SPECIAL KÄHLER-RICCI POTENTIALS ON COMPACT KÄHLER MANIFOLDS

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Abstract. A special Kähler-Ricci potential on a Kähler manifold is any nonconstant C^∞ function τ such that $J(\nabla\tau)$ is a Killing vector field and, at every point with $d\tau \neq 0$, all nonzero tangent vectors orthogonal to $\nabla\tau$ and $J(\nabla\tau)$ are eigenvectors of both $\nabla d\tau$ and the Ricci tensor. For instance, this is always the case if τ is a nonconstant C^∞ function on a Kähler manifold (M,g) of complex dimension $m > 2$ and the metric $\tilde g = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein. (When such τ exists, (M,g) may be called *almost-everywhere conformally Einstein.*) We provide a complete classification of compact Kähler manifolds with special Kähler-Ricci potentials and use it to prove a structure theorem for compact Kähler manifolds of any complex dimension $m > 2$ which are almost-everywhere conformally Einstein.

§0. Introduction

This paper, although self-contained, can also be viewed as the second in a series of three papers that starts with [8] and ends with [9].

We call τ a *special Kähler-Ricci potential* on a Kähler manifold (M,g) if

(0.1) τ is a nonconstant Killing potential on (M,g) and, at every point with $d\tau \neq 0$, all nonzero tangent vectors orthogonal to $v = \nabla\tau$ and to $u = Jv$ are eigenvectors of both $\nabla d\tau$ and the Ricci tensor r.

(Cf. [8], §7; for more on Killing potentials, see §4 below.) The word 'potential' reflects the fact that (0.1) is closely related, although not equivalent, to the requirement that $\nabla d\tau + \chi\,\mathrm{r} = \sigma g$ for some C^∞ functions χ, σ (see [8], beginning of §7). This requirement is reminiscent of Kähler-Ricci solitons (some of which, in fact, do satisfy (0.1), cf. [10] and Remark 10.1 below); while, in complex dimensions $m > 2$, it implies that τ arises from a *Hamiltonian* 2*-form* on the underlying Kähler manifold ([2], §1.4). See also [5]. What further sparked our interest in (0.1) was its being, in cases such as (0.4) below, a consequence of the following assumption:

(0.2) (M,g) is a Kähler manifold of complex dimension m and τ is a nonconstant C^∞ function on M such that the conformally related metric $\tilde g = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein.

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When $m > 2$, (0.2) implies the seemingly stronger condition

(0.3) $\quad M, g, m, \tau$ satisfy (0.2) and $d\tau \wedge d\Delta\tau = 0$ everywhere in M

(see [8], Proposition 6.4), so that locally, at points with $d\tau \neq 0$, the Laplacian of τ is a function of τ. Therefore, (0.3) is of independent interest, as opposed to just (0.2), only for Kähler *surfaces* ($m = 2$). In [8], Corollary 9.3, we found that

(0.4) $\quad$ Condition (0.2) with $m \geq 3$, or (0.3) with $m = 2$, implies (0.1).

The first main result of this paper is a complete classification of compact Kähler manifolds (M, g), in all complex dimensions $m \geq 1$, with functions τ satisfying (0.1). Aside from the trivial case $m = 1$ (see Example 9.1), we show in Theorem 29.2 that, for each fixed $m \geq 2$, such manifolds form two separate families: in one, described in §17, M is a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle (of a very specific kind) over a compact Kähler manifold which is Einstein unless $m = 2$, while the other has M biholomorphic to $\mathbf{C}\mathrm{P}^m$ (see §18). Pairs g, τ with (0.1) on a fixed compact complex manifold M turn out to form an infinite-dimensional moduli space. Namely, $Q = g(\nabla\tau, \nabla\tau)$ then is a C^∞ function of the real variable $\tau \in [\min\tau, \max\tau]$, satisfying the positivity and boundary conditions (17.1) in §17; conversely, any given assignment $\tau \mapsto Q$ with (17.1) is realized in this way on each of the complex manifolds M just mentioned. See §17, §18.

Theorem 29.2 combined with (0.4) leads to our second main result, consisting of four statements that together form a structure theorem for, and a partial classification of, those quadruples M, g, m, τ with compact M which satisfy (0.2) with $m \geq 3$, or (0.3) with $m = 2$. Specifically, in §33 we divide all such quadruples into four disjoint "types" (a), (b), (c1), (c2). We then prove that type (b) is empty (Theorem 33.2), and verify (in Corollary 35.1) that type (c2) leads to a certain necessary condition which, as we show in a subsequent paper [9], is never satisfied; therefore, type (c2) eventually turns out to be empty as well. Next, in Theorem 33.3, we completely classify type (a); every M occurring in it is a flat holomorphic $\mathbf{C}\mathrm{P}^1$ bundle. Finally, our Theorem 34.3 reduces the classification of type (c1) (in which M always is a *nonflat* holomorphic $\mathbf{C}\mathrm{P}^1$ bundle) to the question of finding all rational functions that lie in a specific three-dimensional vector space depending on m and satisfy an analogue of (17.1). An answer to this last question, with examples (not limited to those of [3]) is, again, postponed to [9], as it requires extensive additional arguments based on entirely different methods.

The text is organized as follows. Sections 1 – 7 cover preliminary material. A discussion of basic properties of special Kähler-Ricci potentials on Kähler manifolds in §8 is followed by constructions of examples in sections 9 – 18.

Critical manifolds of special Kähler-Ricci potentials, their dimensions, geodesics normal to them, as well as their normal connections, curvature properties, and normal exponential mappings are studied in sections 19, 23 – 28 and 37.

Further properties of critical manifolds established in §21 allow us to show, in sections 20 and 22, that the assignment $\tau \mapsto Q$ mentioned above satisfies conditions (17.1), which we then use in §29 to prove Theorem 29.2. A similar local result is obtained in §36. The remaining sections 30 – 35 deal with quadruples M, g, m, τ having the property (0.2) with $m \geq 3$, or (0.3) with $m = 2$.

§1. Preliminaries

Except in §2, the symbol ∇ will stand either for the Levi-Civita connection of a given Riemannian metric g, or for the g-gradient. For a C^1 vector field v on a Riemannian manifold (M,g) we will write

$$\nabla v : TM \to TM \qquad \text{with} \quad (\nabla v)w \,=\, \nabla_w v\,, \tag{1.1}$$

treating the covariant derivative ∇v as a vector-bundle morphism sending each $w \in T_xM$, $x \in M$, to $\nabla_w v \in T_xM$. For the second covariant derivative $\nabla d\tau$ of a C^2 function τ and vector fields u, w on a Riemannian manifold we have

$$(\nabla d\tau)(u,w) \,=\, g(u, \nabla_w v) \,=\, g(\nabla_u v, w)\,, \qquad \text{where} \quad v \,=\, \nabla\tau\,. \tag{1.2}$$

Thus, if τ is a C^2 function on a Riemannian manifold,

(1.3) the eigenvalues and eigenvectors hskip4.2ptof the symmetric 2-tensor $\nabla d\tau$ are, at any point, the same as those of ∇v with (1.1), for $v = \nabla\tau$.

The tensor product of 1-forms ξ, ξ' acts on tangent vectors u, v by

$$(\xi \otimes \xi')(u,v) \,=\, \xi(u)\xi'(v)\,. \tag{1.4}$$

We use the symbol J for the complex-structure tensor of any complex manifold M. Thus, J is a real vector-bundle morphism $TM \to TM$ with $J^2 = -1$. At the same time, we treat TM as a complex vector bundle with the multiplication by i provided by J. A Kähler manifold (M,g) is, as usual, a complex manifold M with a Riemannian metric g which makes the complex-structure tensor J skew-adjoint and parallel. The Kähler form ω of (M,g) then is given by

$$\omega(u,v) \,=\, g(Ju,v)\,, \qquad \text{for} \ \ u,v \in T_xM, \ \ x \in M. \tag{1.5}$$

Remark 1.1. Let φ be a C^{k+1} function, $0 \le k \le \infty$, of a real variable s, defined on an interval containing 0 (possibly as an endpoint), and such that $\varphi(0) = 0$. Then $\varphi(s)/s$ can be extended to a C^k function of s defined on the same interval, including $s = 0$. In fact, integrating $d[\varphi(s\sigma)]/d\sigma$ we obtain the Taylor formula $\varphi(s) = sH(s)$, where $H(s) = \int_0^1 \dot\varphi(s\sigma)\,d\sigma$ with $\dot\varphi = d\varphi/ds$.

Remark 1.2. If τ, Q be C^∞-differentiable even functions of a real variable s, defined on a neighborhood of 0 in $\mathbf{R}$, and $d^2\tau/ds^2 \ne 0$ at $s = 0$, then Q restricted to some neighborhood of 0 is a C^∞ function of τ, that is, a composite in which τ is followed by a C^∞ function of the variable τ, defined on a suitable interval.

In fact, let $\zeta = s^2$. By induction on $k \ge 0$, any even C^{2k} function of s is a C^k function of ζ. Namely, for $k = 0$ this is true since $\sqrt{\zeta}$ is a continuous function of $\zeta \ge 0$. Assuming our claim for a given $k \ge 0$, let f be an even C^{2k+2} function of s. The C^{2k+1} function $\dot f = df/ds$ then is odd, and so $\dot f(s)/s$ is an even C^{2k} function of s, also at $s = 0$ (Remark 1.1). Hence, by the inductive assumption, $2\,df/d\zeta = \dot f(s)/s$ is a C^k function of ζ, i.e., f is of class C^{k+1} in ζ.

Consequently, τ and Q, are C^∞ functions of $\zeta = s^2$. However, the limit of $2d\tau/d\zeta = \dot\tau/s$ as $s \to 0$ (or, $\zeta \to 0$) equals $\ddot\tau(0) \ne 0$, so that the assignment $\zeta \mapsto \tau$ is a C^∞ diffeomorphism for $\zeta \ge 0$ near 0.

§2. Connections and curvature

Our sign convention for the curvature tensor R of a (linear) connection ∇ in any real/complex vector bundle $\mathcal{E}$ over a manifold is

$$R(u,v)w \,=\, \nabla_v\nabla_u w \,-\, \nabla_u\nabla_v w \,+\, \nabla_{[u,v]}w\,, \tag{2.1}$$

where u,v are C^2 vector fields tangent to the base and w is a C^2 section of $\mathcal{E}$.

Remark 2.1. The *curvature form* of a connection ∇ in a complex line bundle over any manifold is the complex-valued 2-form $\varOmega$ with $R(u,v)w = i\varOmega(u,v)w$ for u,v,w,R as in (2.1). Any local C^∞ section w without zeros gives rise to the *connection form* $\varGamma$ defined by $\nabla_v w = \varGamma(v)w$, and (2.1) easily yields $\varOmega = i\,d\varGamma$.

Given a vector bundle $\mathcal{L}$ over a manifold N, we will use the same symbol $\mathcal{L}$ for its total space, so that

$$\mathcal{L} \,=\, \{(y,z) : y\in N,\ \ z\in\mathcal{L}_y\} \qquad \text{and} \qquad N\,\subset\,\mathcal{L}\,, \tag{2.2}$$

where N is identified with the zero section formed by all $(y,0)$ with $y\in N$. We similarly treat each fibre $\mathcal{L}_y$ as a subset of $\mathcal{L}$, identifying it with $\{y\}\times\mathcal{L}_y$. In this way N and all $\mathcal{L}_y$ are submanifolds of $\mathcal{L}$ with its obvious manifold structure. Being a vector space, every fibre $\mathcal{L}_y$ is naturally identified, for any $z\in\mathcal{L}_y$, with the tangent space $\mathcal{V}_{(y,z)}\subset T_{(y,z)}\mathcal{L}$ of the submanifold $\{y\}\times\mathcal{L}_y$ at (y,z). These $\mathcal{V}_{(y,z)}$ form a vector subbundle $\mathcal{V}$ of $T\mathcal{L}$ called the *vertical distribution* of $\mathcal{L}$. If $\mathcal{L}$ is a complex line bundle over N, any fixed real number $a\neq 0$ gives rise to the vertical vector fields v,u on $\mathcal{L}$, that is, sections of $\mathcal{V}$, with

$$v(y,z) \,=\, az\,, \qquad u(y,z) \,=\, iaz \qquad \text{for all} \quad (y,z)\in\mathcal{L}\,. \tag{2.3}$$

Remark 2.2. Any Riemannian/Hermitian fibre metric $\langle\,,\rangle$ in a real/complex vector bundle $\mathcal{L}$ over a manifold N is determined by its *norm function* $\mathcal{L}\to[0,\infty)$, denoted r or s, which assigns $|z| = \langle z,z\rangle^{1/2}$ to each $(y,z)\in\mathcal{L}$. As $\mathcal{V}_{(y,z)} = \mathcal{L}_y$ (see above), $\langle\,,\rangle$ may also be treated as a fibre metric in the vertical subbundle $\mathcal{V}$ of $T\mathcal{L}$, and then, if $\mathcal{L}$ is a complex line bundle, for v,u given by (2.3) we have $\langle v,v\rangle = \langle u,u\rangle = a^2r^2$ and $\mathrm{Re}\,\langle v,u\rangle = 0$.

