BESTVINA'S NORMAL FORM COMPLEX AND THE HOMOLOGY OF GARSIDE GROUPS

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ABSTRACT. A Garside group is a group admitting a finite lattice generating set $\mathcal{D}$. Using techniques developed by Bestvina for Artin groups of finite type, we construct $K(\pi, 1)$s for Garside groups. This construction shows that the (co)homology of any Garside group G is easily computed given the lattice $\mathcal{D}$, and there is a simple sufficient condition that implies G is a duality group. The universal covers of these $K(\pi, 1)$s enjoy Bestvina's weak non-positive curvature condition. Under a certain tameness condition, this implies that every solvable subgroup of G is virtually abelian.

1. INTRODUCTION

The main goal of this note is to establish the following result:

Main Theorem. *Garside groups admit finite $K(\pi, 1)$s.*

A Garside group G is the group of fractions of a Garside monoid G^+, where G^+ contains a 'Garside element' Δ whose divisors form a finite lattice $\mathcal{D}$ that generates G^+. (We give a complete definition in §2.) These groups, first introduced by Dehornoy and Paris in [17], generalize Artin groups of finite type, and they share many formal properties with Artin groups of finite type.

The construction of these $K(\pi, 1)$s is concrete and is based on the lattice $\mathcal{D}$. That is, given the lattice $\mathcal{D}$, our proof of the Main Theorem is constructive, and one can easily compute the (co)homology of the associated Garside group (see Theorem 3.6). Topological properties of subcomplexes of the geometric realization $|\mathcal{D}|$ give information on the end connectivity of any Garside group, which yields a simple condition implying that a given Garside group is a duality group (see 4.2 and 4.4).

The construction of these $K(\pi, 1)$s mimics a construction given by Bestvina in the case of Artin groups of finite type [3]. This construction is also used by Brady, as well as Brady and Watt, in their constructions of new $K(\pi, 1)$s for Artin groups of finite type ([7] and [10]). The argument given here indicates that this construction is "functorial," in that it works for any group admitting a lattice generating set, not just the Artin groups of finite type. In particular we extend Bestvina's results on a weak nonpositive curvature condition for Artin groups of finite type to all Garside groups. Under a mild tameness condition on the Garside element (Definition 6.10), this implies that every solvable subgroup of the Garside group is virtually a finitely generated abelian group (Corollary 6.13).

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2. Garside groups and the Deligne normal forms

In terms of the word problem and normal forms, Garside groups behave much like Artin groups of finite type [13]. Here we collect relevant information about the word problem for Garside groups. Most of this has to do with issues in the positive monoid G^+.

Let M be a monoid and let the *indivisible* elements (often called 'atoms') be those $m \in M$ such that $m \neq 1$ and if $m = ab$ then either $a = 1$ or $b = 1$. Let $||m||$ be the supremum of the lengths of all expressions of m in terms of indivisible elements. The monoid M is *atomic* if it is generated by its indivisible elements, and the norm $||m||$ of any element is finite. In an atomic monoid one can define a *partial order* via left (or right) divisibility: $a < b$ if $ac = b$ for some $c \in M - \{1\}$; $a \leq b$ if $ac = b$ for some $c \in M$. Conversely, if M is a finitely generated monoid where the partial order given by divisibility contains no infinite descending chains, then M is atomic, with $||a||$ being the length of the longest expression.

Definition 2.1 (Garside monoid)**.** An atomic monoid M is *Garside* if it satisfies

1. left and right cancellation laws hold in M,
2. any two elements of M admit a least common multiple and a greatest common divisor on both the left and the right,
3. there exists an element Δ such that the left and right divisors of Δ are the same, there are finitely many of them, and they form a set of generators for M.

The element Δ is called a *Garside element.* We denote the set of divisors of Δ by $\mathcal{D}$, and call this generating set the *simple divisors.*

Garside monoids satisfy Ore's criterion, hence they embed in their group of fractions, and thus we may define a *Garside group* to be the group of fractions of a Garside monoid.

Remark 2.2. Many of the properties of these groups were described in the case of the braid groups by Garside in [18]; hence the name "Garside group". When first introduced in [17], however, these groups were called "small Gaussian", while "Garside" was used for a slightly more restrictive condition (which required that Δ be the least common multiple of the indivisible elements). As it has developed that the definition given above is more natural and more useful, the name "Garside" has now been generally adopted for this class of monoids and their groups of fractions [15].

The classic examples of Garside groups are the Artin groups of finite type, where the Garside element is commonly denoted Δ. These groups also admit presentations based on other monoids, such as the ones explored by Bessis, Brady and Watt, where the Garside element is not the usual Δ, but rather a Coxeter element δ ([1], [7], and [10]). This collection contains many other groups (see [13], citegargp, [18], [17] and [20]).

We denote an Garside group by G and let G^+ be its positive monoid. The inverse of an element $g \in G$ will be denoted by $\overline{g}$. We note that the partial order given by left (or right) divisibility extends to the group by defining $g < h$ when $\overline{g}h \in G^+ - \{1\}$, for $g, h \in G$. (Because Garside monoids are cancellative, this partial order is equivalent to the partial order defined above when restricted to the positive monoid.)

Let $g \in G^+$. The *left front* of g is defined to be the left gcd of g and Δ, $\mathrm{LF}(g) = g \wedge \Delta$. If $\mu \in \mathcal{D}$ satisfies $\mu < g$ then $\mu < \mathrm{LF}(g)$. One can use left fronts to define a normal form in G^+, and this normal form is commonly referred to as the *left greedy normal form.*

Proposition 2.3. (3.5 in [13]) *Let G^+ be an Garside monoid. Then g may be uniquely represented as a product of simple divisors, $g = \mu_1\mu_2 \cdots \mu_n$, where $\mu_i = LF(\mu_i \cdots \mu_n)$.*

Proposition 2.4. (3.10 in [13]) *For all $g, h \in G^+$, $LF(gh) = LF(gLF(h))$.*

There is an analogously defined *right greedy* normal form in which one begins by taking the (oddly named) *right front* $\mathrm{RF}(g)$ to be the right gcd of the monoid element g and the Garside element Δ. Then as in Proposition 2.3 each $g \in G^+$ can be uniquely represented as the product of simple divisors $g = \mu_1 \cdots \mu_n$ where $\mu_i = \mathrm{RF}(\mu_1 \cdots \mu_i)$. We make reference to this right greedy normal form in our discussion of ascending and descending links in the sections that follow.

Proposition 2.5. (2.2 and 2.3 in [13]) *Let G^+ be an Garside monoid with Garside element Δ. Then there is a permutation σ of the set of simple divisors $\mathcal{D}$ such that*

$$\Delta\mu = \sigma(\mu)\Delta$$

for all $\mu \in \mathcal{D}$. In particular, there is an m such that Δ^m is central. Further, if $\mu \in \mathcal{D}$ can be expressed as the product of n indivisibles, then $\sigma(\mu)$ can also be expressed as the product of n indivisibles.

