)$.

A result of this type was established under weaker conditions on u and v in [EHL2].

3 Technical results.

Here we give some results of a technical nature which will prove very useful in the sequel. We begin with some information about the function A.

Lemma 3.1 *Let $I = (c,d) \subseteq (a,b)$ and suppose that u and v are constant functions over I. Then*

$$A(I,u,v) = |I||u||v|A((0,1),1,1)$$

Proof. By definition,

$$\begin{aligned} A(I,u,v) &= \sup_{f\in L^p(I)} \inf_{\alpha\in\mathbf{R}} \|Tf - \alpha v\|_{p,I}/\|f\|_{p,I} \\ &= \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbf{R}} \|v\left(\int_c^{\cdot} uf dt - \alpha\right)\|_{p,I} \\ &= |v||u| \sup_{\|f\|_{p,I}\le 1} \inf_{\alpha\in\mathbf{R}} \|\int_c^{\cdot} f dt - \alpha\|_{p,I} \\ &= |v||u||I| \sup_{\|f\|_{p,(0,1)}\le 1} \inf_{\alpha\in\mathbf{R}} \|\int_0^{\cdot} f dt - \alpha\|_{p,(0,1)} \end{aligned}$$

□

Next, we investigate the dependence of $A(I,u,v)$ on u and v.

Lemma 3.2 *Let $I = (c,d) \subset (a,b)$ and suppose that $v \in L^p(I)$ and $u_1, u_2 \in L^{p'}(I)$. Then*

$$|A(I,u_1,v) - A(I,u_2,v)| \le \|u_1 - u_2\|_{p',I}\|v\|_{p,I}.$$

Proof. Without loss of generality we may suppose that $A(I,u_1,v) \geq A(I,u_2,v)$. Then

$$\begin{aligned} A(I,u_1,v) &= \sup_{\|f\|_{p,I}\leq 1} \inf_{\alpha\in\mathbf{R}} \|v\left[\int_c^{\cdot} (u_1-u_2+u_2)fdt-\alpha\right]\|_{p,I} \\ &\leq \sup_{\|f\|_{p,I}\leq 1} \inf_{\alpha\in\mathbf{R}} \Big[\|v\int_c^{\cdot}(u_1-u_2)fdt\|_{p,I} \\ &\qquad + \|v(\int_c^{\cdot} u_2 fdt-\alpha)\|_{p,I}\Big] \\ &\leq \sup_{\|f\|_{p,I}\leq 1} \inf_{\alpha\in\mathbf{R}} \Big[\|v\|_{p,I}\|u_1-u_2\|_{p',I} \\ &\qquad + \|v(\int_c^{\cdot} u_2 f-\alpha)\|_{p,I}\Big] \\ &\leq \|v\|_{p,I}\|u_1-u_2\|_{p',I} + A(I,u_2,v). \end{aligned}$$

The result follows. □

Lemma 3.3 *Let $I=(c,d)\subset(a,b)$ and suppose that $u\in L^{p'}(I)$ and $v_1,v_2\in L^p(I)$. Then*

$$|A(I,u,v_1)-A(I,u,v_2)| \leq \|u\|_{p',I}\|v_1-v_2\|_{p,I}.$$

Proof. We may suppose that $A(I,u,v_1)\geq A(I,u,v_2)$. Then

$$\begin{aligned} A(I,u,v_1) &= \sup_{\|f\|_{p,I}\leq 1} \inf_{\alpha\in\mathbf{R}} \|v_1\left[\int_c^{\cdot} ufdt-\alpha\right]\|_{p,I} \\ &= \sup_{\|f\|_{p,I}\leq 1} \inf_{|\alpha|\leq\|u\|_{p',I}\|f\|_{p,I}} \|v_1\left[\int_c^{\cdot} ufdt-\alpha\right]\|_{p,I} \\ &\leq \sup_{\|f\|_{p,I}\leq 1} \inf_{|\alpha|\leq\|u\|_{p',I}} \Big[\|(v_1-v_2)[\int_c^{\cdot} ufdt-\alpha]\|_{p,I} \\ &\qquad + \|v_2[\int_c^{\cdot} ufdt-\alpha]\|_{p,I}\Big] \\ &\leq \|v_1-v_2\|_{p,I}\|u\|_{p',I} + A(I,u,v_2). \end{aligned}$$

The proof is complete. □

We now turn to the approximation of functions from L^p and $L^{p'}$ by step-functions.