Remark 2.3. Suppose that $M,\hat M$ are locally trivial fibre bundles over a manifold N and a C^∞ diffeomorphism $\varPhi : M\to\hat M$ is fibre-preserving, i.e., $\hat\pi\circ\varPhi = \pi$, where $\pi,\hat\pi$ denote the bundle projections $M\to N$ and $\hat M\to N$. Also, let $\mathcal{H},\hat{\mathcal{H}}$ be some fixed "horizontal" distributions in M and $\hat M$, that is, vector subbundles of TM and $T\hat M$ with $TM = \mathcal{H}\oplus\mathcal{V}$ and $T\hat M = \hat{\mathcal{H}}\oplus\hat{\mathcal{V}}$ for the vertical distributions $\mathcal{V},\hat{\mathcal{V}}$ (tangent to the fibres). If $\varPhi$ sends $\mathcal{H}$ onto $\hat{\mathcal{H}}$, then the differential $d\varPhi_x$ of $\varPhi$ at any $x\in M$, restricted to $\mathcal{H}_x$, preserves any fibre metric or complex-bundle structure obtained in both $\mathcal{H}$ and $\hat{\mathcal{H}}$ by pulling back a fixed analogous object in TN via $\pi : M\to N$ and $\hat\pi : \hat M\to N$.

In fact, $d\varPhi_x$ acts as the identity mapping between $\mathcal{H}_x$ and $\hat{\mathcal{H}}_{\varPhi(x)}$ identified, via the differential of π or $\hat\pi$, with T_yN at $y = \pi(x)$.

Remark 2.4. Let $\mathcal{L}$ be a C^∞ complex line bundle over a complex manifold N, and let $\mathcal{H}$ be the horizontal distribution of a fixed C^∞ linear connection in $\mathcal{L}$ whose curvature form $\varOmega$ (Remark 2.1) is real-valued and skew-Hermitian in the sense that $\varOmega(Jv,v') = -\varOmega(v,Jv')$ for all $y \in N$ and $v,v' \in T_yN$. Then $\mathcal{L}$ admits a unique structure of a holomorphic line bundle over N such that $\mathcal{H}$ is J-invariant as a subbundle of $T\mathcal{L}$, where $J : T\mathcal{L} \to T\mathcal{L}$ now denotes the complex structure tensor on the total space $\mathcal{L}$.

In fact, let $\varGamma$ be the connection form (Remark 2.1) corresponding to a C^∞ local trivializing section w of $\mathcal{L}$, defined on a contractible open set $N' \subset N$. Using w to identify the portion $\mathcal{L}'$ of $\mathcal{L}$ lying over N' with $N' \times \mathbf{C}$, and writing down the parallel-transport equation in terms of $\varGamma$, we see that, for any $(y,z) \in \mathcal{L}'$ and $(w,\zeta) \in T_{(y,z)}\mathcal{L}'$, the $\mathcal{H}$ component of (w,ζ) relative to the decomposition $T\mathcal{L} = \mathcal{H} \oplus \mathcal{V}$ equals $(w, -\varGamma(w)z)$. Thus, w is holomorphic for a holomorphic-bundle structure in $\mathcal{L}'$ for which $\mathcal{H}$ is J-invariant if and only if $\varGamma$ is of type $(1,0)$, i.e., the bundle morphism $\varGamma : TN' \to N' \times \mathbf{C}$ is complex-linear.

Our $\varGamma$, with $d\varGamma = -i\varOmega$ (Remark 2.1) need not be of type $(1,0)$. However, a $(1,0)$ form $\tilde\varGamma$ with $d\tilde\varGamma = -i\varOmega$ exists on N' since $\varOmega$ is a closed real-valued form of type $(1,1)$, and so, choosing a function $\varphi : N' \to \mathbf{R}$ with $i\varOmega = \partial\overline{\partial}\varphi$, we may set $\tilde\varGamma = \partial\varphi$. As $d(\varGamma - \tilde\varGamma) = 0$, we have $\tilde\varGamma = \varGamma + d\varPhi$ for some C^∞ function $\varPhi : N' \to \mathbf{C}$. Now $\mathcal{H}$ is J-invariant for the holomorphic-bundle structure in $\mathcal{L}'$ obtained by declaring the section $\tilde w = e^{\varPhi} w$ holomorphic. (Note that the connection form corresponding to $\tilde w$ is $\tilde\varGamma$.) Any other C^∞ section of $\mathcal{L}'$ without zeros having a $(1,0)$ connection form equals $e^{\varPsi}\tilde w$, where $\varPsi : N' \to \mathbf{C}$ is holomorphic since $d\varPsi$ is of type $(1,0)$. The structure in question is therefore unique.

§3. Tautological bundles

Remark 3.1. Given a Hermitian inner product $\langle\,,\rangle$ in a complex vector space V with $\dim V < \infty$ and a real constant $a \neq 0$, we will denote v, u the vector fields on V given by $x \mapsto ax$ and $x \mapsto aix$, and define two distributions $\mathcal{V}, \mathcal{H}$ on $V \smallsetminus \{0\}$ by $\mathcal{V} = \operatorname{Span}\{v,u\}$ and $\mathcal{H} = \mathcal{V}^\perp$. Clearly, they do not depend on the choice of a.

For a complex vector space V with $1 \le \dim_{\mathbf{C}} V = m < \infty$, let $N \approx \mathbf{C}\mathrm{P}^{m-1}$ be the projective space of V, and let $\mathcal{L}$ be the tautological bundle over N. Thus, N is a complex manifold with the underlying set formed by all complex lines through 0 in V, and $\mathcal{L}$ is the complex line bundle over N whose fibre over any point (i.e., line) is the line itself. We will write $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$, that is, identify $\mathcal{L} \smallsetminus N$ (which is an open set in the total space $\mathcal{L}$, cf. (2.2)) with $V \smallsetminus \{0\}$ using the natural biholomorphism given, in the notation of (2.2), by $(y,z) \mapsto z$.

A fixed Hermitian inner product $\langle\,,\rangle$ in V gives rise to two objects in $\mathcal{L}$. The first is a Hermitian fibre metric, also denoted $\langle\,,\rangle$, and obtained by restricting the inner product to the fibres of $\mathcal{L}$. The second is a canonical connection in $\mathcal{L}$ making the fibre metric $\langle\,,\rangle$ parallel, whose horizontal distribution, restricted to $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$, is $\mathcal{H}$, defined in Remark 3.1. This canonical connection is obtained by projecting the standard flat connection in the product bundle $\mathcal{E} = N \times V$ onto the $\mathcal{L}$ summand of the direct-sum decomposition $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^\perp$. Note that its horizontal distribution is contained in $\mathcal{H}$ (and hence coincides with $\mathcal{H}$, for dimensional reasons), since any horizontal C^1 curve $t \mapsto x(t) \in V \smallsetminus \{0\}$ has, by definition, $dx/dt \in \mathcal{H}_{x(t)}$ at every t.

For $m \geq 2$ the curvature form $\varOmega$ of this canonical connection (see Remark 2.1) equals $-2\omega_{\mathrm{FS}}$, where ω_{FS} is the Kähler form, defined as in (1.5), of the quotient (Fubini-Study) metric on the projective space N.

In fact, both forms are invariant under the obvious action on $N \approx \mathbf{C}\mathrm{P}^{m-1}$ of the group $G \approx \mathrm{U}(m)$ of all unitary automorphisms of V. (The original action of G on $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$ amounts to a lift of its action on N to $\mathcal{L}$, preserving both the fibre metric $\langle\,,\rangle$ and the canonical connection, as $\mathcal{H}$ in Remark 3.1 is G-invariant.) Moreover, both forms are real-valued and skew-Hermitian (cf. Remark 2.4), so that ω_{FS} (or, $\varOmega$) is related, as in (1.5), to the Fubini-Study metric (or, respectively, to some twice-covariant symmetric tensor field b on N). Therefore, $\varOmega$ is a constant times ω_{FS}, since the same is true for b and the metric (as G leaves them invariant and acts on N with an irreducible isotropy representation). This constant is -2, as one sees integrating both forms over a fixed complex projective line $S \subset N$, formed by all complex lines through 0 in some complex plane $W \subset V$. Namely, $\int_S \varOmega$ is 2π times the integral of $c_1(\mathcal{L})$ over the cycle $[S]$, and, since $\mathcal{L}$ restricted to S is the tautological bundle of S, its Chern number, i.e., the latter integral, equals -1. (Note that the dual $\mathcal{L}^*$ of $\mathcal{L}$ restricted to S has the Chern number $+1$, as its admits a holomorphic section with one simple zero, obtained by restricting a nonzero linear functional $W \to \mathbf{C}$ to the fibres.) Finally, as $\int_S \omega_{\mathrm{FS}}$ is the area of S with its own Fubini-Study metric, it equals π, since S is an orientable Riemannian surface having a positive constant Gaussian curvature and the diameter $\pi/2$.

Remark 3.2. The restriction to $\mathcal{H}$ of the pullback of the Fubini-Study metric on N under the standard projection $V \smallsetminus \{0\} \to N$ coincides with the Euclidean metric $\mathrm{Re}\langle\,,\rangle$ divided by the norm-squared function on $V \smallsetminus \{0\}$. Namely, for any complex-linear operator $\varTheta : V \to V$, the linear vector field $x \mapsto \varTheta x$ on $V \smallsetminus \{0\}$ is projectable onto N (since so is its flow, consisting of linear automorphisms), and such fields, or their projections onto N, realize all vectors tangent at any point to $V \smallsetminus \{0\}$ (or, N). The function $x \mapsto \langle \varTheta x, \varTheta x\rangle / \langle x, x\rangle$ then is also projectable onto N, i.e., homogeneous of degree zero. Fixing $x \in V \smallsetminus \{0\}$ and $w \in \mathcal{H}_x$, and then choosing $\varTheta$ with $w = \varTheta x$, we now see that our claim holds at x, since, according to the definition of the Fubini-Study metric, it holds at all x with $\langle x, x\rangle = 1$.

§4. Killing fields and Killing potentials

This section contains well-known facts, listed here for easy reference. More details can be found, for instance, in [8], §5.

A real-valued C^∞ function τ on a Kähler manifold (M, g) is called a *Killing potential* if $u = J(\nabla\tau)$ is a Killing field on (M, g).

As usual, a *Killing vector field* on a Riemannian manifold (M, g) is any C^∞ vector field u such that ∇u is skew-adjoint at every point (cf. (1.1)).

Remark 4.1. A Killing field u on (M, g) (or, an isometry $\varPhi : M \to M$) is uniquely determined by $u(x)$ and $(\nabla u)(x)$ (or, by $\varPhi(x)$ and $d\varPhi_x$) at any given point $x \in M$. In fact, using normal coordinates one sees that $\Phi(x)$ and $d\Phi_x$ determine Φ along any broken geodesic emanating from x, and the same is true for the local isometries forming the local flow of u. This implies a *unique continuation property*: an isometry, or a Killing field, is uniquely determined by its restriction to any nonempty open set.

We call a (real) C^∞ vector field v on a complex manifold *holomorphic* if $\mathbf{L}_v J = 0$, where $\mathbf{L}$ is the Lie derivative. For a C^∞ vector field v on a *Kähler* manifold,

$$(4.1)\qquad \begin{aligned} &\text{a)}\quad v \text{ is holomorphic if and only if } [J, \nabla v] = 0, \\ &\text{b)}\quad \nabla u = J \circ (\nabla v), \text{ with the convention (1.1), if } u = Jv. \end{aligned}$$

Lemma 4.2. *For a C^∞ function τ on a Kähler manifold (M, g), the following three conditions are equivalent*: (i) *τ is a Killing potential*; (ii) *The gradient $v = \nabla\tau$ is a holomorphic vector field*; (iii) *$b = \nabla d\tau$ is Hermitian, that is, $b(Jw, w') = -b(w, Jw')$ for all $x \in M$ and $w, w' \in T_x M$.* ■

Next, there is a well-known local one-to-one correspondence between Killing potentials defined up to an additive constant, and holomorphic Killing vector fields:

Lemma 4.3. *Let (M, g) be a Kähler manifold. For every Killing potential τ on (M, g), the Killing field $J(\nabla\tau)$ is holomorphic. Conversely, if $H^1(M, \mathbf{R}) = \{0\}$, then every holomorphic Killing vector field on (M, g) has the form $J(\nabla\tau)$ for a Killing potential τ, which is unique up to an additive constant.* ■

Remark 4.4. Let τ be a nonconstant Killing potential on a Kähler manifold (M, g). Then $\nabla d\tau \neq 0$ wherever $d\tau = 0$ (and hence $d\tau \neq 0$ on a dense open subset of M, which also follows from Lemma 4.2(ii)). In fact, if $\nabla d\tau$ and $d\tau$ both vanished at some point, so would $v = \nabla\tau$ and ∇v (by (1.2)), as well as $u = Jv$ and ∇u (by (4.1.b)). The Killing field $u = J(\nabla\tau)$ thus would vanish identically on M (see Remark 4.1), contradicting nonconstancy of τ.