Definition 2.6 (Complements)**.** If $\mu \in \mathcal{D}$ then by definition there is an element $\mu^* \in G^+$ such that $\mu\mu^* = \Delta$. We call μ^* the *right complement* of μ, and note that because μ^* is a right divisor of Δ, $\mu^* \in \mathcal{D}$, and thus $\mathcal{D}$ is closed under right complements.

There is also a left complement of μ, ${}^*\mu$, where ${}^*\mu\mu = \Delta$. If we right multiply this last equation by μ^* we get ${}^*\mu\Delta = \Delta\mu^*$, hence ${}^*\mu = \sigma(\mu^*)$. While this formula shows that one does not have to use the notation ${}^*\mu$, we do use this notation since it is easier to understand than $\sigma(\mu^*)$.

Since $\mathcal{D}$ is a generating set for the positive monoid G^+, and every element of $\mathcal{D}$ divides Δ, in order to represent elements of the group G, it suffices to invert the Garside element. Thus we get a set of normal forms for an Garside group that closely parallel those of Deligne for Artin groups of finite type.

Theorem 2.7. *Every element $g \in G$ can be expressed uniquely as $g = \mu_1 \cdots \mu_k \Delta^n$ ($n \in \mathbb{Z}$) where the prefix $\mu_1 \cdots \mu_k$ is in G^+, it is in left greedy normal form, and $\mu_1 \neq \Delta$.*

Definition 2.8 (Deligne normal forms)**.** We refer to the normal form given in Theorem 2.7 as the *Deligne normal form* of the Garside group G. We denote the Deligne normal form of an element $g \in G$ by $\mathrm{DNF}(g)$. Thus DNF can be thought of as a map from G to the free monoid $\{\mathcal{D} \cup \{\Delta^{-1}\}\}^*$, providing a (set theoretic) section of the natural surjection $\{\mathcal{D} \cup \{\Delta^{-1}\}\}^* \twoheadrightarrow G$.

In the next section we'll be working with the Cayley graph of a Garside group with respect to the set of simple divisors $\mathcal{D}$. We'll make use of the following lemma.

Lemma 2.9. *Let $\mu_1 \cdots \mu_n$ be a word in $\mathcal{D}^*$ in left greedy normal form, where $\mu_1 \neq \Delta$. If $\eta \in \mathcal{D}$, then the left greedy normal form for $\mu_1 \cdots \mu_n \eta$ begins with at most one Δ.*

Proof. It is clear that right multiplication by η can produce no Δ or one Δ, so it suffices to show that $\Delta^2 \not\leq \mu_1 \cdots \mu_n \eta$. If $\Delta^2 \leq \mu_1 \cdots \mu_n \eta$, then $\Delta^2 b = \mu_1 \cdots \mu_n \eta$ for some $b \in G^+$. Thus, shifting one of the Δs past b we get

$$\Delta \sigma(b) \Delta = \mu_1 \cdots \mu_n \eta,$$

hence

$$\Delta \sigma(b) \ {}^*\eta = \mu_1 \cdots \mu_n$$

and therefore $\Delta \leq \mu_1 \cdots \mu_n$, which is a contradiction. □

3. The $K(\pi, 1)$ Complexes

Recall that a *flag complex* X is a simplicial complex where every complete subgraph on n-vertices in $X^{(1)}$ is the 1-skeleton of an $(n-1)$-simplex in X. Thus a flag complex is determined by its 1-skeleton.

Fix an Garside group G, whose positive monoid is G^+, with Garside element Δ, and simple divisors $\mathcal{D}$. Let $\mathcal{C}_\mathcal{D}$ be the Cayley graph with respect to the lattice generating set $\mathcal{D}$. That is, $\mathcal{C}_\mathcal{D}$ has vertices corresponding to the elements of G and directed, labelled edges corresponding to right multiplication by elements of $\mathcal{D}$. Let $\widehat{\mathcal{X}}_\mathcal{D}$ be the flag complex induced by the graph $\mathcal{C}_\mathcal{D}$; thus each complete subgraph of $\mathcal{C}_\mathcal{D}$ is the 1-skeleton of a simplex in $\widehat{\mathcal{X}}_\mathcal{D}$. The action of G on $\mathcal{C}_\mathcal{D}$ by left multiplication extends to an action of G on $\widehat{\mathcal{X}}_\mathcal{D}$. This action is free since it is free on the set of vertices and G is torsion-free [14]. We establish our Main Theorem by proving

Theorem 3.1. *The complex $\widehat{\mathcal{X}}_\mathcal{D}$ is a contractible G-complex, and $G \backslash \widehat{\mathcal{X}}_\mathcal{D}$ is a finite $K(G, 1)$.*

In order to establish Theorem 3.1 we show that $\widehat{\mathcal{X}}_\mathcal{D}$ admits a product structure. Let $\mathcal{X}_\mathcal{D}$ be the full subcomplex of $\widehat{\mathcal{X}}_\mathcal{D}$ induced by vertices associated with the set of all $g \in G$ where the Deligne normal form of g contains no Δs. The complex $\mathcal{X}_\mathcal{D}$ can also be thought of as a *coset complex* where the vertices correspond to cosets of $\langle \Delta \rangle$. Coset representatives can be taken from the positive monoid G^+, and represented by the prefixes that occur in the Deligne normal forms. There is an edge between two vertices in this complex if their coset representatives differ by right multiplication by some $\mu \in \mathcal{D} - \Delta$, and $\mathcal{X}_\mathcal{D}$ is the flag complex induced by this graph.

Lemma 3.2. *The complex $\widehat{\mathcal{X}}_\mathcal{D}$ is homeomorphic to the product $\mathcal{X}_\mathcal{D} \times \mathbb{R}$.*

Proof. A k-simplex in $\mathcal{X}_\mathcal{D}$ corresponds to a collection of elements $a_i \in G^+$ where $\Delta \not< a_i$, $a_0 < \cdots < a_k$ and $a_k < a_0 \Delta$. Similarly, a k-simplex in $\widehat{\mathcal{X}}_\mathcal{D}$ consists of elements $a_i \in G$ where $a_0 < \cdots < a_k$ (using the extension of the partial order to G) and $a_k \leq a_0 \Delta$. We give $\mathbb{R}$ the standard simplicial structure where the vertices correspond to the integers.