Suppose $u \in L^{p'}(a,b)$ and $v \in L^p(a,b)$ and let $\alpha > 0$. We define $m_\alpha \in \mathbf{N}$ by the following requirements:

There exist two step-functions, u_α and v_α, each with m_α steps, say,

$$u_\alpha(x) := \sum_{j=1}^{m_\alpha} \xi_j \chi_{w_\alpha(j)}(x), \qquad v_\alpha(x) := \sum_{j=1}^{m_\alpha} \psi_j \chi_{w_\alpha(j)}(x), \tag{3. 1}$$

where $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a family of non-overlapping intervals covering (a,b), such that for

$$\alpha_u := \|u - u_\alpha\|_{p',(a,b)} \qquad \text{and} \qquad \alpha_v := \|v - v_\alpha\|_{p,(a,b)}$$

we have

(i)

$$\max(\alpha_u, \alpha_v) \le \alpha; \tag{3. 2}$$

and

(ii) for any step-functions u'_α, v'_α with less than m_α steps, say n_α steps, $n_\alpha < m_\alpha$,

$$\max(\|u - u'_\alpha\|_{p',(a,b)}, \|v - v'_\alpha\|_{p,(a,b)}) > \alpha.$$

Thus, m_α is the minimum number of steps needed to approximate u in $L^{p'}$ and v in L^p with the required accuracy. Note that, plainly,

$$\|u - u_\alpha\|_{p',(a,b)} \le \alpha, \qquad \|v - v_\alpha\|_{p,(a,b)} \le \alpha.$$

The best way to choose ξ_i and ψ_i for given $\{w_\alpha\}_{j=1}^{m_\alpha}$ is by finding ξ_i and ψ_i such that:

$$\int_{w_\alpha(i)} |u(t) - \xi_i|^{p'-1} \operatorname{sgn}(u(t) - \xi_i)dt = 0$$

and

$$\int_{w_\alpha(i)} |v(t) - \psi_i|^{p-1} \operatorname{sgn}(v(t) - \psi_i)dt = 0$$

(see [S], Theorem 1.11).

It turns out that the relationship between α and m_α is crucial for us; we next address this matter.

Lemma 3.4 *Suppose $u \in C(a,b) \cap L^{p'}(a,b)$ and $v \in C(a,b) \cap L^{p}(a,b)$, at least one of them, say u, being non-constant. Then, when α decreases to 0, m_α increases to ∞.*

Proof. We show that given $m \in N$ there exists $\alpha > 0$ having $m_\alpha > m$. The fact that u is continuous and non-constant on (a,b) guarantees the existence of pairwise disjoint subintervals $I_1, I_2, \ldots, I_{2m}$ of (a,b) on each of which u is non-constant.

Fix $\alpha > 0$ satisfying $\sum_{j=1}^{m} \|u - u_{I_{k_j}}\|_{p',I_{k_j}}^{p'} > \alpha^{p'}$ for every set of m intervals from among $I_1, I_2, \ldots, I_{2m}$. Now, to any partition, $\{w_\alpha(j)\}_{j=1}^{m}$, of (a,b) into m non-overlapping subintervals there correspond $I_{k_1}, I_{k_2}, \ldots, I_{k_m}$ such that each I_{k_j} is subset of some $w_\alpha(i)$ and hence

$$\sum_{j=1}^{m} \|u - u_{w_\alpha(j)}\|_{p',w_\alpha(j)}^{p'} \geq \sum_{j=1}^{m} \|u - u_{w_\alpha(j)}\|_{p',I_{k_j}}^{p'} > \alpha^{p'}.$$

Therefore $m_\alpha > m$. □

Lemma 3.5 *Suppose $u \in C(a,b) \cap L^{p'}(a,b)$ and $v \in C(a,b) \cap L^{p}(a,b)$, at least one of them, say u, being non-constant. Fix $\alpha > 0$ and set $\Lambda_\alpha = \{\beta; 0 < \beta \leq \alpha$ and $m_\beta = m_\alpha\}$. Then, Λ_α is an interval with $\gamma = \inf \Lambda_\alpha$ and $\gamma \in \Lambda_\alpha$.*

Proof. Clearly, Λ_α is nonempty, since $\alpha \in \Lambda_\alpha$. Again, $m_{\lambda_1} \geq m_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$, so Λ_α is convex and hence an interval, possibly equal to $\{\alpha\}$.