§5. Critical manifolds

The following two lemmas are well-known. For details, see, e.g., [8], §12.

Lemma 5.1. *Let u be a Killing field on a Riemannian manifold (M, g), and let $y \in M$ be a point such that $u(y) = 0$. Then, for every sufficiently small $\mathrm{d} > 0$, the flow of u restricted to the radius d open ball U centered at y consists of "global" isometries $U \to U$, while the exponential mapping $\exp_y$ is defined everywhere in the open ball U' of radius d in $T_y M$, centered at 0, and $\exp_y$ maps U' diffeomorphically onto U.* ■

Lemma 5.2. *For a Killing vector field u on a Riemannian manifold (M, g), let $N(u) = \{y \in M : u(y) = 0\}$ be the set of all zeros of u. If $u \neq 0$ somewhere in M, then, for every connected component N of $N(u)$, with ∇u as in* (1.1),

(a) *N is contained in an open set that does not intersect any other component.*
(b) *$N \subset M$ is a closed set and a submanifold with the subset topology.*
(c) *The submanifold N is totally geodesic in (M, g) and $\dim M - \dim N \geq 2$.*
(d) *For any $y \in N$ we have $T_y N = \mathrm{Ker}\,[(\nabla u)(y)] = \{w \in T_y M : \nabla_w u = 0\}$.*

Furthermore, the set $M' = M \smallsetminus N(u)$ *is connected, open and dense in* M. ■

Let $\tau : M \to \mathbf{R}$ be a C^1 function on a manifold M. If all connected components N of the set $\mathrm{Crit}(\tau)$ of its critical points happen to satisfy conditions (a), (b) of Lemma 5.2, we will refer to them as the *critical manifolds* of τ.

Remark 5.3. Let $\tau : M \to \mathbf{R}$ be a nonconstant Killing potential (§4) on a Kähler manifold (M, g), and let $M' \subset M$ be the open set on which $d\tau \neq 0$. Then

(i) M' is connected and dense in M.

(ii) The connected components of $\mathrm{Crit}(\tau)$ are totally geodesic submanifolds of (M, g) (the *critical manifolds* of τ), satisfying (a) – (d) in Lemma 5.2.

(iii) Every critical manifold N of τ is a complex submanifold of M, and $T_yN = \mathrm{Ker}\,[(\nabla v)(y)] = \{w \in T_yM : \nabla_w v = 0\}$ for any $y \in N$.

(All three conclusions are well-known.) In fact, $u = J(\nabla\tau)$ is a Killing field, and so (i), (ii) are obvious from Lemma 5.2 with $N(u) = \mathrm{Crit}(\tau)$, i.e., $M' = M \smallsetminus N(u)$. Finally, N in (iii) is a complex submanifold since every T_yN, $y \in N$, is J-invariant (by Lemma 5.2(d), as J and ∇u commute, cf. Lemma 4.3 and (4.1.a)), while, for $v = \nabla\tau$, (4.1.b) gives $\nabla u = J \circ (\nabla v)$, so that $\mathrm{Ker}\,[(\nabla u)(y)] = \mathrm{Ker}\,[(\nabla v)(y)]$ for $y \in M$, and the formula for T_yN follows from Lemma 5.2(d).

§6. Geodesic vector fields

Let ∇ be a fixed connection in the tangent bundle TM of a manifold M, for instance, the Levi-Civita connection of some Riemannian metric on M. By a *geodesic vector field* on M we mean any C^∞ vector field v on M such that

$$\nabla_v v = \psi v \quad \text{for some function} \quad \psi : M \to \mathbf{R}. \tag{6.1}$$

The function ψ is not even required to be continuous: its values at points where $v = 0$ are not determined by (6.1), and may be completely arbitrary.

Remark 6.1. Condition (6.1) holds if and only if every integral curve $s \mapsto x(s)$ of v is a (re-parameterized) geodesic of ∇. In fact, since $\dot x(s) = v(x(s))$ (where $\dot x = dx/ds$), it follows that $\nabla_{\dot x}\dot x$ at any s equals $\nabla_v v$ at $x(s)$, while curves $s \mapsto x(s)$ obtained from geodesics by diffeomorphic changes of parameter are characterized by $\nabla_{\dot x}\dot x = \psi\dot x$ with ψ, this time, denoting a function of s.

Remark 6.2. Let ∇ be a connection in the tangent bundle TM. If v is a C^∞ vector field with (6.1) and $X \subset M$ is a geodesic segment such that $v(x)$ is tangent to X at some point $x \in X$ and $v \neq 0$ at all points of X, then v is tangent to X at every point of X. In fact, both X and the underlying set $\tilde X$ of the maximal integral curve of v containing x are geodesics (Remark 6.1), tangent to each other at x, and so $x \in X' \subset \tilde X$ for some nontrivial subsegment X' of X. Choosing X' to be the maximal subsegment with this property, we must have $X' = X$, for otherwise an endpoint x' of X' would be an interior point of X and $v(x') \neq 0$ would be tangent to X at x', thus allowing X' to be extended past x', contrary to maximality.

Remark 6.3. Given a connection ∇ in the tangent bundle TM, we use the standard symbol $\exp_x : U_x \to M$ for the geodesic exponential mapping of ∇ at any point $x \in M$. Here U_x is a neighborhood of the zero vector in T_xM, obtained as a union of maximal line segments emanating from zero on which $\exp_x$ is defined. Thus, $s \mapsto x(s) = \exp_x sw$ is the geodesic with $x(0) = x$ and $\dot{x}(0) = w \in T_xM$. A related mapping is $\mathrm{Exp} : U^{\mathrm{Exp}} \to M$ with $\mathrm{Exp}(x, w) = \exp_x w$, defined on the subset $U^{\mathrm{Exp}} = \bigcup_{x\in M}[\{x\} \times U_x]$ of the total space $TM = \{(x, w) : x \in M, \ \ w \in T_xM\}$, containing the zero section $M \subset TM$ (cf. (2.2)). The set U^{Exp} is open in TM, and Exp is of class C^∞ (see, e.g., [11], p. 147).

Lemma 6.4. *Suppose that ∇ is a connection in the tangent bundle TM of a manifold M and v is a C^∞ vector field on M with* (6.1)*, while $X \subset M$ is a geodesic segment containing an endpoint y with $v(y) = 0$. If $\nabla_w v = aw$ for some nonzero vector w tangent to X at y and some $a \in \mathbf{R} \smallsetminus \{0\}$, then*

(a) *There exists a nontrivial compact subsegment X' of X, containing y, and such that $v(x) \neq 0$ for all $x \in X' \smallsetminus \{y\}$.*
(b) *For any subsegment $X' \subset X$ with the properties listed in* (a) *we have $v(x) \in T_xX$ at every $x \in X'$.*

Proof. Let $s \mapsto x(s)$ be a geodesic parameterization of X with $x(0) = y$, defined on a subinterval of $[0, \infty)$. Thus, $\nabla_{\dot{x}}\dot{x} = 0$, where $\dot{x} = dx/ds$. A fixed 1-form ξ of class C^∞ on a neighborhood U of y such that $a\xi(w) > 0$ at y, for $w = \dot{x}(0)$, gives rise to a C^∞ function $\varphi = \xi(v) : U \to \mathbf{R}$ with $\varphi = 0$ wherever $v = 0$ in U and $d[\varphi(x(s))]/ds > 0$ for all $s \geq 0$ near 0 (as $d_w\varphi = a\xi(w)$), which proves (a).

For X' as in (a), let $\ell > 0$ be such that $x(\ell)$ is an endpoint of X', and let $s \mapsto w(s) \in T_{x(s)}M$ be the vector field along X' given by $w(0) = \dot{x}(0)$ and $w(s) = v(x(s))/f(s)$ for $s \in (0, \ell]$, where $f : [0, \ell] \to \mathbf{R}$ is any fixed C^1 function with $f(0) = 0$, $\dot{f}(0) = a$ and $|f| > 0$ on $(0, \ell]$. Thus, $w(s) \neq 0$ for all $s \in [0, \ell]$ due to our choice of X' and ℓ. Also, setting $\tilde{v}(s) = v(x(s))$ we have $\tilde{v}(0) = 0$, while $\nabla_{\dot{x}}\tilde{v}$ at $s = 0$ equals $\nabla_w v = aw$, with $w = \dot{x}(0)$. This, along with l'Hospital's rule, shows that the mapping $[0, \ell] \ni s \mapsto (x(s), w(s))$, valued in the total space TM (see Remark 6.3), is continuous, also at $s = 0$.

For any fixed $s \in [0, \ell]$, let $r \mapsto x_s(r) \in M$ be the geodesic with $x_s(s) = x(s)$ and $d[x_s(r)/dr]_{r=s} = w(s)$, defined on the maximal possible interval containing s. Then, for any sufficiently small $\varepsilon \in (0, \ell]$,

(i) $r \mapsto x_s(r)$ is defined on an interval containing $[0, \ell]$, for every $s \in [0, \varepsilon]$.
(ii) $v \neq 0$ at $x_s(r)$ for any s, r with $0 < s \leq r \leq \varepsilon$.

In fact, if there were no $\varepsilon \in (0, \ell]$ with (i), we could find values of $s \in (0, \ell]$ arbitrarily close to 0 such that one of the points $(x(s), -sw(s))$, $(x(s), (\ell - s)w(s))$ lies in the complement $TM \smallsetminus U^{\mathrm{Exp}}$, with U^{Exp} as in Remark 6.3. Since $TM \smallsetminus U^{\mathrm{Exp}}$ is a closed set, it would then also contain the limit of one of these points as $s \to 0$, i.e., $(y, 0)$ or $(y, \ell\dot{x}(0))$, contradicting either the fact that $M \subset U^{\mathrm{Exp}}$, or our choice of ℓ.

Also, $d[\varphi(x_s(r))]/dr] > 0$ for all sufficiently small $r, s \in [0, \ell]$ since, due to our choice of φ, this is the case for $s = 0$ and $r \geq 0$ close to 0. (Note that

$x_0(r) = x(r)$.) As $\varphi > 0$ on a nontrivial subsegment of X' containing y, except for the point y at which $\varphi = 0$, by making $\varepsilon > 0$ with (i) smaller we thus have $\varphi > 0$ (and hence $v \neq 0$) at $x_s(r)$ for any s, r with $s \in (0, \varepsilon]$ and $s \leq r \leq s + \varepsilon$, which gives (ii).

By (i), (ii) and Remark 6.2, for every $s \in (0, \varepsilon)$, the geodesic $[s, \ell] \ni r \mapsto x_s(r)$ is a (re-parameterized) integral curve of v and, in particular, v is tangent to it at the point $x_s(\varepsilon)$. Taking the limit as $s \to 0$, we now see that v is tangent to the limiting geodesic, i.e., to X', at $x(\varepsilon)$. Applying Remark 6.2 to $x = x(\varepsilon)$, we obtain (b) for our X', which completes the proof. ∎

Lemma 6.5. *Let a Killing field u on a Riemannian manifold (M, g) vanish at a point y and let $z \in T_yM$ lie in the set U_y defined as in* Remark 6.3 *for the Levi-Civita connection ∇. Then, at the point $x = \mathrm{Exp}(y, z)$, the vector $u(x)$ is the image of $\nabla_z u \in T_yM = T_z(T_yM)$ under the differential of $\exp_y$ at z.*

In fact, the local isometries $\varPhi^t$ forming the local flow of u are all defined, for t near 0 in $\mathbf{R}$, on some open set in M containing the compact geodesic segment $X = \{\exp_y sz : 0 \leq s \leq 1\}$. Since they keep y fixed and map geodesics onto geodesics, we have $\varPhi^t(\exp_y z) = \exp_y(d\varPhi^t_y z)$. Our claim follows if we apply d/dt and let $t \to 0$, since $(\nabla u)(y) : T_yM \to T_yM$ (cf. (1.1)) is the infinitesimal generator of the one-parameter group $t \mapsto d\varPhi^t_y$ in T_yM. ∎

§7. Morse-Bott functions

By a *Morse-Bott function* on a manifold M (cf. [4]) we mean any C^∞ function $\tau : M \to \mathbf{R}$ such that every connected component N of the set $\mathrm{Crit}(\tau)$ of its critical points satisfies conditions (a), (b) of Lemma 5.2 and, for every $y \in N$, the nullspace of the Hessian $\mathrm{Hess}_y\tau$ coincides with T_yN. (Since the nullspace contains T_yN for any submanifold $N \subset \mathrm{Crit}(\tau)$, the last requirement amounts to $\mathrm{rank}\,\mathrm{Hess}_y\tau = \dim M - \dim N$.) The connected components of $\mathrm{Crit}(\tau)$ then are called the *critical manifolds* of τ, cf. §5.