There are projections from the 0-skeleton of $\widehat{\mathcal{X}}_{\mathcal{D}}$ onto the 0-skeleta of $\mathcal{X}_{\mathcal{D}}$ and $\mathbb{R}$ defined via the Deligne normal forms. Let $g \in G$ have Deligne normal form $\mathrm{DNF}(g) = \mu_0 \cdots \mu_k \Delta^n$. Define π_+ to be the map sending g to $\mu_0 \cdots \mu_k$ and π_Δ the map sending g to n. Both maps can be thought of as (set theoretic) retractions of $\widehat{\mathcal{X}}_{\mathcal{D}}^{(0)}$; π_+ retracts the vertices of $\widehat{\mathcal{X}}_{\mathcal{D}}$ onto the vertices of $\mathcal{X}_{\mathcal{D}}$ while π_Δ can be viewed as retracting the vertices of $\widehat{\mathcal{X}}_{\mathcal{D}}$ onto the vertices corresponding to the infinite cyclic subgroup $\langle \Delta \rangle < G$.

We form continuous maps from $\widehat{\mathcal{X}}_{\mathcal{D}}$ to $\mathcal{X}_{\mathcal{D}}$ and $\mathbb{R}$ by extending π_+ and π_Δ linearly over simplices. Since $\mathcal{X}_{\mathcal{D}}$ is a flag complex, it suffices to show that π_Δ and π_+ take edges of $\widehat{\mathcal{X}}_{\mathcal{D}}$ to edges or vertices of their target spaces. For the rest of this discussion, fix an edge e in $\widehat{\mathcal{X}}_{\mathcal{D}}$ and let the bounding vertices of e correspond to the group elements $a\Delta^n$ and $a\Delta^n\mu$ where $a \in G^+$, $\Delta \not\leq a$, and $\mu \in \mathcal{D}$. The Deligne normal form for $a\Delta^n\mu$ is then either $\mathrm{DNF}(a\sigma^n(\mu))\Delta^n$, or $\mathrm{DNF}(b)\Delta^{n+1}$ if it's the case that $a = b({}^*[\sigma^n(\mu)])$. (The element $a\sigma^n(\mu)$ is divisible by at most one Δ by Lemma 2.9.)

The map π_Δ takes e of $\widehat{\mathcal{X}}_{\mathcal{D}}$ to the edge $[n, n+1]$ (when right multiplying by μ introduces a Δ) or it collapses e to the vertex corresponding to n (when ${}^*[\delta^n(\mu)]$ is not a right divisor of a). Hence the map π_Δ extends to a map from $\widehat{\mathcal{X}}_{\mathcal{D}}$ onto $\mathbb{R}$.

Similarly the map π_+ extends to edges. Let e be as before, so that π_+ maps the vertex $a\Delta^n$ to a and

$$\pi_+(a\Delta^n\mu) = \begin{cases} a\sigma^n(\mu) & \text{if } a \text{ is not right divisible by } {}^*[\sigma^n(\mu)] \\ b = a\left({}^*[\sigma^n(\mu)]\right)^{-1} & \text{otherwise} \end{cases} .$$

So π_+ takes e to an edge of $\mathcal{X}_{\mathcal{D}}$, except in the case where $\mu = \Delta$, in which case ${}^*[\delta^n(\mu)] = {}^*\Delta = 1$, and π_+ collapses e to a vertex.

Since π_+ and π_Δ extend to $\widehat{\mathcal{X}}_{\mathcal{D}}^{(1)}$, and all three spaces are flag complexes, these maps extend to all of $\widehat{\mathcal{X}}_{\mathcal{D}}$. Thus we have a continuous map $\pi : \widehat{\mathcal{X}}_{\mathcal{D}} \to \mathcal{X}_{\mathcal{D}} \times \mathbb{R}$ given on the level of vertices by $\pi(g) = (\pi_+(g), \pi_\Delta(g))$. It's clear that π is surjective.

There is also a natural return map $\varpi : \mathcal{X}_{\mathcal{D}} \times \mathbb{R} \to \widehat{\mathcal{X}}_{\mathcal{D}}$ that on the level of vertices is described by $(a, n) \mapsto a\Delta^n$. The cells in $\mathcal{X}_{\mathcal{D}} \times \mathbb{R}$ are of the form $\sigma \times I$, where σ is a k-simplex in $\mathcal{X}_{\mathcal{D}}$, and I is a vertex or edge. By definition the simplex σ corresponds to a collection of elements $a_i \in G^+$ where $\Delta \not\leq a_i$, $a_0 < \cdots < a_k$ and $a_k < a_0\Delta$. If $I = [n, n+1]$ then there is then a simplicial subdivision of the cell $\sigma \times I$ induced by taking the simplices

$$\left\{\{a_0\Delta^n < \cdots < a_k\Delta^n < a_0\Delta^{n+1}\}, \ldots, \{a_k\Delta^n < a_0\Delta^{n+1} < \cdots < a_k\Delta^{n+1}\}\right\} .$$

With this subdivision, $\mathcal{X}_{\mathcal{D}} \times \mathbb{R}$ is a simplicial complex (in fact a flag complex), and so in order to extend ϖ to a map of complexes, it suffices to check that the ϖ can be extended to a simplicial map between the 1-skeleta.

In the simplicial subdivision of $\mathcal{X}_{\mathcal{D}} \times \mathbb{R}$, there is an edge between (a, n) and (b, m) if $|m - n| \leq 1$ and $a\Delta^n < b\Delta^m \leq a\Delta^{n+1}$, or similarly $b\Delta^m < a\Delta^n \leq b\Delta^{m+1}$. Because the two cases are symmetric we may use the previous set of inequalities without loss of generality, and we note that either $m = n$ or $m = n + 1$.

If $m = n$ then by dividing out Δ^n we see that $a < b \leq a\Delta$ hence $b = a\mu$ for some $\mu \in \mathcal{D}$. Thus (b, m) maps to the vertex associated with $a\mu\Delta^n = a\Delta^n\sigma^{-n}(\mu)$ which is joined to the vertex associated with $a\sigma^n$ by an edge labelled $\sigma^{-n}(\mu)$.

If $m = n + 1$ then by dividing out Δ^n we see that $a < b\Delta \leq a\Delta$ hence $b = a\mu^{-1}$ for some $\mu \in \mathcal{D}$. Thus $b\Delta = a\mu^*$ and the pair (b, m) maps to the vertex associated with $a\mu^*\Delta^n = a\Delta^n\sigma^{-n}(\mu^*)$ and so there is an edge joining $\varpi(a,n)$ and $\varpi(b,m)$ when (a,n) is joined to (b,m) in the simplicial decomposition of $\mathcal{X}_\mathcal{D} \times \mathbb{R}$.

Since the composition $\varpi \circ \pi$ is the identity on $\widehat{\mathcal{X}}_\mathcal{D}$, the two complexes are homeomorphic. □

Remark 3.3. The complex $\mathcal{X}_\mathcal{D}$, viewed as a coset complex, admits a left G action. The kernel of this action is $\langle\Delta^m\rangle$, where m is the smallest power of Δ which is central, and hence it descends to an action by the quotient group $G_\Delta = G/\langle\Delta^m\rangle$. The action of G_Δ on $\mathcal{X}_\mathcal{D}$ has finite stabilizers, namely conjugates of the subgroup $\langle\Delta\rangle/\langle\Delta^m\rangle$.