It follows from Lemma 3.4 that $\gamma > 0$. Now, if $\Lambda_\lambda = \{\alpha\}$, so that $\gamma = \alpha$, we are done. Otherwise, there exists a sequence $\{\alpha_n\}$ in Λ_α with $\alpha_n \searrow \gamma$. Let $u_{\alpha_n} = \sum_{j=1}^{m_\alpha} u_{w_{\alpha_n}(j)} \chi_{w_{\alpha_n}(j)}$ and $v_{\alpha_n} = \sum_{j=1}^{m_\alpha} v_{w_{\alpha_n}(j)} \chi_{w_{\alpha_n}(j)}$, as in (3. 1), so that

$$\max(\|u - u_{\alpha_n}\|_{p',(a,b)}, \|v - v_{\alpha_n}\|_{p,(a,b)}) \leq \alpha_n.$$

Assume the notation has been chosen to ensure the end points of $w_{\alpha_n}(j) = (c_n^j, d_n^j)$ satisfy $a = c_n^1 < d_n^j \leq c_n^{j+1} < d_n^{m_\alpha} = b$, $j = 1, 2, \ldots, m_\alpha - 1$.

There exists a sequence $n(k), k = 1, 2, \ldots$ of positive integers and numbers $c^1, c^2, \ldots, c^{m_\alpha}, d^1, d^2, \ldots, d^{m_\alpha}$ such that

$$\lim_k c^j_{n(k)} = c^j, \qquad \lim_k d^j_{n(k)} = d^j, \qquad j = 1, 2, \ldots, m_\alpha,$$

and

$$a = c^1 \le d^j \le c^{j+1} \le d^{m_\alpha} = b, \qquad j = 1, 2, \ldots, m_\alpha.$$

Observe that, setting

$$u_\gamma = \sum_{j=1}^{m_\alpha} u_{(c^j,d^j)} \chi_{(c^j,d^j)} \qquad \text{and} \qquad v_\gamma = \sum_{j=1}^{m_\alpha} v_{(c^j,d^j)} \chi_{(c^j,d^j)},$$

we have

$$\max(\|u - u_\gamma\|_{p',(a,b)}, \|v - v_\gamma\|_{p,(a,b)}) = \gamma,$$

which forces $m_\gamma = m_\alpha$. $\square$

Lemma 3.6 *Suppose that $u \in L^{p'}(a,b) \cap C(a,b)$ and $v \in L^p(a,b) \cap C(a,b)$ are not equal to zero on (a,b), indeed, assume at least one of u and v be non-constant on (a,b). Then, there exists $\alpha_0 > 0$ such that given any α, $0 < \alpha < \alpha_0$, there exists a β, $0 < \beta < \alpha$, with $m_\beta = m_\alpha + 1$ or $m_\beta = m_\alpha + 2$.*

Proof. Say u is non-constant on (a,b). We take α_0 to be the positive distance of u from the closed set $\{k\chi_I; k \in \mathbf{R}, 0 < |I| < \infty\}$ in $L^{p'}(a,b)$. Observe that $m_\alpha \ge 2$ whenever $0 < \alpha < \alpha_0$.

Fix $\alpha, 0 < \alpha < \alpha_0$. By Lemma 3.5, $m_\gamma = m_\alpha$, where $\gamma = \inf \Lambda_\alpha$. Hence, there exists a partition $\{w_\gamma(j)\}_{j=1}^{m_\gamma}$ of (a,b) whose corresponding step functions, $u_\gamma = \sum_{j=1}^{m_\alpha} u_{w_\gamma(j)} \chi_{w_\gamma(j)}$ and $v_\gamma = \sum_{j=1}^{m_\alpha} v_{w_\gamma(j)} \chi_{w_\gamma(j)}$, satisfy

$$\max(\|u - u_\gamma\|_{p',(a,b)}, \|v - v_\gamma\|_{p,(a,b)}) = \gamma.$$

If $\|u - u_\gamma\|_{p',(a,b)} > \|v - v_\gamma\|_{p,(a,b)}$ then for some some j_0, $1 \le j_0 \le m_\alpha$,

$$\|u - u_{w_\gamma(j_0)}\|^{p'}_{p',w_\gamma(j_0)} > 0.$$

It is possible to find a point c in the interval $w_\gamma(j_0) = (d, e)$ such that

$$\|u - u_{w_\gamma(j_0)}\|^{p'}_{p', w_\gamma(j_0)} > \|u - u_{(d,c)}\|^{p'}_{p',(d,c)} + \|u - u_{(c,e)}\|^{p'}_{p',(c,e)}.$$