Example 7.1. Every Killing potential τ on a Kähler manifold (M, g) (§4) is a Morse-Bott function: this is clear when τ is constant, while for nonconstant τ it follows from Remark 5.3(ii), (iii) along with (1.2). (Note that $\mathrm{Hess}_y\tau = (\nabla d\tau)(y)$ whenever $y \in \mathrm{Crit}(\tau)$.)

Lemma 7.2. *Let τ be a Morse-Bott function on a manifold M. Every point y of any critical manifold N of τ at which $\mathrm{Hess}_y\tau$ is positive/negative semidefinite then has a neighborhood U such that $\tau > \tau(y)$ everywhere in $U \smallsetminus N$ or, respectively, $\tau < \tau(y)$ everywhere in $U \smallsetminus N$.*

Proof. We may assume that $M = \mathbf{R}^n$, $y = (0, \ldots, 0)$ and $N = \{(0, \ldots, 0)\} \times \mathbf{R}^{n-r}$. Everywhere in N we thus have $\partial_a\tau = \partial_\lambda\tau = 0$ and hence $\partial_\lambda\partial_\mu\tau = \partial_\lambda\partial_a\tau = \partial_a\partial_\lambda\tau = 0$ for all $a, b \in \{1, \ldots, r\}$ and $\lambda, \mu \in \{r+1, \ldots, n\}$, where $\partial_j = \partial/\partial x^j$. Since $\mathrm{rank}\,\mathrm{Hess}_y\tau = r$ and $\dim N = n - r$, semidefiniteness of $\mathrm{Hess}_y\tau$ now implies that, at $y = (0, \ldots, 0)$, the $r \times r$ matrix $[\partial_a\partial_b\tau]$ is positive/negative definite.

For any fixed vectors $\mathbf{u} = (u^{r+1}, \ldots, u^n)$ close to $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{w} = (w^1, \ldots, w^r)$ with $|\mathbf{w}| = 1$, we will write $\dot\tau = d[\tau(x(s))]/ds$, where the curve

$s \mapsto x(s) \in U$ is given by $x(s) = (s\mathbf{w}, \mathbf{u})$, that is, $x^a(s) = sw^a$, $x^\lambda(s) = u^\lambda$ for s near $0 \in \mathbf{R}$ and a, b, λ, μ as before. If no neighborhood U of $\mathbf{0}$ in $\mathbf{R}^n$ (i.e., of y in M) had the required property, we could find sequences $\mathbf{u}_k, \mathbf{w}_k$ of such vectors with $\mathbf{u}_k \to \mathbf{0}$ and $\mathbf{w}_k \to \mathbf{w} \neq \mathbf{0}$ as $k \to \infty$, with $\pm[\tau(x_k(s_k)) - \tau(y)] \leq 0$ for $x_k(s) = (s\mathbf{w}_k, \mathbf{u}_k)$ and some sequence s_k converging to 0 in $\mathbf{R}$. However, as $\dot\tau = w^a \partial_a \tau$ (summed over $a = 1, \ldots, r$), at $s = 0$ we have $\tau = \tau(y)$ and $\dot\tau = 0$. The mean value theorem for $s \mapsto \tau(x_k(s))$ and $\dot\tau_k = d[\tau(x_k(s))]/ds$ then would give $\pm\ddot\tau_k \leq 0$ at $s = \tilde s_k$ for some $\tilde s_k$ with $\tilde s_k \to 0$ as $k \to \infty$. Since $\ddot\tau = w^a w^b \partial_a \partial_b \tau$, it would follow that, in the limit, $\pm w^a w^b \partial_a \partial_b \tau \leq 0$ at $y = \mathbf{0}$, contrary to positive/negative definiteness of $[\partial_a \partial_b \tau]$. This completes the proof. ■

Lemma 7.3. *Let there be given a C^∞ vector field v without zeros on a manifold M', an interval $\mathcal{I}$ of the variable τ, a sequence $\varPhi_k : \mathcal{I} \to M'$, $k = 1, 2, \ldots$, of integral curves of v, and a family $\{P[\tau] : \tau \in \mathcal{I}\}$ of compact sets $P[\tau] \subset M'$ with $\varPhi_k(\tau) \in P[\tau]$ for every $k = 1, 2, \ldots$ and $\tau \in \mathcal{I}$. If, for some $\tau \in \mathcal{I}$, the sequence $\varPhi_k(\tau)$ converges in M', then there exists an integral curve $\varPhi : \mathcal{I} \to M'$ of v such that $\varPhi_k(\tau) \to \varPhi(\tau)$ as $k \to \infty$ for every $\tau \in \mathcal{I}$.*

Proof. The set $\mathcal{J}$ of those $\tau \in \mathcal{I}$ for which $\varPhi(\tau) = \lim_{k\to\infty} \varPhi_k(\tau)$ exists is relatively open in $\mathcal{I}$ and the restriction of the mapping $\varPhi : \mathcal{J} \to M'$ just defined to any interval contained in $\mathcal{J}$ is an integral curve of v. This becomes clear if one fixes $\tau \in \mathcal{J}$ and chooses local coordinates at $\varPhi(\tau)$ in which v is a coordinate vector field.

All we still need to show is that $\mathcal{J}_0 = \mathcal{I}$, where the interval $\mathcal{J}_0$ is any fixed connected component of $\mathcal{J}$. To this end, let us assume, on the contrary, that $\tau' \in \mathcal{I} \smallsetminus \mathcal{J}_0$ is an endpoint of $\mathcal{J}_0$. Since $P[\tau']$ is compact, the $\varPhi_k(\tau')$ have a convergent subsequence. Let $\tilde{\mathcal{J}}$ be the analogue of $\mathcal{J}$ for that subsequence, and let $\tilde{\mathcal{J}}_0$ be the connected component of $\tilde{\mathcal{J}}$ containing τ'. Applying the last paragraph to our subsequence and $\tilde{\mathcal{J}}$, we see that some integral curve $\varPhi' : \tilde{\mathcal{J}}_0 \to M'$ of v is the pointwise limit, on $\tilde{\mathcal{J}}_0$, of the $\varPhi_k$ in our subsequence. Hence $\varPhi'$ must agree with $\varPhi$, the pointwise limit of the original sequence $\varPhi_k$, on the (obviously nonempty) open interval $\mathcal{J}_0 \cap \tilde{\mathcal{J}}_0$. All convergent subsequences of the original sequence $\varPhi_k(\tau')$ therefore have the same limit $\varPhi'(\tau') = \lim_{\tau\to\tau'} \varPhi(\tau)$, i.e. (as $\varPhi_k(\tau') \in P[\tau']$ and $P[\tau']$ is compact), the sequence $\varPhi_k(\tau')$ itself converges. Hence τ' lies in $\mathcal{J}_0$ instead of $\mathcal{I} \smallsetminus \mathcal{J}_0$. This contradiction completes the proof. ■

Lemma 7.4. *Let τ be a C^∞ function on a manifold M' such that the τ-preimage of every real number is compact and τ has no critical points.*

(i) *There exist a compact manifold P and a diffeomorphic identification $M' = P \times (\tau_-, \tau_+)$ under which τ appears as the projection onto the (τ_-, τ_+) factor, τ_- and τ_+ being the infimum and supremum of τ.*
(ii) *The τ-preimage of every real number is both compact and connected.*

Proof. Let us choose a Riemannian metric g on M' with $g(v, v) = 1$, where $v = \nabla\tau$ is the g-gradient of τ. (A unique metric with this property exists in every conformal class.) Also, let U be the union of all maximal integral curves of $v = \nabla\tau$

that intersect a fixed connected component P of a (nonempty) τ-preimage of a given real number $\hat{\tau}$.

For any $x \in P$, we denote $\Phi(x,\,\cdot\,) : (\tau_-(x), \tau_+(x)) \to U$ the maximal (parametrized) integral curve of v with $\Phi(x, \hat{\tau}) = x$, for $\hat{\tau}$ as above. Since $d_v\tau = g(v.\nabla\tau) = 1$, the natural parameter of every integral curve of v coincides, up to an additive constant, with τ. Thus, $\Phi(x,\,\cdot\,)$ is parameterized by τ, i.e., for any real number $\tau' \in (\tau_-(x), \tau_+(x))$, the value of τ at $\Phi(x, \tau')$ is τ'. If we now denote U' the union of all $\{x\} \times (\tau_-(x), \tau_+(x))$ with $x \in P$, then U' is an open set in $P \times (\tau_{\mathrm{inf}}, \tau_{\mathrm{sup}})$ and the mapping Φ defined above is a C^∞ diffeomorphism $U' \to U$. Note that U is an open subset of M', as it is the union of the images under the local flow of v of sufficiently small neighborhoods in M' of points of P.

For any given closed interval $[\tau_0, \tau_1]$ containing $\hat{\tau}$, the set of those $x \in P$ for which $[\tau_0, \tau_1] \subset (\tau_-(x), \tau_+(x))$ is both open and closed in P. In fact, its openness follows from that of U' (see above), while its closedness is immediate from Lemma 7.3 applied to any open interval $\mathcal{I} \subset [\tau_0, \tau_1]$, the compact sets $P[\tau']$ being the τ-preimages of real numbers τ'.

Consequently, as P is connected, $\tau_\pm = \tau_\pm(x)$ do not depend on $x \in P$, i.e., $U' = P \times (\tau_-, \tau_+)$. Thus, $U = M'$, since the open set $U = \Phi(U')$ is also closed in M'. Namely, if a sequence $\Phi(x_k, \tau_k)$ converges to $x \in M'$, we have $\tau_k = \tau(\Phi(x_k, \tau_k)) \to \tau(x)$ as $k \to \infty$ and, since P is compact, the x_k contain a subsequence converging to some $x_\infty \in P$, with $x = \Phi(x_\infty, \tau(x)) \in \Phi(U') = U$. This proves (i). Now (ii) follows, as the preimages in question are connected (being either empty or diffeomorphic to P), which completes the proof. ■

Corollary 7.5. *Let τ be a Morse-Bott function on a compact manifold M such that $\mathrm{Hess}_y\tau$ is semidefinite for every $y \in \mathrm{Crit}(\tau)$, and the real codimensions of all critical manifolds of τ are greater than one. Then τ has exactly two critical manifolds, which are the τ-preimages of its extremum values $\tau_+ = \tau_{\mathrm{max}}$ and $\tau_- = \tau_{\mathrm{min}}$, and the τ-preimage of every real number is both compact and connected.*

Proof. Since M is compact, τ has only finitely many critical manifolds, due to their discreteness property analogous to (a) in Lemma 5.2. None of them disconnects M, even locally, in view of the codimension condition, and τ restricted to each of them is constant. Therefore, the open set $M' = M \smallsetminus \mathrm{Crit}(\tau)$ of all noncritical points of τ is connected and dense in M, and its τ-image $\tau(M')$ is connected, open in $\mathbf{R}$, and dense in $[\tau_-, \tau_+]$, so that $\tau(M') = (\tau_-, \tau_+)$.

Moreover, the function $\tau : M' \to \mathbf{R}$ satisfies the hypotheses of Lemma 7.4. In fact, any sequence of points in M' that lies in the τ-preimage of a given real number has a subsequence converging to a limit $y \in M$, and then $y \notin \mathrm{Crit}(\tau)$, for otherwise our semidefiniteness assumption, combined with Lemma 7.2, would lead to a contradiction. Hence the assertion of Lemma 7.4(ii) holds for $\tau : M' \to \mathbf{R}$.

The only critical values of $\tau : M \to \mathbf{R}$ are $\tau_\pm$. In fact, let $y \in \mathrm{Crit}(\tau)$. Denseness of M' in M gives $x_k \to y$ as $k \to \infty$ for some sequence x_k in M'. If we had $\tau(y) \in (\tau_-, \tau_+)$, the sequence in $P \times (\tau_-, \tau_+)$ corresponding to the x_k under the identification of Lemma 7.4(i) (applied to $\tau : M' \to \mathbf{R}$, with $\tau(M') = (\tau_-, \tau_+)$) would have a convergent subsequence, i.e., a subsequence of the x_k would have a limit in M', even though $x_k \to y \notin M'$.