The projection π_+ of $\widehat{\mathcal{X}}_\mathcal{D}$ onto $\mathcal{X}_\mathcal{D}$ is equivariant with respect to the G-actions, but the inclusion of $\mathcal{X}_\mathcal{D}$ as a subcomplex of $\widehat{\mathcal{X}}_\mathcal{D}$ is not.

Lemma 3.4. *The complex $\mathcal{X}_\mathcal{D}$ is contractible.*

Proof. We use the norm $||a||$ — where $||a||$ is the maximum length of a representative of a as a product of indivisible elements — to define a Morse function $\nu : \mathcal{X}_\mathcal{D} \to [0,\infty)$. On the level of vertices, which are represented by the coset representatives, $\nu(a) = ||a||$; the map is then extended linearly over the simplices. To see that this map is non-constant on edges, consider the two possibilities: for $a \in G^+$ with $\Delta \not\leq a$, and $\mu \in \mathcal{D} - \{\Delta\}$, the left greedy normal form of $a\mu$ begins with zero or one Δ. If it contains no Δ, then $a\mu$ is another coset representative and $\nu(a\mu) = ||a\mu|| \geq ||a|| + ||\mu|| > ||a||$. Otherwise $a = b\sigma(\mu^*)$ for some coset representative $b \in G^+$. and right multiplication by μ corresponds to an edge from b to a. Again, $\nu(a) = ||b\sigma(\mu^*)|| \geq ||b|| + ||\sigma(\mu^*)|| > ||b||$.

In order to establish that $\mathcal{X}_\mathcal{D}$ is contractible it suffices to show that the descending links of vertices are contractible (cf. [3]). Identifying the vertices with elements $a \in G^+$ whose left greedy normal form does not begin with Δ, one can describe the descending link of a as the subcomplex induced by the vertices $\{b \mid ||b|| < ||a|| \text{ and } b\mu = a \text{ for some } \mu \in \mathcal{D}\}$. Thus the vertices of $\mathrm{Lk}_\downarrow(a)$ correspond to those $\mu \in \mathcal{D}$ where $\Delta \leq a\mu$. If we express a in *right* greedy normal form, so $a = \mu_1 \cdots \mu_k$ where $\mu_k = \mathrm{RF}(a)$ is the right front of a, then $\Delta \leq a\mu$ if and only if $\Delta = \mathrm{RF}(a\mu) = \mathrm{RF}(\mu_k\mu)$. This occurs precisely when $\mu_k^* \leq \mu$. Thus the descending link is the subcomplex spanned by the simple divisors $\mu \in \mathcal{D} - \{\Delta\}$ that are greater than the right complement of the right front of a, and the vertex corresponding to $\mathrm{RF}(a)^*$ is a cone point for $\mathrm{Lk}_\downarrow(a)$. □

Because $\mathcal{X}_\mathcal{D}$ is contractible, $\widehat{\mathcal{X}}_\mathcal{D} \simeq \mathcal{X}_\mathcal{D} \times \mathbb{R}$ is also contractible. The G action on $\widehat{\mathcal{X}}_\mathcal{D}$ is free, and the quotient $G\backslash\widehat{\mathcal{X}}_\mathcal{D}$ is finite, hence G admits a finite $K(\pi,1)$. In principle one can compute the homology of G from this $K(\pi,1)$. This process can be made quite concrete if one knows the lattice $\mathcal{D}$.

Definition 3.5. Let $\mathcal{D}_n$ denote the set of all ordered n-tuples of elements of $\mathcal{D}$, $[\mu_1|\cdots|\mu_n]$, such that the product $\mu_1\cdots\mu_n$ is an element of $\mathcal{D}$. Note that the largest n for which this set is non-empty is the maximal length of Δ with respect to the indivisible elements, that is $||\Delta||$.

Theorem 3.6. *Let G be a Garside group with simple divisors $\mathcal{D}$, and let $B_\mathcal{D}$ be the quotient of $\widehat{\mathcal{X}}_\mathcal{D}$ under the action of G. Then $B_\mathcal{D}$ has one k-cell for every element in $\mathcal{D}_k$. In particular, if $d = ||\Delta||$, then there is a free resolution of $\mathbb{Z}$ as a trivial*

$\mathbb{Z}G$-module of the form

$$0 \to \mathbb{Z}G^{|\mathcal{D}_d|} \to \cdots \to \mathbb{Z}G^{|\mathcal{D}_2|} \to \mathbb{Z}G^{|D|} \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0 \ .$$

Proof. One can describe the complex $B_{\mathcal{D}} = G \backslash \widehat{\mathcal{X}}_{\mathcal{D}}$ using a variation on the standard bar construction. Since the 1-skeleton of $\widehat{\mathcal{X}}_{\mathcal{D}}$ is the Cayley graph of G with respect to $\mathcal{D}$, the complex $B_{\mathcal{D}}$ contains a single vertex, and a loop for each $\mu \in \mathcal{D}$. Inductively one proceeds by considering each factorization $\mu = \mu_1\mu_2 \cdots \mu_k$ where μ and each μ_i are in $\mathcal{D}$. Such a factorization gives rise to a k-simplex $[\mu_1|\mu_2|\cdots|\mu_k]$ whose codimension 1 faces attach to the $(k-1)$ cells of $B_{\mathcal{D}}^{(k-1)}$ corresponding to $[\mu_2|\cdots|\mu_k]$, $[\mu_1|\cdots|\mu_i\mu_{i+1}|\cdots|\mu_k]$, and $[\mu_1|\cdots|\mu_{k-1}]$. Thus the resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module induced by the universal cover $\widehat{\mathcal{X}}_{\mathcal{D}}$ resonates with the standard bar resolution of a group, only instead of using the entire multiplication table as in the bar resolution, one restricts to the partial multiplication in the lattice $\mathcal{D}$. □

Question 3.7. Given an Garside group G, is it always possible to find an Garside monoid M where G is the group of fractions of M, and the Garside element Δ satisfies $||\Delta|| = \mathrm{cd}(G)$? One suspects not, but this is possible for Artin groups of finite type, as we show in §5.

4. The end connectivity of Garside groups

Just as one can compute the (co)homology of an Garside group using the poset $\mathcal{D}$, one can also get concrete information about the end connectivity of $\mathcal{X}_{\mathcal{D}}$ by looking at the (co)homology of certain subcomplexes of the geometric realization of the poset of simple divisors $|\mathcal{D}|$. Recall that a contractible, locally finite complex X is *n-connected at infinity* if given any compact subcomplex $C \subset X$, there is a subcomplex $D \subset X$ such that every map $\phi : S^i \to X - D$ extends to a map $\widehat{\phi} : B^{i+1} \to X - C$, for $-1 \le i \le n$. The property of being 0-connected at infinity is usually called "one ended" and 1-connected at infinity is usually referred to as "simply connected at infinity". Similarly X is *n-acyclic at infinity* if given any compact subcomplex $C \subset X$, there is a subcomplex $D \subset X$ such that i-cycles supported outside of D bound $(i+1)$-chains supported outside of C. If G admits a finite $K(\pi, 1)$, then the end connectivity of the universal covers of all finite $K(\pi, 1)$s is the same, and the end connectivity is therefore a property of the group G.