Let $w'_\gamma(j) = w_\gamma(j)$, $j = 1, 2, \ldots, j_0 - 1, j_0 + 1, \ldots, m_\alpha$, $w'_\gamma(j_0) = (d, c)$ and $w'_\gamma(m_\alpha + 1) = (c, e)$. Then, $\{w'_\gamma(j)\}_{j=1}^{m_\alpha+1}$ is a partition of (a, b) with associated step functions $u'_\gamma = \sum_{j=1}^{m_\alpha+1} u_{w'_\gamma(j)} \chi_{w'_\gamma(j)}$ and $v'_\gamma = \sum_{j=1}^{m_\alpha+1} v_{w'_\gamma(j)} \chi_{w'_\gamma(j)}$ such that

$$\max(\|u - u'_\gamma\|_{p',(a,b)}, \|v - v'_\gamma\|_{p,(a,b)}) = \beta < \gamma,$$

and so $m_\beta = m_\alpha + 1$.

Similarly, when $\|v - v_\gamma\|_{p,(a,b)} > \|u - u_\gamma\|_{p,(a,b)}$, there is a $\beta \in (0, \alpha)$ with $m_\beta = m_\alpha + 1$.

Suppose, then, $\|v - v_\gamma\|_{p,(a,b)} = \|u - u_\gamma\|_{p',(a,b)} = \gamma > 0$. As before, we can find an interval $w_\gamma(j_0) = (d_0, e_0)$ and a point c_0 such that

$$\|u - u_{w_\gamma(j_0)}\|^{p'}_{p', w_\gamma(j_0)} > \|u - u_{(d_0,c_0)}\|^{p'}_{p',(d_0,c_0)} + \|u - u_{(c_0,e_0)}\|^{p'}_{p',(c_0,e_0)},$$

and an interval $w_\gamma(j_1) = (d_1, c_1)$ and a point c_1 such that

$$\|v - v_{w_\gamma(j_1)}\|^{p}_{p, w_\gamma(j_1)} > \|v - v_{(d_1,c_1)}\|^{p}_{p,(d_1,c_1)} + \|v - v_{(c_1,e_1)}\|^{p}_{p,(c_1,e_1)}.$$

Now, if it is possible to have $j_0 = j_1$ and $c_0 = c_1$ we can get $\beta \in (0, \alpha)$ with $m_\beta = m_\alpha + 1$. Otherwise, we can only conclude there is a $\beta \in (0, \alpha)$ for which m_β is one of $m_\alpha + 1$ and $m_\alpha + 2$. □

Lemma 3.7 *Let* $-\infty \le a < b \le \infty$ *and suppose that* $u' \in L^{p'/(p'+1)}(a, b) \cap C(a, b)$. *For each small* $h > 0$ *define*

$$x_1 = -\frac{1}{h}, x_{i+1} := x_i + h \text{ for } i \in 1, \ldots, [2/h^2];$$

put $J_i = (a, b) \cap (x_i, x_{i+1})$, $i \in 1, \ldots, [2/h^2]$.

Then

$$\int_a^b |u'(t)|^{p'/(p'+1)} dt \quad = \quad \lim_{h \to 0} \sum_{i=1}^{[2/h^2]} |J_i| \max_{x \in J_i} |u'(x)|^{p'/(p'+1)}$$

$$= \lim_{h\to 0} \sum_{j=1}^{[2/h^2]} |J_i| \min_{x\in J_i} |u'(x)|^{p'/(p'+1)}.$$

Proof. Simply use the definition of the integral. □

We are now prepared to establish an important estimate for $\limsup_{\alpha\to 0_+} \alpha m_\alpha$.

Theorem 3.8 *Suppose $u \in L^{p'}(a,b)$, $v \in L^p(a,b)$ and $u' \in L^{p'/(p'+1)}(a,b) \cap C(a,b)$, $v' \in L^{p/(p+1)}(a,b) \cap C(a,b)$. Then,*

$$\limsup_{\alpha\to 0_+} \alpha m_\alpha \le c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)}).$$

Proof. The result is trivial if both u and v are constant so we assume that at least one of them, say u, is not.

Given $\beta, 0 < \beta < \inf_{c\in\mathbf{R}} \|u-c\|_{p',(a,b)}$, let $w_\beta(i) = (a_i, a_{i+1})$, $i = 1, 2, \ldots, n_\beta^u$, be a partition of (a,b) satisfying

$$\|u - u_{w_\beta(i)}\|_{p',w_\beta(i)} = \beta, \qquad i = 1, 2, \ldots, n_\beta^u - 1,$$

and $\|u - u_{w_\beta(i)}\|_{p',w_\beta(i)} \le \beta$, $i = n_\beta^u$. Fix λ, $0 < \lambda < 1$, and define the $[\lambda n_\beta^u]$ points x_k by the rule that if (a,b) is bounded, then

$$x_k := a + \frac{b-a}{\lambda n_\beta^u} k, \qquad k = 1, 2, \ldots, [\lambda n_\beta^u];$$

if $(a,b) = (-\infty, \infty)$, then, with $h = (\frac{2}{\lambda n_\beta^u})^{1/2}$,

$$x_1 = -\frac{1}{h}, \quad x_{k+1} = x_k + h, \qquad k = 1, 2, \ldots, [\lambda n_\beta^u];$$

for other types of intervals we proceed in the same sort of way.