We thus have connectedness of the τ-preimages of all real $\tau' \notin \{\tau_+, \tau_-\}$. To see that the τ-preimages $P[\tau_\pm]$ of $\tau_\pm$ are connected as well, let us fix $\tau_0 \in \{\tau_-, \tau_+\}$ and denote $N_1, \ldots, N_l$ the connected components of $P[\tau_0]$. Also, let $U_1, \ldots, U_l$ be pairwise disjoint open sets in M with $N_j = U_j \cap \mathrm{Crit}(\tau)$ for $j = 1, \ldots, l$. The τ-preimage $P[\tau']$ of every $\tau' \in (\tau_-, \tau_+)$ sufficiently close to τ_0 must now be contained in the union $U = U_1 \cup \cdots \cup U_l$, or else there would be a sequence x_k in $M' \smallsetminus U$ with $\tau(x_k) \to \tau_0$ as $k \to \infty$, a subsequence of which would have a limit that lies in $P[\tau_0]$, yet not in the set U containing $P[\tau_0]$. However, $P[\tau']$ obviously intersects *each* of the sets $U_1, \ldots, U_l$ whenever $\tau' \in (\tau_-, \tau_+)$ is sufficiently close to τ_0. Since such $P[\tau']$ are connected (see above) and $U_1, \ldots, U_l$ are pairwise disjoint and open, we must have $l = 1$. This completes the proof. ■

§8. Special Kähler-Ricci potentials

Except for Remarks 8.3 – 8.4, the material in this section also appears in [8].

Given a special Kähler-Ricci potential $\tau : M \to \mathbf{R}$ on a Kähler manifold (M, g), as in (0.1), let $M' \subset M$ be the open set on which $d\tau \neq 0$, and let the vector fields v, u on M and distributions $\mathcal{H}, \mathcal{V}$ on M' be defined by

$$\mathcal{V} = \mathrm{Span}\,\{v, u\} \quad \text{and} \quad \mathcal{H} = \mathcal{V}^\perp\,, \quad \text{with} \quad v = \nabla\tau \quad \text{and} \quad u = Jv\,. \tag{8.1}$$

Setting $Q = g(\nabla\tau, \nabla\tau)$, we thus have

$$g(v, v) = g(u, u) = Q\,, \qquad g(v, u) = 0 \qquad \text{everywhere in} \quad M. \tag{8.2}$$

In view of the eigenvector clause of (0.1), and since (8.6.b) implies (8.6.a) (see Remark 8.1 below), there exist C^∞ functions $\phi, \psi, \lambda, \mu : M' \to \mathbf{R}$ with

$$\begin{aligned} &\mathrm{r} = \lambda g \quad \text{on} \quad \mathcal{H}, && \mathrm{r} = \mu g \quad \text{on} \quad \mathcal{V}, \\ &\nabla d\tau = \phi g \quad \text{on} \quad \mathcal{H}, && \nabla d\tau = \psi g \quad \text{on} \quad \mathcal{V}, \\ &\mathrm{r}(\mathcal{H}, \mathcal{V}) = (\nabla d\tau)(\mathcal{H}, \mathcal{V}) = \{0\} && \text{for} \quad \mathcal{H}, \mathcal{V} \quad \text{as in} \quad (8.1). \end{aligned} \tag{8.3}$$

The last line states that $\mathcal{H}$ is both r-orthogonal and $\nabla d\tau$-orthogonal to $\mathcal{V}$. We set $\phi = \lambda = 0$ if $\dim_{\mathbf{C}} M = 1$, For a vector field w on M', (1.3) now gives

$$\nabla_w v \ \text{equals} \ \phi w \ \text{(or,} \ \psi w\text{) whenever} \ w \ \text{is a section of} \ \mathcal{H} \ \text{(or, of} \ \mathcal{V}\text{)}. \tag{8.4}$$

Also, according to [8], Lemmas 7.5 and 11.1(b), on M' we have

$$\begin{aligned} &\text{i)} \quad dQ = 2\psi\, d\tau\,, \quad \text{i.e.,} \quad \nabla Q = 2\psi v\,, \quad \text{and} \quad \nabla\phi = 2(\psi - \phi)\phi v/Q\,. \\ &\text{ii)} \quad Y = 2\psi + 2(m-1)\phi\,, \quad \text{where} \quad Y = \Delta\tau\,. \end{aligned} \tag{8.5}$$

Relations $dQ = 2\psi\, d\tau$ and (8.5.ii) also follow from (8.3), as $d_w Q = d_w[g(v, v)] = 2g(\nabla_w v, v) = 2\psi g(w, v) = 2\psi d_w\tau$, by (8.2), (8.4), (8.1), for any vector field w.

As $dY = -2\,\mathrm{r}(\nabla\tau, \cdot\,)$ for any Killing potential τ, the Ricci tensor r, and $Y = \Delta\tau = \langle g, \nabla d\tau\rangle$ ([6]; see also [8], formula (5.4)), (8.1) and (8.3) give $dY = -2\mu\, d\tau$.

Remark 8.1. For a distribution $\mathcal{V}$ on a Riemannian manifold (M,g) and a symmetric twice-covariant tensor b at a point $x \in M$, consider the conditions

(8.6)
a) All nonzero vectors in $\mathcal{V}_x$ and $\mathcal{H}_x = \mathcal{V}_x^\perp$ are eigenvectors of b.
b) All nonzero vectors in $\mathcal{H}_x = \mathcal{V}_x^\perp$ are eigenvectors of b.

Let $\mathcal{V}$ now be a *J-invariant* distribution of complex dimension one on a Kähler manifold (M,g), and let $x \in M$. For a symmetric twice-covariant tensor b at x which is also Hermitian (cf. Lemma 4.2(iii)), condition (8.6.b) then implies (8.6.a). In fact, the operator $B : T_xM \to T_xM$ with $b(w,w') = g(Bw,w')$ for all $w,w' \in T_xM$ is self-adjoint, commutes with J, and $B\mathcal{V}_x^\perp \subset \mathcal{V}_x^\perp$. Hence $B\mathcal{V}_x \subset \mathcal{V}_x$. Choosing $v \in \mathcal{V}_x \smallsetminus \{0\}$ and $\lambda \in \mathbf{R}$ with $Bv = \lambda v$, we thus have $Bu = \lambda u$ for $u = Jv$ (as $BJv = JBv$), which yields (8.6.a) since $\dim_{\mathbf{R}}\mathcal{V}_x = 2$.

Lemma 8.2. *Given τ with* (0.1) *on a Kähler manifold (M,g), let $Q = g(\nabla\tau,\nabla\tau)$ and let ϕ be as in* (8.3)*. Then either $\phi = 0$ identically on M', or $\phi \neq 0$ everywhere in M', where $M' \subset M$ is the open set on which $d\tau \neq 0$. In the latter case, there exists a constant c with $Q/\phi = 2(\tau - c)$ and $\tau \neq c$ everywhere in M'.*

For a proof, see [8], Lemma 12.5. Note that relation $\tau \neq c$ on M' is obvious from $Q/\phi = 2(\tau - c)$. ■

Remark 8.3. Let a function τ satisfy (0.1) on a Kähler manifold (M,g). We define $\varepsilon \in \{-1,0,1\}$ by $\varepsilon = 0$ when $\phi = 0$ identically on M' and $\varepsilon = \operatorname{sgn}(\tau - c)$ when $\phi \neq 0$ everywhere in M' (with M', ϕ, c as in Lemma 8.2). Note that in the latter case $\varepsilon = \pm 1$ is uniquely defined, since $\tau \neq c$ everywhere in M' (Lemma 8.2) and M' is connected (Remark 5.3(i)).

Remark 8.4. Let $Q = g(\nabla\tau,\nabla\tau)$ and let ϕ, ψ be as in (8.3) for a special Kähler-Ricci potential τ on a Kähler manifold (M,g) (see (0.1)), and let $v = \nabla\tau$.

(i) Writing $\dot f = d[f(x(s))]/ds$ for a fixed C^1 curve $s \mapsto x(s) \in M$ and a C^1 function f defined in M, we have $\dot f = d_{\dot x} f = g(\nabla f, \dot x)$, where $\dot x = dx/ds$. Consequently, $g(v,\dot x) = \dot\tau$ and, by (8.5.i), $\dot Q = 2\psi\dot\tau$.

(ii) Given $y \in M$ with $v(y) = 0$, let $s \mapsto x(s) \in M$ be a C^2 curve such that $x(0) = y$ and $\dot x(0)$ is an eigenvector of $(\nabla d\tau)(y)$ for an eigenvalue $a \neq 0$. (Such a exists since $\nabla d\tau \neq 0$ at y by (0.1) and Remark 4.4.) Then $\dot\tau = 0$ and $\ddot\tau = a|\dot x|^2 \neq 0$ at $s = 0$ (notation of (i)), so that $\dot\tau \neq 0$ for all $s \neq 0$ close to 0. In fact, $g(v,\dot x) = \dot\tau$ (see (i)) and applying d/ds we obtain $\ddot\tau = g(\nabla_{\dot x} v, \dot x) + g(v, \nabla_{\dot x}\dot x)$, which at $s = 0$ equals $(\nabla d\tau)(\dot x,\dot x)$ (by (1.2) with $v(y) = 0$).

Remark 8.5. Let $Q = g(\nabla\tau,\nabla\tau)$ for a function τ with (0.1) on a Kähler manifold (M,g), and let $M' \subset M$ be the open set given by $d\tau \neq 0$. Then

(a) Every point of M' has a neighborhood on which Q is a C^∞ function of τ such that $dQ/d\tau = 2\psi$, with ψ given by (8.3).

(b) Q is constant along every connected component of the preimage of any real number under $\tau : M' \to \mathbf{R}$.

In fact, (a) is obvious as $dQ = 2\psi\, d\tau$ (cf. (8.5.i)); using local coordinates in M' having τ as one of the coordinate functions, one sees that the assignment $\tau \mapsto Q$ is (locally, in M') of class C^∞. Thus, Q is locally constant on each of the connected components in (b), and assertion (b) follows.

§9. The simplest examples

Example 9.1. A nonconstant C^∞ function τ on a Kähler manifold (M, g) of complex dimension 1 satisfies (0.1) if and only if it is a Killing potential, as the remainder of (0.1) then is vacuously true. Consequently, in complex dimension 1 there is, locally, a one-to-one correspondence between special Kähler-Ricci potentials (defined up to an additive constant), and nontrivial Killing fields. This is clear from Lemma 4.3 and the fact that, in complex dimension 1, every Killing field u is holomorphic (by (4.1.a), since skew-adjointness of ∇u, cf. §4, then gives $\nabla u = \psi J$ for some function ψ).

Similarly, for a nonconstant C^∞ function τ on a Kähler manifold (M, g) of any complex dimension m, one has (0.1) whenever g is an Einstein metric and $\nabla d\tau = \psi g$ for some function ψ. For instance, this is the case when g is the standard Euclidean metric on $M = \mathbf{C}^m$, with the norm-squared function τ and $\psi = 2$, or with a real-linear function τ and $\psi = 0$. (Cf. Example 9.4 below.)

Example 9.2. Let (S, γ) be an oriented Riemannian surface admitting a nonconstant Killing potential $\tau : S \to \mathbf{R}$, and let (N, h) be a Kähler manifold of complex dimension $m - 1 \geq 1$ with the Ricci tensor $\mathrm{r}^{(h)} = \lambda h$ for some function $\lambda : N \to \mathbf{R}$, so that (N, h) is Kähler-Einstein (and λ is constant) unless $m = 2$. Note that, for an oriented surface (S, γ) with $H^1(S, \mathbf{R}) = \{0\}$, such τ exists if and only if (S, γ) admits a nontrivial Killing field (Lemma 4.3 and Example 9.1).

Treated as a function on $M = N \times S$ constant along the N factor, τ then is a special Kähler-Ricci potential on the Kähler manifold (M, g) of complex dimension $m \geq 2$ obtained as the Riemannian product of (N, h) and (S, γ).

In fact, let $\mathcal{H}, \mathcal{V}$ be the N and S factor distributions on M. Conditions (8.3) are obviously satisfied by the function λ with $\mathrm{r}^{(h)} = \lambda h$ and the Gaussian curvature μ of γ, along with $\phi = 0$ and $\psi = \Delta\tau/2$ (the Laplacian of $\tau/2$ in (S, γ)), as relation $\nabla u = \psi J$ in Example 9.1 gives $\nabla d\tau = \psi\gamma$, by (1.2) with g replaced by γ. This implies (0.1), as τ is a Killing potential on (M, g) due to being one when viewed as a function on (S, γ).

Lemma 9.3. *Let two vector fields v, u on a Riemannian manifold (M, g) be linearly independent at every point, and let $\mathcal{H}$ be a distribution on M with $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \mathrm{Span}\,\{v, u\}$. If G is a group of isometries of (M, g) such that, at every $x \in M$, the action on T_xM, via differentials, of the isotropy subgroup of G at x leaves $v(x)$, $u(x)$ and $\mathcal{H}_x$ invariant and acts transitively on the unit sphere in $\mathcal{H}_x$, then $\mathcal{H} = \mathcal{V}^\perp$, and any G-invariant symmetric twice-covariant tensor field b on M satisfies condition* (8.6.b) *at every point $x \in M$.*

In fact, for such b and x the functions $b(v(x), w)$, $b(u(x), w)$ and $b(w, w)$ of $w \in \mathcal{H}_x$ are constant on the unit sphere; the first two are also linear, so they must be identically zero. This yields (8.6.b) and, applied to $b = g(x)$, gives $\mathcal{H} = \mathcal{V}^\perp$. ■

Example 9.4. Let our complex manifold be any G-invariant nonempty open connected set U in a complex vector space V of complex dimension $m \geq 2$, carrying a fixed Hermitian inner product $\langle\,,\rangle$, where $G \approx \mathrm{U}(m)$ is the group of all unitary automorphisms $V \to V$. For any G-invariant Kähler metric g on U, a special Kähler-Ricci potential τ on (U, g) (cf. (0.1)) can be obtained as follows.