In §3 we establish that $\mathcal{X}_{\mathcal{D}}$ is contractible by examining the descending links of vertices with respect to the Morse function $\nu : \mathcal{X}_{\mathcal{D}} \to [0, \infty)$. The ascending links of vertices can be used to describe the connectivity at infinity of $\mathcal{X}_{\mathcal{D}}$. Using this idea, Bestvina was able to show that if A is an Artin group of finite type, with cohomological dimension n, then A is $(n-2)$-connected at infinity [3]. (As we discuss below, this implies that A is a duality group.) Arbitrary Garside groups are not highly connected at infinity, and in particular, they are not all duality groups. The main result in this section is that the lattice $\mathcal{D}$ determines the connectivity at infinity of $\mathcal{X}_{\mathcal{D}}$ and hence it determines the connectivity at infinity of the associated Garside group.

Let $\mathcal{PD}$ be the subposet $\mathcal{D} - \{\Delta\}$, and for any element $\mu \in \mathcal{D}$ let $\mathcal{PD}_{\not\geq\mu}$ be the subposet consisting of all elements which do not have μ as a left divisor

$$\mathcal{PD}_{\not\geq\mu} := \{\eta \in \mathcal{D} \mid \mu \not\leq \eta\} \ .$$

The ascending links of vertices can be described in terms of the geometric realizations of these subposets. Namely, if $a \in G^+$, then just as $\mathrm{Lk}_\downarrow(a)$ is the subposet of elements greater than or equal to the right complement of the right front $\mathrm{RF}(a)^*$; the ascending link is then $\mathrm{Lk}_\uparrow(a) \simeq |\mathcal{PD}_{\not\geq \mathrm{RF}(a)^*}|$.

Theorem 4.1. *Let G be an Garside group with simple divisors $\mathcal{D}$, and let $\mathcal{X}_\mathcal{D}$ be the coset complex. If the reduced homology groups $\widetilde{H}_i(\mathcal{PD})$ and $\widetilde{H}_i(\mathcal{PD}_{\not\geq\mu})$, for $i \leq n$, are trivial, then $\mathcal{X}_\mathcal{D}$ is n-acyclic at infinity.*

Proof. We quickly sketch this argument since it is similar to the one used by Bestvina for Artin groups of finite type [3] and by Bestvina and Feighn for $\mathrm{Out}(F_n)$ [4].

One starts by listing the coset representatives $\{1 = a_0, a_1, \dots, a_m, \dots\}$ such that $||a_i|| \leq ||a_j||$ if $i < j$. Define K_m to be the subcomplex of $\mathcal{X}_\mathcal{D}$ induced by the vertices corresponding to $\{a_0, \dots, a_m\}$ and $K_{-1} = \emptyset$. The complement $\mathcal{X}_\mathcal{D} - K_{m-1}$ is then the complement $\mathcal{X}_\mathcal{D} - K_m$ with the ascending link $\mathrm{Lk}_\uparrow(a_m)$ coned off. (The vertex corresponding to a_m forms the cone point.)

For $i > 0$ the ascending link of the vertex associated to a_i is homeomorphic to the complex $|\mathcal{PD}_{\not\geq \mathrm{RF}(a_i)^*}|$; the ascending link of $a_0 = 1$ is $|\mathcal{PD}|$. Let $C(\mathrm{Lk}_\uparrow(a))$ denote the cone on the ascending link. Applying Mayer-Vietoris to the union

$$\mathcal{X}_\mathcal{D} - K_{m-1} = \{\mathcal{X}_\mathcal{D} - K_m\} \cup_{\mathrm{Lk}_\uparrow(a_m)} C(\mathrm{Lk}_\uparrow(a_m))$$

yields

$$\cdots \to \widetilde{H}_i(\mathrm{Lk}_\uparrow(a_m)) \to \widetilde{H}_i(\mathcal{X}_\mathcal{D} - K_m) \to \widetilde{H}_i(\mathcal{X}_\mathcal{D} - K_{m-1}) \to \cdots$$

where the term corresponding to $C(\mathrm{Lk}_\uparrow(a_m))$ has been removed since cones are contractible. Because the complex $\mathcal{X}_\mathcal{D}$ is contractible, and each $\widetilde{H}_i(\mathrm{Lk}_\uparrow(a_m))$ is assumed to be trivial (for $i \leq n$), we see by induction that $\widetilde{H}_i(\mathcal{X}_\mathcal{D} - K_m) = 0$ for all $i \leq n$ and all m. □

Corollary 4.2. *The Garside group G is $(n+1)$-connected at infinity if the reduced homology groups $\widetilde{H}_i(\mathcal{PD})$ and $\widetilde{H}_i(\mathcal{PD}_{\not\geq\mu})$, for $i \leq n$, are trivial.*

Proof. Since $\widehat{\mathcal{X}}_\mathcal{D}$ is a free, cocompact G-space, the end connectivity of G is the same as the end connectivity of $\widehat{\mathcal{X}}_\mathcal{D}$.

The complex $\widehat{\mathcal{X}}_\mathcal{D}$ can be filtered by subcomplexes of the form $\widehat{K}_m = K_m \times [-m, m]$ where K_m is the compact subcomplex of $\mathcal{X}_\mathcal{D}$ described in the proof of Theorem 4.1. The complement $\widehat{\mathcal{X}}_\mathcal{D} - \widehat{K}_m$ decomposes into the union of two contractible pieces $U_m = \{\mathcal{X}_\mathcal{D} \times [-m, \infty)\} - \widehat{K}_m$ and $V_m = \{\mathcal{X}_\mathcal{D} \times (-\infty, m]\} - \widehat{K}_m$. The intersection $U_m \cap V_m$ deformation retracts onto $\mathcal{X}_\mathcal{D} - K_m$, hence by Mayer-Vietoris $\widetilde{H}^i(\widehat{\mathcal{X}}_\mathcal{D} - \widehat{K}_m) \simeq \widetilde{H}^{i-1}(\mathcal{X}_\mathcal{D} - K_m)$. It follows by Theorem 4.1 that if the homology of $|\mathcal{PD}|$ and each $|\mathcal{PD}_{\not\geq\mu}|$ is trivial for $i \leq n$, then $\widetilde{H}^i(\widehat{\mathcal{X}}_\mathcal{D} - \widehat{K}_m) = 0$ for $i \leq n+1$ and all m.