From the union of the points $a_1, a_2, \ldots, a_{n_\beta^u} + 1$ and $x_1, x_2, \ldots, x_{[\lambda n_\beta^u]}$, arrange them in the ascending order and denote the resulting points by $b_j, j = 1, 2, \ldots, J(\beta) + 1$, so that $n_\beta^u \le J(\beta) \le n_\beta^u + [\lambda n_\beta^u]$. Put $I_j^\beta = (b_j, b_{j+1}), j = 1, 2, \ldots, J(\beta)$. We observe there are at least $n_\beta^u - [\lambda n_\beta^u]$ intervals I_j^β with

$$I_j^\beta = w_\beta(i)$$

for some i.

Now,

$$\sum_{j=1}^{J(\beta)} \|u - u_{I_j^\beta}\|_{p',I_j^\beta}^{p'/(p'+1)} \leq \sum_{j=1}^{J(\beta)} |I_j^\beta| \max_{x\in I_j^\beta} |u'(x)|^{p'/(p'+1)}.$$

Again, setting $N = \#(\{j : I_j^\beta = w_\beta(i) \text{ for some } i < n_\beta^u\})$, we have $N \geq n_\beta^u - [\lambda n_\beta^u] - 1$ and

$$\beta^{p'/(p'+1)}(n_\beta^u - [\lambda n_\beta^u] - 1) \leq \beta^{p'/(p'+1)} N \leq \sum_{j=1}^{J(\beta)} \|u - u_{I_j^\beta}\|_{p',I_j^\beta}^{p'/(p'+1)}$$

$$\leq \sum_{j=1}^{J(\beta)} |I_j^\beta| \max_{x\in I_j^\beta} |u'(x)|^{p'/(p'+1)}.$$

Thus, by Lemma 3.7,

$$\limsup_{\beta\to 0_+} \beta^{p'/(p'+1)}(n_\beta^u - [\lambda n_\beta^u]) \leq \int_a^b |u'(x)|^{p'/(p'+1)} dx. \tag{3. 3}$$

Similarly, if neither v is constant, there exists, for $0 < \beta < \inf_{c\in\mathbf{R}} \|v - c\|_{p,(a,b)}$, a partition $\{w'_\beta(i)\}_{i=1}^{n_\beta^v}$ such that

$$\|v - v_{w'_\beta(i)}\|_{p,w'_\beta(i)} = \beta, \qquad i = 1, 2, \ldots, n_\beta^v - 1,$$

$$\|v - v_{w'_\beta(i)}\|_{p,w'_\beta(i)} \leq \beta, \qquad i = n_\beta^v,$$

and

$$\limsup_{\beta\to 0_+} \beta^{p/(p+1)}(n_\beta^v - [\lambda n_\beta^v]) \leq \int_a^b |v'(x)|^{p/(p+1)} dx. \tag{3. 4}$$

Put $\alpha = \max[(\beta^{p'}(n_\beta + [\lambda n_\beta]))^{1/p'}, (\beta^p(n_\beta + [\lambda n_\beta]))^{1/p}], 0 < \beta < \min[\inf_{c\in\mathbf{R}} \|u - c\|_{p',(a,b)}, \inf_{c\in\mathbf{R}} \|v - c\|_{p,(a,b)}]$, where $n_\beta = n_\beta^u + n_\beta^v$ if v is not constant and $n_\beta = n_\beta^u$ if it is. Note that (3. 3) and (3. 4) imply $\alpha \to 0_+$ as $\beta \to 0_+$.