Formula $u(x) = aix$, with any fixed real constant $a \neq 0$, defines a G-invariant holomorphic Killing field u on (U, g). (Thus, u is an infinitesimal generator of the center subgroup of G.) We now choose $\tau : M \to \mathbf{R}$ to be a Killing potential with $u = Jv$ for $v = \nabla\tau$, where ∇ is the g-gradient; by Lemma 4.3, such τ exists and is unique up to an additive constant.

Since our v, u are the same as in Remark 3.1, Lemma 9.3 now shows that $\mathcal{V}, \mathcal{H}$ defined in Remark 3.1 are g-orthogonal to each other, and the tensors $b = \mathrm{r}$ and $b = \nabla d\tau$ satisfy (8.6.b) at every $x \in U \smallsetminus \{0\}$, which yields (0.1).

Example 9.5. For an integer $m \geq 2$ and a fixed point $y \in \mathbf{C}\mathrm{P}^m$, let g be any G-invariant Kähler metric on $\mathbf{C}\mathrm{P}^m$, where G is the group of all biholomorphisms $\mathbf{C}\mathrm{P}^m \to \mathbf{C}\mathrm{P}^m$ that keep y fixed and preserve the standard (Fubini-Study) metric. Then $(\mathbf{C}\mathrm{P}^m, g)$ admits a function τ with (0.1), obtained as follows.

As $G \approx \mathrm{U}(m)$ (see below), its center, isomorphic to $\mathrm{U}(1)$, is generated by a nontrivial holomorphic Killing field u on $(\mathbf{C}\mathrm{P}^m, g)$, unique up to a factor; we choose a Killing potential $\tau : \mathbf{C}\mathrm{P}^m \to \mathbf{R}$ with $u = J(\nabla\tau)$, where ∇ is the g-gradient. Such τ exists and is unique up to an additive constant (Lemma 4.3).

The usual identification of $\mathbf{C}^m$ with an open dense set in $\mathbf{C}\mathrm{P}^m$, such that $y = 0 \in \mathbf{C}^m \subset \mathbf{C}\mathrm{P}^m$, makes G act on $\mathbf{C}^m$ as the matrix group $\mathrm{U}(m)$. Restricting g to $\mathbf{C}^m$ we then obtain a $\mathrm{U}(m)$-invariant Kähler metric on $\mathbf{C}^m$. That τ is a special Kähler-Ricci potential on $(\mathbf{C}\mathrm{P}^m, g)$ is now immediate from Example 9.4: since $\mathbf{C}^m$ is dense in $\mathbf{C}\mathrm{P}^m$, the eigenvector clause of (0.1) holds on $\mathbf{C}\mathrm{P}^m$ as well.

Remark 9.6. More precisely, on $\mathbf{C}\mathrm{P}^{m-1} = \mathbf{C}\mathrm{P}^m \smallsetminus \mathbf{C}^m$ (the hyperplane at infinity) the eigenvector clause of (0.1) holds vacuously, since u (and hence $d\tau$) vanishes there; namely, the flow of u in $\mathbf{C}^m$ consists of multiples of the identity, all of which leave every line through 0 invariant, i.e., keep every point at infinity fixed. Since 0 is the only point they keep fixed in $\mathbf{C}^m$, it follows that the set of critical points of τ, i.e., zeros of u, is the union of two critical manifolds (cf. Remark 5.3(ii)): the one-point set $\{y\}$ and the hyperplane at infinity.

§10. A local model

All special Kähler-Ricci potentials τ on Kähler manifolds arise (locally, at points with $d\tau \neq 0$) from the construction described below; see also §36.

Given a positive C^∞ function Q of a real variable τ, defined on an open interval $\mathcal{I}'$, and a real constant $a \neq 0$, a C^∞ function r of the variable $\tau \in \mathcal{I}'$ with

$$dr/d\tau = ar/Q \quad \text{and} \quad r > 0 \quad \text{on} \quad \mathcal{I}' \tag{10.1}$$

exists and is unique up to a constant factor, constituting a diffeomorphism

$$\mathcal{I}' \ni \tau \mapsto r \in (r_-, r_+) \quad \text{with} \quad 0 \leq r_- < r_+ \leq \infty. \tag{10.2}$$

Let there be given the following set of data:

$$\mathcal{I}', \tau, Q, r; \qquad a, \varepsilon, c; \qquad m, N, h; \qquad \mathcal{L}, \mathcal{H}, \langle\,,\rangle. \tag{10.3}$$

Here $\mathcal{I}' \subset \mathbf{R}$ is an open interval, $\tau \in \mathcal{I}'$ is a real variable, Q is a positive C^∞ function Q of the variable $\tau \in \mathcal{I}'$, while $a \neq 0$ is a real constant, r is a fixed function of $\tau \in \mathcal{I}'$ satisfying (10.1), and ε, c are constants such that either $\varepsilon = 0$, or $c \notin \mathcal{I}'$ and $\varepsilon = \operatorname{sgn}(\tau - c) = \pm 1$ for all $\tau \in \mathcal{I}'$. (When $\varepsilon = 0$, we leave c undefined.) Next, $m \geq 2$, is an integer, (N, h) is a Kähler manifold of complex dimension $m - 1$ (which we assume to be Einstein unless $m = 2$), while $\mathcal{L}$ is a C^∞ complex line bundle over N, with a Hermitian fibre metric $\langle\,,\rangle$, and $\mathcal{H}$ is the horizontal distribution of a connection in $\mathcal{L}$ making $\langle\,,\rangle$ parallel, whose curvature form (Remark 2.1) equals $-2\varepsilon a$ times the Kähler form of (N, h) (cf. (1.5)).

Using the diffeomorphism (10.2), we treat functions of $\tau \in \mathcal{I}'$ as functions of $r \in (r_-, r_+)$. This includes τ itself, Q and f defined by

$$f = 1 \quad (\text{when } \varepsilon = 0), \quad \text{or} \quad f = 2|\tau - c| \quad (\text{when } \varepsilon = \pm 1). \tag{10.4}$$

Let U now be the open subset of $\mathcal{L} \smallsetminus N$ given by $r_- < r < r_+$, where r this time denotes the norm function of $\langle\,,\rangle$ (see Remark 2.2); thus, r, τ, Q, f can also be regarded as C^∞ functions $U \to \mathbf{R}$. We define a metric g on U by

$$g = f\pi^*h \text{ on } \mathcal{H}, \quad g = (ar)^{-2} Q \operatorname{Re}\langle\,,\rangle \text{ on } \mathcal{V}, \quad g(\mathcal{H}, \mathcal{V}) = \{0\}, \tag{10.5}$$

where $\pi : \mathcal{L} \to N$ is the bundle projection, $\mathcal{V}$ denotes the vertical distribution in $\mathcal{L}$, and $\operatorname{Re}\langle\,,\rangle$ is the standard Euclidean metric on each fibre of $\mathcal{L}$. (The last relation in (10.5) means that $\mathcal{H}$ is g-orthogonal to $\mathcal{V}$.)

Moreover (see Remark 2.4), $\mathcal{L}$ has a unique structure of a holomorphic line bundle over N such that $\mathcal{H}$ is J-invariant. This turns the open submanifold U of $\mathcal{L}$ into a complex manifold of complex dimension m. According to [8], beginning of §16 and Remark 16.1, for $\phi, \psi, \lambda, \mu, f$ as in (8.3) and (10.4),

(a) g is a Kähler metric on U,
(b) τ is a special Kähler-Ricci potential on (U, g), in the sense of (0.1),
(c) Q treated as a function $U \to \mathbf{R}$ is given by $Q = g(\nabla\tau, \nabla\tau)$.
(d) If $\varepsilon = 0$ then $\phi = 0$ identically, while if $\varepsilon = \pm 1$ then $\phi \neq 0$ everywhere and c in (10.3) is the same as in Lemma 8.2, i.e., $2\phi = Q/(\tau - c)$.
(e) Let $\kappa : N \to \mathbf{R}$ be a function such that h has the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$. Identifying κ with the composite $\kappa \circ \pi$, we may treat it as a function on U, constant unless $m = 2$. Then $\lambda = (\kappa - \varepsilon Y)/f$, with $Y = \Delta\tau = \langle g, \nabla d\tau\rangle$.

Remark 10.1. Let the data (10.3) be chosen so that h is an Einstein metric, $\varepsilon = \pm 1$, and Q satisfies the differential equation $pQ' - Q + (m-1)pQ/(\tau - c) = \varepsilon p\kappa - 2\sigma(\tau - c)$, where $Q' = dQ/d\tau$ and κ is the constant in (e), while p, σ are constants with $p \neq 0$. Also, let ∇, r be the Levi-Civita connection and Ricci tensor of the metric g with (10.5) on the manifold U constructed above.

These (U, g) and $\tau : U \to \mathbf{R}$ then satisfy the condition $\nabla d\tau + p\,\mathrm{r} = \sigma g$.

In fact, $2\psi = dQ/d\tau = Q'$ and $2\phi = Q/(\tau-c)$ by (8.5.i) and (d), so that (8.5.ii) gives $Y = Q'+(m-1)Q/(\tau-c)$, i.e., our assumption yields $pY-Q = \varepsilon p\kappa-2\sigma(\tau-c)$. Applying $d/d\tau$ we get $-2(\psi+p\mu) = -2\sigma$, as $dY/d\tau = -2\mu$ (see the line preceding Remark 8.1) and $Q' = 2\psi$, while $-2(\phi+p\lambda) = (pY-Q-\varepsilon p\kappa)/(\tau-c) = -2\sigma$ since $2\phi = Q/(\tau-c)$ and $2\lambda = (\varepsilon\kappa-Y)/(\tau-c)$, by (e), as $f = 2\varepsilon(\tau-c)$ (cf. (10.4), with $\varepsilon = \operatorname{sgn}(\tau-c)$). Our claim now follows from (8.3) with $\psi+p\mu = \sigma$ and $\phi+p\lambda = \sigma$.

Note that, whenever such a triple U, g, τ can be "compactified" as in §17, it becomes an example of a Kähler-Ricci soliton ([14], [15], [7]). See [10] for details.

§11. The case of open spherical shells

Suppose that we are given the data

$$\mathcal{I}', \tau, Q, r; \qquad a, \varepsilon, c; \qquad m, V, \langle\,,\rangle \tag{11.1}$$

which consist of an open interval $\mathcal{I}' \subset \mathbf{R}$, a positive C^∞ function Q of the variable $\tau \in \mathcal{I}'$, real constants a, c and $\varepsilon = \pm 1$ such that $\varepsilon a > 0$ and $\varepsilon(\tau-c) > 0$ for all $\tau \in \mathcal{I}'$, a fixed function r of $\tau \in \mathcal{I}'$ satisfying (10.1), a complex vector space V of complex dimension $m \geq 2$, and a Hermitian inner product $\langle\,,\rangle$ in V. The symbol r also denotes the norm function $V \to [0,\infty)$, with $z \mapsto \langle z, z\rangle^{1/2}$.

Let U be the open spherical shell in V lying between spheres of radii r_+, r_- centered at 0, i.e., given by $r_- < r < r_+$, with $r_\pm$ as in (10.2). Our τ and Q, being C^∞ functions of $r \in (r_-, r_+)$, thus become C^∞ functions $U \to \mathbf{R}$. We now define a Riemannian metric g on U by

$$|a|r^2 g = 2|\tau-c|\operatorname{Re}\langle\,,\rangle \text{ on } \mathcal{H}, \quad a^2r^2 g = Q\operatorname{Re}\langle\,,\rangle \text{ on } \mathcal{V}, \quad g(\mathcal{H},\mathcal{V}) = \{0\} \tag{11.2}$$

for $\mathcal{H}, \mathcal{V}$ as in Remark 3.1, $\operatorname{Re}\langle\,,\rangle$ being the standard Euclidean metric on V.

Lemma 11.1. *The above construction of U, g and $\tau : U \to \mathbf{R}$ is a special case of that in* §10. *Namely, our U, g, τ are the same as those obtained from the data* (10.3) *with $\mathcal{I}', \tau, Q, r, a, \varepsilon, c, m$ as in* (11.1)*, $N, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ defined as in* §3 *for $m, V, \langle\,,\rangle$ in* (11.1)*, and h equal to $1/|a|$ times the Fubini-Study metric on the projective space N. The latter data* (10.3) *satisfy all the conditions listed in the paragraph following* (10.3)*, while equality between $U \subset V \smallsetminus \{0\}$ and $U \subset \mathcal{L} \smallsetminus N$ makes sense due to the biholomorphic identification $V \smallsetminus \{0\} = \mathcal{L} \smallsetminus N$ of* §3.