Since U_m and V_m are contractible, van Kampen's Theorem implies that $\widehat{\mathcal{X}}_\mathcal{D} - \widehat{K}_m$ is simply connected, assuming that the complexes $|\mathcal{PD}|$ and $|\mathcal{PD}_{\not\geq\mu}|$ are connected. Thus by the Hurewicz Theorem, the complements $\widehat{\mathcal{X}}_\mathcal{D} - \widehat{K}_m$ are actually $(n+1)$-connected, and therefore G is $(n+1)$-connected at infinity. □

Definition 4.3 (Duality groups)**.** A group G of type FP (eg, a group with a finite $K(\pi, 1)$-space) is a *duality group* if it is n-dimensional and $(n-2)$-acyclic at infinity. Equivalently, it is an n-dimensional duality group if its cohomology with group ring coefficients is torsion free and concentrated in dimension n. The term 'duality' is appropriate since in an n-dimensional duality group there is a natural isomorphism between the group's homology and cohomology giving $H_i(G, M) \simeq H^{n-i}(G, H^n(G; \mathbb{Z}G) \otimes M)$ for all i and all G-modules M. The module $H^n(G, \mathbb{Z}G)$ is called the dualizing module. (See [12] for further background on duality groups.)

Proposition 4.4. *Let G be a Garside group with simple divisors $\mathcal{D}$. If the cohomology of $|\mathcal{PD}|$ and each $|\mathcal{PD}_{\not\geq \mu}|$ is torsion free and concentrated in dimension $n-2$, then G is an n-dimensional duality group.*

Proof. The cohomology group $H^i(G, \mathbb{Z}G)$ is isomorphic to the direct limit of the induced system $H^{i-1}(\widehat{\mathcal{X}}_{\mathcal{D}} - \widehat{K}_m)$. By the proof of Corollary 4.2 we see that

$$\widetilde{H}^i(G, \mathbb{Z}G) = \lim_{\rightarrow} \left\{ \widetilde{H}^{i-2}(\mathcal{X}_{\mathcal{D}} - K_m) \right\} = \widetilde{H}^{i-1}_c(\mathcal{X}_{\mathcal{D}})$$

Given that the cohomology of each complex $|\mathcal{PD}|$ and $|\mathcal{PD}_{\not\geq \mu}|$ is concentrated in dimension $n-2$, it follows from the Mayer-Vietoris sequence in cohomology that there is a short exact sequence

$$0 \to \widetilde{H}^{i-2}(\mathrm{Lk}_{\uparrow}(a_n)) \to \widetilde{H}^{i-2}(\mathcal{X}_{\mathcal{D}} - K_n) \to \widetilde{H}^{i-2}(\mathcal{X}_{\mathcal{D}} - K_{n-1}) \to 0\ .$$

Since the ascending links are finite, $\widetilde{H}^{n-2}(\mathrm{Lk}_{\uparrow}(a_m))$ is free abelian, and a quick induction shows that

$$\widetilde{H}^{i-2}(\mathcal{X}_{\mathcal{D}} - K_n) \simeq \bigoplus_{i=0}^{n} \widetilde{H}^{i-2}(|\mathcal{PD}_{\not\geq \mathrm{RF}(a_i)^*}|)$$

and so taking the direct limit we get

$$\widetilde{H}^i(G, \mathbb{Z}G) = \widetilde{H}^{i-1}_c(\mathcal{X}_{\mathcal{D}}) = \bigoplus_{i=0}^{\infty} \widetilde{H}^{i-2}(|\mathcal{PD}_{\not\geq \mathrm{RF}(a_i)^*}|)$$

hence $H^i(G, \mathbb{Z}G)$ is torsion free and concentrated in dimension n if the reduced cohomology of $|\mathcal{PD}|$ and each $|\mathcal{PD}_{\not\geq \mu}|$ is torsion free and concentrated in dimension $(n-2)$. □

5. Examples

The standard examples of Garside groups are the Artin groups of finite type which are associated to the finite Coxeter groups. Recall that a Coxeter system $(\mathcal{W}, S)$ consists of a finite set $S = \{s_1, s_3, \ldots, s_n\}$ and a group $\mathcal{W}$ with presentation

$$\mathcal{W} = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m(i,j)} = 1 \rangle$$

where $m(i,j) \in \{2, 3 \ldots, \infty\}$. The associated Artin group $\mathcal{A}$ is the group with presentation

$$\mathcal{A} = \langle s_1, s_2, \ldots, s_n \mid \mathit{prod}(i,j) = \mathit{prod}(j,i) \rangle$$

where $\mathit{prod}(i,j)$ is the alternating product $s_i s_j s_i \ldots$ of length $m(i,j)$. The pair $(\mathcal{A}, S)$ is called an Artin system. If $\mathcal{W}$ is finite, $\mathcal{A}$ is said to be of *finite type*. The map $\mathcal{A} \to \mathcal{W}$ sending each s_i to the generator of the same name is a quotient homomorphism.

For finite type $\mathcal{A}$, the monoid of positive words defined by the presentation above is Garside. The indivisible elements are the generators S, and the lattice $\mathcal{D}$ is in one to one correspondence with the Coxeter quotient $\mathcal{W}$, with Δ corresponding to the maximal length element of $\mathcal{W}$.

Our construction of $\widehat{\mathcal{X}}_{\mathcal{D}}$, and especially $\mathcal{X}_{\mathcal{D}}$, is exactly that of Bestvina for finite type Artin groups based on these monoids [3]. However, there are other natural Garside monoids whose groups of fractions are the Artin groups of finite type. The most well-known of these is the Birman-Ko-Lee braid monoid [5] which was shown by Bessis, Digne, and Michel to be a Garside monoid [2]. This has been generalized by Bessis, Brady, and Brady and Watt, to all Artin groups of finite type ([1], [7], and [10]). We briefly describe these monoids.

Let $(\mathcal{W}, S)$ be a finite Coxeter system and let $\mathcal{A}$ be the associated Artin group. Let R be the set of all reflections in $\mathcal{W}$, that is, all conjugates of $s_1, \dots s_n$. For $w \in \mathcal{W}$, let $|w|_R$ denote the minimal length of an expression for w as a product of reflections. If $uv = w$ in W and $|u|_R + |v|_R = |w|_R$, we write $u \leq_l w$ and $v \leq_r w$. These define partial orderings on $\mathcal{W}$. Let $\delta \in \mathcal{W}$ be the product $\delta = s_1 \cdots s_n$. Then δ is called a Coxeter element of $\mathcal{W}$. (The choice of ordering of the generators does not matter; permuting the generators gives a conjugate Coxeter element.) The set

$$\mathcal{D}_\delta = \{w \in \mathcal{W} \mid w \leq_l \delta\} = \{w \in \mathcal{W} \mid w \leq_r \delta\}$$

contains R and is a lattice with respect to both partial orderings. Define M_δ to be the monoid with generators $\mathcal{D}_\delta$ and relations $uv = w$ whenever $uv = w$ in $\mathcal{W}$, and $u \leq_l w \leq_l \delta$. This monoid, which Bessis calls the *dual monoid* is small Gaussian and its group of fractions is isomorphic to $\mathcal{A}$.