Taking the refinement of the partition $\{I_j^\beta\}_{j=1}^{J(\beta)}$ and the analogous one for v (if necessary) we get a partition of (a, b), of at most $n_\beta + [\lambda n_\beta]$ subintervals, whose corresponding step-functions u_α and v_α satisfy

$$\max[\|u - u_\alpha\|_{p',(a,b)}, \|v - v_\alpha\|_{p,(a,b)}] \leq \beta \max[(n_\beta^u)^{1/p'}, (n_\beta^v)^{1/p}] \leq \alpha.$$

This means

$$m_\alpha \leq n_\beta + [\lambda n_\beta];$$

hence

$$\begin{aligned}
\limsup_{\alpha\to 0_+}(\alpha m_\alpha) &\leq \limsup_{\alpha\to 0_+}(\alpha m_\alpha) \\
&\quad + \limsup_{\alpha\to 0_+}(\alpha m_\alpha) \\
&\leq \limsup_{\beta\to 0_+}\left[\beta^{p'}(n_\beta - [\lambda n_\beta])^{1/p'}\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{1/p'}\right] \\
&\quad + \limsup_{\beta\to 0_+}\left[\beta^{p}(n_\beta - [\lambda n_\beta])^{1/p}\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right)^{1/p}\right] \\
&\leq \Bigg(\limsup_{\beta\to 0_+}\left[\beta^{p'}(n_\beta - [\lambda n_\beta])^{1/p'}\right] \\
&\quad + \limsup_{\beta\to 0_+}\left[\beta^{p}(n_\beta - [\lambda n_\beta])^{1/p}\right]\Bigg)\left(\frac{n_\beta + [\lambda n_\beta]}{n_\beta - [\lambda n_\beta]}\right) \\
&\leq c(p,p')(\|u'\|^{(p'+1)/p'}_{p'/(p'+1),(a,b)} + \|v'\|^{(p+1)/p}_{p/(p+1),(a,b)})\frac{(1+\lambda)}{(1-\lambda)}.
\end{aligned}$$

Since λ may be chosen arbitrarily small, we obtain

$$\limsup_{\alpha\to 0_+}\alpha m_\alpha \leq c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)}),$$

as asserted. □

4 The Main theorem.

In this section we give the remainder estimate promised in the Introduction. To begin, we prove

Theorem 4.1 *Let* $-\infty \leq a < b \leq \infty$*, let* $u \in L^{p'}(a,b)$*,* $v \in L^p(a,b)$ *and suppose that* $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b])$*,* $v' \in L^{p/(p+1)}(a,b) \cap C([a,b])$*. Then*

$$\limsup_{\varepsilon\to 0_+}\left|\alpha_p\int_a^b |u(t)v(t)|dt - \varepsilon N(\varepsilon)\right| N^{1/2}(\varepsilon)$$

$$\leq c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)})\left(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}\right)$$
$$+3\alpha_p\|uv\|_{1,(a,b)},$$

where $\alpha_p = A((0,1),1,1)$ and $c(p,p')$ is a constant depending only on p and p'.

Proof. Let $\alpha > 0$. Then (see (3. 1) and (3. 2)) there are $m_\alpha \in \mathbf{N}$ and step-functions u_α, v_α such that

$$\|u_\alpha - u\|_{p',(a,b)} < \alpha, \qquad \|v_\alpha - v\|_{p,(a,b)} < \alpha;$$

and $\{w_\alpha(j)\}_{j=1}^{m_\alpha}$ is a corresponding family of non-overlapping intervals which cover (a,b). Plainly,

$$|\int_a^b (uv - u_\alpha v_\alpha)dt| \leq \alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)} + \alpha). \tag{4. 1}$$

Let $\varepsilon > 0$ be small and let $\{I_i\}_{i=1}^{N(\varepsilon)}$ be the non-overlapping intervals which occur in the definition of $N(\varepsilon)$.

Put $J_1 = \{j; I_i \subset w_\alpha(j)$ for some $i\}$, $J_2 = \{j; w_\alpha(j) \subset I_i$ for some $i\}$, $J_3 = \{j; w_\alpha(j) \not\subset I_i \not\subset w_\alpha(j)$, for all $i\}$, $L_1 = \{i; I_i \subset w_\alpha(j)$ for some $j\}$ and $L_2 = \{i;$ for all $j, I_i \not\subset w_\alpha(j)\}$. Then we see from Lemma 3.1 that

$$\begin{aligned} \alpha_p \int_a^b u_\alpha v_\alpha dt &= \alpha_p \left(\sum_{j\in J_1} + \sum_{j \in J_2} + \sum_{j\in J_3} \right) \xi_j \psi_j |w_\alpha(j)| \\ &\leq \sum_{i \in L_1} A(I_i, u_\alpha, v_\alpha) \\ &\quad + 2\sum_{i\in L_2} A(I_i, u_\alpha, v_\alpha) \\ &\quad + \sum_{j \in J_2} \alpha_p \xi_j \psi_j |w_\alpha(j)|. \end{aligned} \tag{4. 2}$$