In particular, g is a Kähler metric on U with the complex structure of an open submanifold of V, and τ is a special Kähler-Ricci potential on (U, g).

In fact, in §3 we verified that the data (10.3) described in the lemma satisfy the assumptions made in §10. (Note that $\varOmega = -\omega_{\mathrm{FS}}$ equals $-2\varepsilon a$ times the Kähler form of (N, h), as $\varepsilon a = |a|$.) The last two relations of (10.5), for our g, are immediate from (11.2). Finally, the first equality in (10.5) follows from (10.4) and (11.2) with $|a|r^2\pi^*h = \operatorname{Re}\langle\,,\rangle$ on $\mathcal{H}$ (see Remark 3.2) and with the identification $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$, which also proves our claim about the complex structure. ∎

The triples U, g, τ constructed here also form a special case of those described in Example 9.4. In fact, g is a Kähler metric, $u = J(\nabla\tau)$ is a holomorphic Killing field (Lemmas 11.1 and 4.3), and g, τ are both invariant under the unitary group $G \approx \mathrm{U}(m)$, since so are the Euclidean metric $\mathrm{Re}\langle\,,\rangle$, its norm function r, and τ (which is a function of r). Since u is G-invariant, it generates the center of G.

§12. Duality and metrics on annuli

Let V^* be the dual space of a complex vector space V of complex dimension 1. We define the *inversion biholomorphism* $V \smallsetminus \{0\} \to V^* \smallsetminus \{0\}$ to be the assignment $z \mapsto z^{-1}$, where $z^{-1} \in V^*$ is the $\mathbf{C}$-linear functional $V \to \mathbf{C}$ sending z to 1. Any fixed Hermitian inner product $\langle\,,\rangle$ in V gives rise to a Hermitian inner product $\langle\,,\rangle^*$ in V^* such that $\langle z^{-1}, z^{-1}\rangle^* = \langle z, z\rangle^{-1}$ whenever $z \in V \smallsetminus \{0\}$.

Given a septuple $\mathcal{I}', \tau, Q, r, a, V, \langle\,,\rangle$ consisting of

(12.1) an open interval $\mathcal{I}' \subset \mathbf{R}$, a positive C^∞ function $\mathcal{I}' \ni \tau \mapsto Q$, a real constant $a \neq 0$, a function r of $\tau \in \mathcal{I}'$ satisfying (10.1), and a complex vector space V of complex dimension 1 with a Hermitian inner product $\langle\,,\rangle$,

let U be the open annulus in V given by $r_- < r < r_+$. Here r also denotes the norm function of $\langle\,,\rangle$, which allows us to treat r, τ and Q as C^∞ functions $U \to \mathbf{R}$. Formula $\gamma = (ar)^{-2} Q\,\mathrm{Re}\langle\,,\rangle$ now defines a Riemannian metric γ on the annulus U, conformal to the standard Euclidean metric $\mathrm{Re}\langle\,,\rangle$.

Next, let us replace $r, a, V, \langle\,,\rangle$ in these data by r^*, a^* with $r^* = 1/r$, $a^* = -a$ and $V^*, \langle\,,\rangle^*$ as in the beginning of this section, but keep the same $\mathcal{I}', \tau, Q$ and $m = 1$. Since (10.1) implies that $dr^*/d\tau = a^* r^*/Q$, the new data satisfy the same assumptions, and may be used as above to define a metric γ^* on an open annulus $U^* \subset V^* \smallsetminus \{0\}$. Then, with r^* also standing for the norm function of $\langle\,,\rangle^*$,

(a) The assignment $z \mapsto z^{-1}$ is an isometry $(U, \gamma) \to (U^*, \gamma^*)$, under which r treated as a function on U corresponds to the function $1/r^*$ on U^*.
(b) τ as a function of r with (10.1) is related to τ viewed, similarly, as a function of r^*, in such a way that the assignment $r \mapsto r^* = 1/r$ leaves the corresponding value of τ unchanged.

In fact, regarded as functions of τ inverse to those in (b), our r, r^* satisfy $r^* = 1/r$, which implies (b). To verify (a), let us use the multiplicative notation $\zeta z \in \mathbf{C}$ for evaluating functionals $\zeta \in V^*$ on vectors $z \in V$. Thus, $|\zeta z|^2 = \langle z, z\rangle\langle \zeta, \zeta\rangle^*$. (To see this, assume that $z \neq 0$ and write ζ as a scalar times z^{-1}, cf. the beginning of this section.) Setting $\zeta(s) = [z(s)]^{-1}$ for any C^1 curve $s \mapsto z(s) \in U$, and differentiating the resulting relation $\zeta(s)z(s) = 1$, we now see that the differential of $z \mapsto z^{-1}$ at any point $z \in U$ sends each tangent vector $\dot z \in T_z U = V$ to the vector $\dot\zeta = -(\zeta \dot z)\zeta \in V^*$ tangent to U^* at the point $\zeta = z^{-1}$. As $\langle \zeta, \zeta\rangle^* = \langle z, z\rangle^{-1}$ (see above), we thus have $\langle \dot\zeta, \dot\zeta\rangle^* = \langle \dot z, \dot z\rangle / \langle z, z\rangle^2 = \langle \dot z, \dot z\rangle / r^4$, i.e., the pullback under $z \mapsto z^{-1}$ of the Euclidean metric $\mathrm{Re}\langle\,,\rangle^*$ on U^* is $1/r^4$ times the Euclidean metric on U. The pullback of the function r^* on U^*, that is, the composite of $z \mapsto z^{-1}$ followed by r^*, clearly is $1/r$ on U, as claimed in (a). (See the definition of $\langle\,,\rangle^*$.) Also, as Q is a function of τ, (b) remains valid when τ is replaced by Q, so that the pullback of Q is Q. Consequently, the pullback of $\gamma^* = (a^* r^*)^{-2} Q\,\mathrm{Re}\langle\,,\rangle^*$ is $\gamma = (ar)^{-2} Q\,\mathrm{Re}\langle\,,\rangle$, as required.

§13. The inversion biholomorphism

Let $\mathcal{L}^*$ be the dual of a holomorphic line bundle $\mathcal{L}$ over a complex manifold N. As in (2.2), the symbols $\mathcal{L}, \mathcal{L}^*$ also stand for their total spaces, and N is identified with the zero sections $N \subset \mathcal{L}$ and $N^* \subset \mathcal{L}^*$. We now define M to be the complex manifold obtained from the disjoint union $\mathcal{L} \cup \mathcal{L}^*$ by identifying the open subsets $\mathcal{L} \smallsetminus N$ and $\mathcal{L}^* \smallsetminus N^*$ via the *inversion biholomorphism* $\mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N^*$ given by $(y, z) \mapsto (y, z^{-1})$, in the notation of (2.2), where $z^{-1} \in \mathcal{L}_y^*$ is the unique $\mathbf{C}$-linear functional $\mathcal{L}_y \to \mathbf{C}$ that sends z to 1 (cf. §12). This makes M a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over N. If N is compact, so is M, and one then refers to M as the *projective compactification* of $\mathcal{L}$.

Equivalently, we could define M to be the bundle associated with the principal $\mathrm{GL}(1, \mathbf{C})$-bundle of $\mathcal{L}$ via the obvious multiplicative action of $\mathrm{GL}(1, \mathbf{C}) = \mathbf{C} \smallsetminus \{0\}$ on the Riemann sphere $\mathbf{C} \cup \{\infty\}$.

Remark 13.1. For $\mathcal{L}$ and M as above, the inversion biholomorphism clearly sends the horizontal distribution $\mathcal{H}$ of any C^∞ linear connection in the line bundle $\mathcal{L}$ onto the horizontal distribution, also denoted $\mathcal{H}$, of its dual connection in $\mathcal{L}^*$. Thus, $\mathcal{H}$ has an extension from $\mathcal{L} \subset M$ to a C^∞ distribution on M.

Also, any Hermitian fibre metric $\langle\,,\rangle$ in $\mathcal{L}$ gives rise to a Hermitian fibre metric $\langle\,,\rangle^*$ in $\mathcal{L}^*$ obtained, in each fibre, as in §12.

Lemma 13.2. *Let the data* (10.3) *have the properties listed in the paragraph following* (10.3)*. For U, g and $\tau : U \to \mathbf{R}$ determined by them as in* §10*,*

(i) *The same assumptions hold for the new set of data obtained if one leaves $\mathcal{I}', \tau, Q, \varepsilon, c, m, N, h$ unchanged and replaces the function r of the variable τ by $r^* = 1/r$, the constant a by $a^* = -a$, and $\mathcal{L}, \langle\,,\rangle, \mathcal{H}$ by $\mathcal{L}^*, \langle\,,\rangle^*, \mathcal{H}$ described in* Remark 13.1.
(ii) *The construction of* §10 *applied to the new data in* (i) *leads to analogous objects U^*, g^* and $\tau^* : U^* \to \mathbf{R}$ such that the inversion biholomorphism $\mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N^*$ sends U, g, τ onto U^*, g^*, τ^*.*

Proof. The curvature forms of a given connection and its dual differ by sign, since so do their connection forms (see Remark 2.1) relative to local sections without zeros having the form w and w^{-1}. This implies (i). Let r, r^* now also stand for specific functions $\mathcal{L} \to \mathbf{R}$ and $\mathcal{L}^* \to \mathbf{R}$, namely, the norm functions of our fibre metrics (Remark 2.2). That the inversion biholomorphism sends U onto U^* is clear as U, U^* are given by $r_- < r < r_+$ and $r_-^* < r^* < r_+^*$, with $r_\pm^* = 1/r_\pm$, and that it makes τ, r and the restriction of g to the vertical distribution $\mathcal{V}$ in U correspond to their counterparts in U^* is immediate from (a), (b) in §12. By (10.5) and Remark 13.1, the same holds for g and the horizontal distribution $\mathcal{H}$ in U. Finally, again by (10.5), $\mathcal{H}, \mathcal{V}$ are orthogonal to each other both in $\mathcal{L}$ and in $\mathcal{L}^*$. Combined with Remark 13.1, this completes the proof. ∎

§14. A one-sided boundary condition

Let $Q, \tau, \mathcal{I}, \mathcal{I}', \tau_0, a, r$ have the following properties:

(14.1) Q is a C^∞ function of the real variable τ, defined on a half-open interval $\mathcal{I}$, positive on its interior $\mathcal{I}'$, and such that at the only endpoint τ_0 of $\mathcal{I}$ we have $Q = 0$ and $dQ/d\tau = 2a \neq 0$, while r is a positive C^∞ function of $\tau \in \mathcal{I}'$ satisfying equation (10.1),

and let $r_\pm$ be as in (10.2). Then

(a) $r_- = 0$, i.e., $r \to 0$ as $\tau \to \tau_0$, and Q/r^2 has a positive limit as $\tau \to \tau_0$.
(b) τ and Q/r^2 are C^∞ functions of $r^2 \in [0, r_+^2)$ with $Q/r^2 > 0$ at $r = 0$.

In fact, $Q/(\tau - \tau_0)$ is a C^∞ function of $\tau \in \mathcal{I}$ equal to $2a$ at $\tau = \tau_0$ (Remark 1.1), and so (10.1) implies that $2d[\log r]/d\tau = 2a/Q$ equals $1/(\tau - \tau_0)$ plus a C^∞ function of τ, i.e., $\log r^2$ equals $\log|\tau - \tau_0|$ plus a C^∞ function of $\tau \in \mathcal{I}$. Hence $r^2/(\tau - \tau_0)$ *is a* C^∞ *function of* $\tau \in \mathcal{I}$ *with a nonzero value at* τ_0. Now, as $Q/|\tau - \tau_0|$ and $|\tau - \tau_0|/r^2$ both have positive limits as $\tau \to \tau_0$ (the former limit being $2|a|$), the same follows for Q/r^2, which proves (a). In view of (14.1), (10.1) and the phrase italicized above, the assignment $\tau \mapsto r^2$ is a C^∞ diffeomorphism of $\mathcal{I}$ onto $[0, r_+^2)$, sending the endpoint τ_0 to 0, and so (a) implies (b).

§15. Metrics on disk bundles

Let a set (10.3) of data have the properties listed in the paragraph following (10.3), and let (14.1) hold for $Q, \tau, \mathcal{I}, \mathcal{I}', \tau_0, a, r$ consisting of the same $\mathcal{I}', \tau, Q, a$ as in (10.3), a fixed finite endpoint τ_0 of $\mathcal{I}'$, and $\mathcal{I} = \mathcal{I}' \cup \{\tau_0\}$. Thus, we require Q to have a C^∞ extension to $\mathcal{I}$. Let us also assume that, in (10.3), either $\varepsilon = 0$, or $\varepsilon = \pm 1$ and $\tau_0 \neq c$.