Proposition 5.1. *Let $(\mathcal{A}, S)$ be an Artin system of finite type and let M_δ be its associated dual monoid. Then the complex $\widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ is a $K(\mathcal{A}, 1)$ whose dimension $n = |S|$ is the cohomological dimension of $\mathcal{A}$.*

It is easy to see that the cohomological dimension of $\mathcal{A}$ at least n since $\mathcal{A}$ contains a free abelian group of rank n.

Remark 5.2. In the case of 2- and 3-generator finite type Artin groups, the complex $\widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ is the same as the $K(\mathcal{A}, 1)$-complexes constructed by Brady in [8]. Brady shows that in these dimensions, this complex can be given a piecewise Euclidean CAT(0) metric. Recent work of Brady and McCammond, however, shows that the analogous metric in higher dimensions will fail to be CAT(0) for at least some of the Artin groups [19]. While the question of whether Artin groups are CAT(0) is still open, in the next section we are able to show that Garside groups satisfy a weak version of non-positive curvature.

Example 5.3 (Dihedral type)**.** Let $\mathcal{A}$ be an Artin group of finite dihedral type, that is, the associated Coxeter group $\mathcal{W}$ is the dihedral group of order $2n$, D_n. In this case we make take the standard generators of $\mathcal{A}$ to be s and t where the image of s and t in D_n correspond to reflections meeting at an angle of π/n. The Coxeter element $\delta = s \cdot t$ is then a rotation through an angle of $2\pi/n$.

There are n distinct reflections in $\mathcal{W}$. Viewed as elements of $\mathcal{A}$, the set of reflections is

$$R = \{s, t, \bar{t}st, st\bar{s}, \bar{t}\bar{s}tst, sts\bar{t}\bar{s}, \dots\}$$

and the set of divisors is simply $\mathcal{D}_\delta = R \cup \{\delta\}$. The complex $\widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ is thus the product of an n-valent tree and $\mathbb{R}$. The orbit space $\mathcal{A} \backslash \widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ has a single vertex, a loop based at that vertex for each element of $\mathcal{D}_\delta$ and a 2-cell for each element of R:

$\{[s|t], [t|\bar{t}st], [\bar{t}st|\bar{t}\bar{s}tst], \ldots\}$. The link of the vertex in $\mathcal{A}\backslash\widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ is topologically the suspension of n points, one for each element of R. The suspension points correspond to the two ends, $\{\delta^+, \delta^-\}$, of the loop δ, and the arc corresponding to a reflection r is subdivided by two internal vertices r^- and ${}^*r^+$.

Example 5.4 (Braid groups)**.** Let $\mathcal{A}$ be the braid group on $n+1$ strings. Then the reflections R consist of braids $r_{i,j}$, $i < j$, which interchange the ith and jth strings and leave the other strings fixed (with a fixed convention on over- or under-crossing) and the element δ is the braid which crosses the first string over to the last position and shifts all the other strings one place to the left. The complex $\mathcal{A}\backslash\widehat{\mathcal{X}}_{\mathcal{D}_\delta}$ is n-dimensional and has one n-cell for each expression of δ as a product of n-reflections.

6. Curvature

A CAT(0) group is a group which acts properly and cocompactly on a CAT(0) metric space. The existence of such an action has many algebraic consequences for the group (see [11].) It is not known whether all Artin groups—or even those of finite type—are CAT(0) groups. (See [8], [9], and [6] for some partial results on this question.) Bestvina showed that the Artin groups of finite type, modulo their centers, satisfy a combinatorial convexity property which was good enough to get some interesting corollaries. Many of these arguments work in the context of Garside groups as well.

Definition 6.1 (Geodesics)**.** We define a *geodesic* in $\mathcal{X}_\mathcal{D}$ to be any path whose sequence of edge labels corresponds to a Deligne normal form with no Δs. If $\mathrm{DNF}(v) = b_1 b_2 \ldots b_n \Delta^k$ with $b_n \neq \Delta$, then $b_1, b_2, \ldots, b_n$ are the edge labels for the geodesic from 1 to v. The geodesic from v to w is the left translation by v of the geodesic from 1 to $v^{-1}w$.

Definition 6.2 (Distance)**.** We define a *nonsymmetric* distance $d(v, w)$ from a vertex v to a vertex w in $\mathcal{X}_\mathcal{D}$ as follows: if $\mathrm{DNF}(v^{-1}w) = b_1 b_2 \ldots b_n \Delta^k$, where $b_n \neq \Delta$, then $d(v, w) = ||b_1 b_2 \ldots b_n||$.

Lemma 6.3. *The geodesic from v to w is the inverse of the geodesic from w to v.*

Proof. An edge path is a geodesic if and only if every subpath of length two is a geodesic. If a and b are simple divisors, then the path $a, 1, b$ is a geodesic if and only if ${}^*ab$ is in Deligne normal form. We must show that ${}^*ba$ is also in Deligne normal form. If not, then ${}^*ba = {}^*bsa'$, where $s, {}^*bs$, and a' are also simple divisors. Then

$$ {}^*ab = \Delta({}^*ba)^{-1}\Delta = \Delta({}^*bsa')^{-1}\Delta = {}^*(a')({}^*bs)^* = {}^*as({}^*bs)^* $$

and so ${}^*ab$ was not in Deligne normal form. □

Proposition 6.4. *We can orient the edges of $\mathcal{X}_\mathcal{D}$ in terms of the Morse function. Then for any two vertices v and w in $\mathcal{X}_\mathcal{D}$, all edges of the geodesic from v to w whose orientation points toward v occur at the beginning of the path, and these are followed by edges oriented toward w.*

Proof. Assume to the contrary that x, y, z are consecutive vertices on the geodesic from v to w and that $||x|| < ||y|| > ||z||$. Thus there are divisors b and c in $\mathcal{D}$ such that $xb = y$ and $yc = z\Delta$. It follows that the first letter in the left greedy normal form for $x(bc) = z\Delta$ is Δ. But then by Proposition 2.4, $\mathrm{LF}(xb) = \Delta$. But $xb = y$ and $\Delta \not\leq y$. $\square$

Corollary 6.5. *If p is a vertex on the geodesic from v to w and $v \neq p \neq w$, and u is any vertex, then there is a strict inequality:*

$$d(u,p) < \max\{d(u,v), d(u,w)\}.$$

Definition 6.6 (Centers)**.** Let T be a finite set of vertices in $\mathcal{X}_\mathcal{D}$. For a vertex v in $\mathcal{X}_\mathcal{D}$, let $r(T,v) = max_{t\in T} d(t,v)$. The *circumscribed radius* $r(T)$ is the smallest integer r such that $r = r(T,v)$ for some vertex v. Any such vertex is called a *center* of T.