Lemmas 3.2, 3.3 as well as the estimates

$$\begin{aligned} \alpha_p \xi_j \psi_j |w_\alpha(j)| &\leq A(w_\alpha(j), u_\alpha, v_\alpha) \\ &\leq A(w_\alpha(j), u, v) + \|u - u_\alpha\|_{p', w_\alpha(j)} \|v - v_\alpha\|_{p, w_\alpha(j)} \\ &\quad + \|u\|_{p', w_\alpha(j)} \|v - v_\alpha\|_{p, w_\alpha(j)} \\ &\quad + \|u - u_\alpha\|_{p', w_\alpha(j)} \|v\|_{p, w_\alpha(j)} \end{aligned}$$

and $A(w_\alpha(j),u,v) \le A(I_i,u,v) \le \varepsilon$ for $w_\alpha(j) \subset I_i$ now show that the right-hand side of (4. 2) may be estimated from above by

$$\sum_{I_i \subset w_\alpha(j)} A(I_i,u,v) + 2 \sum_{I_i \not\subset w_\alpha(j)} A(I_i,u,v) + \varepsilon m_\alpha$$
$$+3\sum_{i=1}^{N(\varepsilon)} (\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \qquad (4.\ 3)$$
$$+\|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i})$$

To proceed further, note that $A(I_i,u,v) \le \varepsilon$,

$$\#\{i; I_i \subset w_\alpha(j) \text{ for some } j\} \le N(\varepsilon)$$

and

$$\#\{i, \text{ for all } j, I_i \not\subset w_\alpha(j)\} \le m_\alpha.$$

It follows that

$$\begin{aligned} \alpha_p \int_a^b u_\alpha v_\alpha &\le N(\varepsilon)\varepsilon + 3m_\alpha\varepsilon \\ &\quad +3\sum_{i=1}^{N(\varepsilon)} (\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \\ &\qquad +\|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i}) \\ &\le N(\varepsilon)\varepsilon + 3m_\alpha\varepsilon + 2\alpha^2 + 2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}). \qquad (4.\ 4) \end{aligned}$$

On the other hand, since $A(I_i,u,v) = \varepsilon$ for $1 \le i \le N(\varepsilon)-1$ and $N(\varepsilon) - 2m_\alpha \le \#\{i; I_i \subset w_\alpha(j) \text{ for some } j\}$, we see that

$$\begin{aligned} (N(\varepsilon) - 2m_\alpha - 1)\varepsilon &\le \sum_{I_i \subset w_\alpha(j)} A(I_i,u,v) \\ &= \sum_{I_i \subset w_\alpha(j)} A(I_i,u_\alpha,v_\alpha) \\ &\quad + \sum_{I_i \subset w_\alpha(j)} [A(I_i,u,v) - A(I_i,u_\alpha,v_\alpha)] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{I_i \subset w_\alpha(j)} \alpha_p |I_i||\xi_j||\psi_j| \\
&\quad + \sum_{I_i \subset w_\alpha(j)} (\|u-u_\alpha\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} \\
&\qquad +\|u\|_{p',I_i}\|v-v_\alpha\|_{p,I_i} + \|u-u_\alpha\|_{p',I_i}\|v\|_{p,I_i}) \\
&\leq \alpha_p \int_a^b |u_\alpha v_\alpha| dt + \alpha^2 + \alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}) \\
&\leq \alpha_p \int_a^b |uv| dt + 2\alpha^2 \\
&\quad +2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}), \qquad (4.\ 5)
\end{aligned}
$$

the final inequality following from (4. 1). Together with (4. 4) and (4. 1) this shows that

$$
\begin{aligned}
\varepsilon(N(\varepsilon) &- 2m_\alpha - 1) - 2\alpha^2 - 2\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}) \\
&\leq \alpha_p \int_a^b |uv| dt \qquad (4.\ 6) \\
&\leq \varepsilon(N(\varepsilon) + 3m_\alpha) + 3\alpha^2 + 3\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).
\end{aligned}
$$

From Lemma 3.4 we can see that for any small $\varepsilon > 0$, we can find $\alpha > 0$ such that $m_\alpha \geq [N^{1/2}(\varepsilon)] \geq m_\alpha - 2$. Then (4. 6) gives

$$
\begin{aligned}
N^{1/2}(\varepsilon)|\alpha_p \int_a^b |uv| dt - N(\varepsilon)\varepsilon| &\leq 3N(\varepsilon)\varepsilon + 3\alpha^2(N^{1/2}(\varepsilon) - 1) \\
&\quad +3\alpha(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)})N^{1/2}(\varepsilon).
\end{aligned}
$$