Proposition 6.7. *The set of centers of T span a simplex in $\mathcal{X}_\mathcal{D}$.*

Proof. Since $\mathcal{X}_\mathcal{D}$ is a flag complex, it suffices to show that any two centers are connected by a single edge. Suppose v_1 and v_2 are centers and that the geodesic between them passes through a third vertex w. Then for every $t \in T$,

$$d(t,w) < \max\{d(t,v_1), d(t,v_2)\} \leq r(T)$$

which contradicts the minimality of $r(T)$. $\square$

Let m be the smallest power of Δ which is central in G and let $G_\Delta = G/\langle\Delta^m\rangle$. Recall that for $\mu \in \mathcal{D}$, $\sigma(\mu) = \Delta\mu\Delta^{-1} \in \mathcal{D}$.

Corollary 6.8. *Let H be a finite subgroup of G_Δ. Then H fixes the barycenter of some simplex in $\mathcal{X}_\mathcal{D}$, and therefore there are only finitely many conjugacy classes of finite subgroups in G_Δ. Moreover, up to conjugacy, the finite subgroups of G_Δ are of one of two types:*

1. *cyclic generated by $\mu\Delta^j$, where $\mu \in \mathcal{D}$ satisfies $\mu\sigma^j(\mu)\sigma^{2j}(\mu)\cdots\sigma^{(t-1)j}(\mu) = \Delta$ for some $t \leq ||\Delta||$, or*
2. *the direct product of a cyclic group of type (1) and $\langle\Delta^k\rangle$ where Δ^k commutes with μ.*

Proof. The first statement follows immediately from Proposition 6.7. Suppose H stabilizes the simplex $1 < \mu_1 < \mu_2 < \cdots < \mu_{t-1}$ and acts transitively on the vertices. Following Bestvina's argument ([3], Lemma 4.6), one can show that H preserves the cyclic order on the vertices induced by the linear ordering above. Let $h \in H$ be an element which takes the vertex 1 to the vertex $\mu = \mu_1$, so $h = \mu\Delta^j$ for some j. Then for $i = 1, 2, \dots t-1$, $h^i = \mu\sigma^j(\mu)\sigma^{2j}(\mu)\cdots\sigma^{(i-1)j}(\mu)\Delta^{ij}$ and the length ordering implies that $\mu\sigma^j(\mu)\sigma^{2j}(\mu)\cdots\sigma^{(i-1)j}(\mu) = \mu_i$. (If the product on the left produced a Δ at any stage, then some μ_i would have shorter length than the previous one.) It follows that h^t, which fixes the entire simplex, must equal Δ^{tj+1} which gives the equation in (1) above. Finally, note that if $h' \in H$ takes the vertex 1 to the vertex μ_k, then $h'h^{-k}$ fixes the entire simplex and hence must be a power of Δ which commutes with μ. $\square$

Definition 6.9 (Translation Length)**.** For $g \in G$, let $|g|_\mathcal{D}$ denote the word length of g with respect to the generating set $\mathcal{D}$. The *translation length* of g, denoted $\tau(g)$, is defined to be

$$\tau(g) = \lim_{n\to\infty} \frac{|g^n|_\mathcal{D}}{n}$$

In order to use the convexity property of $\mathcal{X}_\mathcal{D}$, we will need a relation between the word length $|g|_\mathcal{D}$ and the distance $d(*, g(*))$ in $\mathcal{X}_\mathcal{D}$, where G is understood to act on $\mathcal{X}_\mathcal{D}$ via the projection onto G_Δ.

Definition 6.10. The Garside element Δ is *tame* if there exists a constant c such that $||\Delta^n|| \leq cn$ for all n.

Note that by definition, the norm $||\ ||$ on G^+ satisfies a reverse triangle inequality: $||ab|| \geq ||a|| + ||b||$. Thus, $||\Delta^n|| \geq n||\Delta||$, but it is not obvious that Δ is tame. On the other hand, we know of no Garside groups for which this condition fails.

Lemma 6.11. *Suppose Δ is tame. Then for all $g \in G$, $|g|_\mathcal{D} \geq \frac{1}{c}\, d(*, g(*))$.*

Proof. The normal form $g = \mu_1\mu_2\cdots\mu_k\Delta^j$ is an expression of minimal length for g as a word in $\mathcal{D}$ ([17]), thus $|g|_\mathcal{D} = k + |j|$. Let $a = \mu_1\mu_2\cdots\mu_k \in G^+$. Then there exists $b \in G^+$ with $ab = \Delta^k$. It follows that $d(*, g(*)) = ||a|| \leq ||a|| + ||b|| \leq ||\Delta^k|| \leq ck \leq c\,|g|_\mathcal{D}$. □

Proposition 6.12. *If G is a Garside group with tame Δ, then the set of translation lengths of elements of G is bounded away from zero.*

Proof. Let $g \in G$. We first consider the case where the image of g in G_Δ has finite order. By Corollary 6.8, there exists an N such that the order of every torsion element in G_Δ divides N. Thus, $g^N = \Delta^{km}$ for some $k \neq 0$ (since G is torsion-free). It follows that $\tau(g) = \frac{|km|}{N} \geq \frac{m}{N}$.

Now suppose the image of g in G_Δ has infinite order. Let T_n be the set of vertices in $\mathcal{X}_\mathcal{D}$, $T = \{*, g(*), g^2(*), \ldots, g^{n-1}(*)\}$. Let r_n be the circumscribed radius of T_n. We claim that for any n, $r_n < r_{n+d}$ where $d = ||\Delta||$. For suppose $r_n = r_{n+d}$. If z is a center of T_{n+d}, then it is also a center for any subset of T_{n+d} whose circumscribed radius is r_n. In particular, z is a center for each of the sets $T_n, g(T_n), g^2(T_n), \ldots, g^d(T_n)$. It follows that $z, g(z), g^2(z), \ldots, g^d(z)$ are all centers of $g^d(T_n)$, hence they span a simplex. But the dimension of the complex $\mathcal{X}_\mathcal{D}$ is $d-1$ so we must have $g^k(z) = z$ for some $k \leq d$. Since vertex stabilizers in G_Δ are finite, this contradicts the assumption that the image of g has infinite order.

Thus we have $r_d < r_{2d} < r_{3d}\ldots.$ and so $r_{nd} \geq n$ for all n. In particular, for some $0 < i \leq d$, we must have $d(*, g^{nd+i}(*)) \geq n$, and hence by Lemma 6.11, $|g^{nd+i}|_\mathcal{D} \geq \frac{n}{c}$. It now follows that $\tau(g) \geq \frac{1}{cd}$. □

Corollary 6.13. *If G is a Garside group with tame Δ, then every solvable subgroup of G, is virtually a finitely generated abelian group.*

Proof. The proof is the same as for Bestvina's Corollaries 4.2 and 4.4. □

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