Let $\varepsilon \to 0_+$; then $m_\alpha \leq N^{1/2}(\varepsilon) + 2 \to \infty$ and so $\alpha \to 0_+$. Hence

$$
\begin{aligned}
&\limsup_{\varepsilon \to 0_+} N^{1/2}(\varepsilon)|\alpha_p \int_a^b |uv| dt - N(\varepsilon)\varepsilon| \\
&\quad \leq 3 \limsup_{\varepsilon \to 0_+} N(\varepsilon)\varepsilon + 3 \limsup_{\varepsilon \to 0_+} \alpha^2 N^{1/2}(\varepsilon) \\
&\qquad +3 \limsup_{\varepsilon \to 0_+} \alpha N^{1/2}(\varepsilon)(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).
\end{aligned}
$$

Since $\lim_{\varepsilon \to 0} \varepsilon N(\varepsilon) = \alpha_p \int_a^b |uv| dt$, by Lemma 2.8, we finally see, with the help of Lemma 3.8, that

$$\limsup_{\varepsilon\to 0_+} \quad N^{1/2}(\varepsilon)|\alpha_p \int_a^b |uv| - N(\varepsilon)\varepsilon|$$

$$\leq \quad 3\alpha_p \int_a^b |uv|dt$$
$$+3c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'\|_{p/(p+1),(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}),$$

as required. □

Armed with this result it is now easy to give the promised remainder estimate for the approximation numbers of $T : L^2(a,b) \to L^2(a,b)$ given by (1. 1).

Theorem 4.2 *Let* $-\infty \leq a < b \leq \infty$*, suppose that* $u \in L^{p'}(a,b)$*,* $v \in L^p(a,b)$ *and let* $u' \in L^{p'/(p'+1)}(a,b) \cap C((a,b))$*,* $v' \in L^{p/(p+1)}(a,b) \cap C((a,b))$*. Then*

$$\limsup_{n\to\infty} n^{1/2} \left| \alpha_p \int_a^b |uv|dt - na_n \right| \leq 3\alpha_p \int_a^b |uv|dt$$

$$+3c(p,p')(\|u'\|_{p'/(p'+1),(a,b)} + \|v'|)\|_{p/(p+1),(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).$$

Proof. Simply use Theorem 4.1, Lemma 2.7, Lemma 2.6 and the fact that

$$\lim_{n\to\infty} n^{1/2} a_n(T) = 0.$$

□

If the interval (a,b) is bounded, it follows immediately from Hölder's inequality that Theorem 4.2 gives rise to

Theorem 4.3 *Let* $-\infty < a < b < \infty$ *and suppose that* $u', v' \in C([a,b])$*. Then*

$$\limsup_{n\to\infty} n^{1/2} |\alpha_p \int_a^b |uv|dt - na_n|$$

$$\leq 3\alpha_p \int_a^b |uv|dt + 3c(p,p')(b-a)(\|u'\|_{p',(a,b)} + \|v'\|_{p,(a,b)})(\|u\|_{p',(a,b)} + \|v\|_{p,(a,b)}).$$

From the following observation we can see that any optimal exponent from Theorem 4.2 has to belong to $[1/2, 1]$.

Observation 4.4 *Let* $-\infty \le a < b \le \infty$.

(i) Let $\alpha < 1/2$. *Then for every* $u \in L^{p'}(a,b)$, $v \in L^p(a,b)$ *with* $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b])$, $v' \in L^{p/(p+1)}(a,b) \cap C([a,b])$ *we have*

$$\limsup_{n\to\infty} n^\alpha \left| \alpha_p \int_a^b |uv|dt - na_n(T) \right| = 0.$$

(ii) Let $\alpha > 1$. *Then there exist* a *and* b, *and functions* u *and* v *satisfying the conditions of Theorem 4.2 on the interval defined by* a *and* b, *such that*

$$\limsup_{n\to\infty} n^\alpha \left| \alpha_p \int_a^b |uv|dt - na_n(T) \right| = \infty.$$

Proof. (i) follows from (4. 6) on putting $m_\alpha = [N^\alpha(\varepsilon)]$ or $[N^\alpha(\varepsilon)] + 1$.

(ii) Take $(a,b) = (0,1)$ and $u = 1$, $v = 1 + x$. Then from (4. 6), with $m_\alpha = [N^\alpha(\varepsilon)]$ a lower bound results which is unbounded as $\varepsilon \to 0$ and the result follows. □

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