The construction of §10 now yields a triple U, g, τ such that $U \subset \mathcal{L}$ is the open set given by $0 < r < r_+$, for r_+ as in (10.2), where r is the norm function of the fibre metric $\langle\,,\rangle$ in $\mathcal{L}$ (Remark 2.2), while g is a Kähler metric on U and τ is a special Kähler-Ricci potential on (U, g), as in (0.1).

These g and τ also have C^∞ extensions to a metric and a function, still denoted g, τ, on the open set $U^{\mathrm{o}} \subset \mathcal{L}$ given by $0 \leq r < r_+$, that is, on the bundle of open disks of radius r_+ in $\mathcal{L}$. The resulting triple U^{o}, g, τ satisfies (0.1) as well.

In fact, by (10.5) our g is a real fibre metric on $TU = \mathcal{H} \oplus \mathcal{V}$ obtained as the direct sum of $f\pi^*h$ in $\mathcal{H}$ and $\theta\,\mathrm{Re}\langle\,,\rangle$ in $\mathcal{V}$, with $\theta = Q/(ar)^2$ and f as in (10.4). The required extensions exist since the distributions $\mathcal{H}, \mathcal{V}$ and the fibre metrics π^*h and $\mathrm{Re}\langle\,,\rangle$ on them are defined and of class C^∞ everywhere in $\mathcal{L}$ (cf. Remark 2.2), while, by (b) in §14, the functions τ, θ, f have C^∞ extensions to U^{o}, which are positive in the case of θ and f (the latter due to our assumption that $\tau_0 \neq c$ unless $\varepsilon = 0$). Now (0.1) for U^{o}, g, τ follows since U is dense in U^{o}.

§16. The case of an open ball

Lemma 16.1. *Let a set* (11.1) *of data satisfying the conditions listed in the paragraph following* (11.1)*, with* $m \geq 2$*, also have the property that* c *is an endpoint of* $\mathcal{I}'$*, while the function* Q *of the variable* $\tau \in \mathcal{I}'$ *has a* C^∞ *extension to the half-open interval* $\mathcal{I} = \mathcal{I}' \cup \{c\}$ *with* $Q = 0$ *and* $dQ/d\tau = 2a \neq 0$ *at* $\tau = c$.

The open spherical shell U *defined in* §11 *then is a punctured ball, i.e., has the inner radius* $r_- = 0$*, while* g *with* (11.2) *and* $\tau : U \to \mathbf{R}$ *admit* C^∞ *extensions* *to a metric/function on the solid ball* $U \cup \{0\}$.

Proof. Let the vector fields v, u on $U \cup \{0\}$ be as in Remark 3.1 (for our a), and let ξ, ξ' be the 1-forms on $U \cup \{0\}$ with $\xi = \mathrm{Re}\langle v, \cdot\rangle$ and $\xi' = \mathrm{Re}\langle u, \cdot\rangle$. Then g is, on U, a combination of $\xi \otimes \xi + \xi' \otimes \xi'$ and $\mathrm{Re}\langle\,,\rangle$ with the coefficients $[Q - 2a(\tau - c)]/(ar)^4$ and $2(\tau - c)/(ar^2)$. In fact, $(\tau - c)/a > 0$ due to the assumptions on ε in the lines following (11.1), so that, using (1.4) and (11.2) we obtain equal values when both tensors are evaluated on two vectors, one of which is in $\mathcal{H}$ and the other in $\mathcal{H}$ or $\mathcal{V}$, and it is also immediate for the vector fields v, v, or v, u, or u, u, as $\langle v, v\rangle = \langle u, u\rangle = a^2 r^2$ and $\mathrm{Re}\langle v, u\rangle = 0$ (see Remark 2.2).

Both coefficients are C^∞ functions of the variable $r^2 \in [0, r_+^2)$. In fact, for $2(\tau - c)/(ar^2)$ this is clear from Remark 1.1 (with $s = r^2$) and §14 with $\tau_0 = c$. Now, Q/r^2 and $2a(\tau - c)/r^2$, treated as C^∞ functions of $r^2 \in [0, r_+^2)$ (see (b) in §14), have the same positive value at $r^2 = 0$, since (10.1) gives $Q/r^2 = 2a\,d\tau/d(r^2)$. Thus, their difference divided by r^2 is a C^∞ function of $r^2 \in [0, r_+^2)$ (Remark 1.1 for $s = r^2$). Positivity of $2a(\tau - c)/r^2$ at $r^2 = 0$ also shows that the limit of g at $0 \in V$ is positive definite, completing the proof. ■

Remark 16.2. Conditions $\varepsilon a > 0$ and $\varepsilon(\tau - c) > 0$ for all $\tau \in \mathcal{I}'$, required in (11.1), follow from each other under the remaining hypotheses of Lemma 16.1. In fact, since c is an endpoint of $\mathcal{I}'$ and $Q > 0$ on $\mathcal{I}'$, while $Q = 0$ at c, the sign of $dQ/d\tau$ at c must be the same as that of $\tau - c$ for $\tau \in \mathcal{I}'$.

Remark 16.3. For reasons mentioned at the end of §11, the triples U, g, τ constructed above form another special subset of those in Example 9.4. Conversely, every triple U, g, τ of Example 9.4, in which U is an open ball, can also be obtained as described in this section. This fact, not needed for our argument, follows if one applies the local-structure theorem established in §36 to $M = U$ and $y = 0$.

§17. Special Kähler-Ricci potentials on $\mathbf{CP}^1$ bundles

Examples of functions τ with (0.1) on compact Kähler manifolds, in all complex dimensions $m \ge 2$, can be constructed as follows. Suppose that

(17.1) $[\tau_{\min}, \tau_{\max}]$ is a nontrivial closed interval of the variable τ with a C^∞ function $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q \in \mathbf{R}$, which is positive on the open interval $(\tau_{\min}, \tau_{\max})$ and vanishes at the endpoints $\tau_{\min}, \tau_{\max}$, while the values of $dQ/d\tau$ at the endpoints are mutually opposite and nonzero.

Next, let some data (10.3) satisfy all conditions listed in the paragraph following (10.3), and have $N, \mathcal{I}', Q, a, \varepsilon, c$ such that N is compact, Q is the restriction of Q in (17.1) to $\mathcal{I}' = (\tau_{\min}, \tau_{\max})$, while $dQ/d\tau = 2a$ at a fixed endpoint τ_0 of $[\tau_{\min}, \tau_{\max}]$, and either $\varepsilon = 0$, or $\varepsilon = \pm 1$ and $c \notin [\tau_{\min}, \tau_{\max}]$.

These assumptions then also hold for a new set of data, analogous to (10.3), which is obtained as in Lemma 13.2(i), so that $r, a, \mathcal{L}, \langle\,,\rangle$ are replaced by some specific $r^*, a^*, \mathcal{L}^*, \langle\,,\rangle^*$. Using §15 with $r_+ = \infty$ (cf. Remark 17.1 below), we see that the construction of §10 applied to our original data (or, the new data) leads to a Kähler metric g on $\mathcal{L}$ (or, g^* on $\mathcal{L}^*$) along with a special Kähler-Ricci potential τ on $(\mathcal{L}, g)$ (or, τ^* on $(\mathcal{L}^*, g^*)$).

¿From Lemma 13.2(ii), with $U = \mathcal{L} \smallsetminus N$ and $U^* = \mathcal{L}^* \smallsetminus N^*$ it now follows that g, τ and g^*, τ^* together form a Kähler metric and a special Kähler-Ricci potential, again denoted g, τ, on the projective compactification M of $\mathcal{L}$ (cf. §13).

Remark 17.1. Assuming (17.1) and choosing r, r_+, r_- with (10.1) - (10.2) for $\mathcal{I}' = (\tau_{\min}, \tau_{\max})$ and a such that $\pm 2a$ are the values of $dQ/d\tau$ at the endpoints of $\mathcal{I}'$, we have $r_- = 0$ and $r_+ = \infty$. In fact, (a) in §14 for these r, a and one endpoint gives $r_- = 0$, while, when applied to $r^* = 1/r$, $a^* = -a$ and the other endpoint, it yields $r^*_- = 0$, i.e., $r_+ = \infty$.

Remark 17.2. The manifold M constructed above contains two holomorphically embedded copies of N, namely the zero sections $N \subset \mathcal{L}$ and $N^* \subset \mathcal{L}^*$. By (c) in §10, $d\tau$ vanishes precisely at the zeros of the function Q prescribed in (17.1) and treated as a function on $U = \mathcal{L} \smallsetminus N$ (or $U^* = \mathcal{L}^* \smallsetminus N^*$), via the dependence of τ on the norm function r (or, r^*). Since (17.1) and (a) in §14 show that Q has in U (or, U^*) the same zeros as r (or, r^*), $d\tau$ vanishes in M along the zero sections only, i.e., $\tau : M \to \mathbf{R}$ has two critical manifolds: N and N^*.

§18. Special Kähler-Ricci potentials on $\mathbf{C}\mathrm{P}^m$

The following construction is a more explicit version of Example 9.5; see Remark 18.1 below.

Let us assume (17.1), and let the conditions listed in the paragraph following (11.1) hold for some data (11.1). Furthermore, let $\mathcal{I}', Q, a, c$ in (11.1) be such that $\mathcal{I}' = (\tau_{\min}, \tau_{\max})$ and Q is the restriction of Q in (17.1) to $\mathcal{I}'$, while c is an endpoint of $\mathcal{I}'$ and $2a$ is the value of $dQ/d\tau$ at $\tau = c$.

According to Lemmas 16.1 and 11.1, the construction of §11 applied to these data is a special case of that in §10, and yields a Kähler metric g on the complex vector space V along with a special Kähler-Ricci potential τ on (V, g). (The solid ball in Lemma 16.1 is V itself, since $r_+ = \infty$ by Remark 17.1.) The objects (10.3) leading, as in §10, to g and τ on $M' = V \smallsetminus \{0\} = \mathcal{L} \smallsetminus N$ are our $\mathcal{I}', \tau, Q, r, a, \varepsilon, c, m$ in (11.1), the projective space N of V, the metric h such that $|a|h$ is the Fubini-Study metric, and the tautological bundle $\mathcal{L}$ over N with the standard fibre metric and the horizontal distribution $\mathcal{H}$ of the canonical connection (§3).

Let N^* stand for N treated as the zero section $N^* \subset \mathcal{L}^*$ in the dual bundle $\mathcal{L}^*$, cf. (2.2). By Lemma 13.2(ii), the biholomorphism $V \smallsetminus \{0\} = \mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N^*$ identifies g, τ on $V \smallsetminus \{0\}$ with g^*, τ^* on $\mathcal{L}^* \smallsetminus N^*$ which are obtained as in §10 from the new data $\mathcal{I}', \tau, Q, 1/r, -a, \varepsilon, c, m, N, h, \mathcal{L}^*, \mathcal{H}, \langle\,,\rangle^*$ analogous to (10.3). Therefore, g, τ give rise to a Kähler metric and a special Kähler-Ricci potential, still denoted g, τ, on the complex manifold M obtained from the disjoint union $V \cup \mathcal{L}^*$ by using the above biholomorphism to identify the open sets $V \smallsetminus \{0\} \subset V$ and $\mathcal{L}^* \smallsetminus N^* \subset \mathcal{L}^*$. (Note that, applying §15 to the new data and the endpoint of $\mathcal{I}'$ other than c, we can extend g^*, τ^* from $\mathcal{L}^* \smallsetminus N^*$ to $\mathcal{L}^*$.)

This M is clearly biholomorphic to the projective space $\mathbf{P} \approx \mathbf{C}\mathrm{P}^m$ formed by all complex lines through 0 in $V \times \mathbf{C}$ (cf. §3). Namely, we have holomorphic embeddings $V \smallsetminus \{0\} \to \mathbf{P}$ and $\mathcal{L}^* \smallsetminus N^* \to \mathbf{P}$ sending any $z \in V \smallsetminus \{0\}$ to the line spanned by $(z, 1) \in V \times \mathbf{C}$, and any $(y, \zeta) \in \mathcal{L}^* \smallsetminus N^*$ (see (2.2)) to the graph of the linear functional ζ on the line $y \subset V$ (cf. §3); the graph is itself a line through zero in $y \times \mathbf{C} \subset V \times \mathbf{C}$, that is, an element of $\mathbf{P}$. The resulting transition mapping, obtained from the former embedding followed by the inverse of the latter, is precisely the biholomorphism $V \smallsetminus \{0\} \to \mathcal{L}^* \smallsetminus N^*$ we just used, as it takes $z \in V \smallsetminus \{0\}$ to (y, ζ) such that $y \subset V$ is a line through zero and the graph of the functional $\zeta : y \to \mathbf{C}$ is spanned by (i.e., contains) $(z, 1)$, which means that z spans y and ζ sends z to 1, i.e., $\zeta = z^{-1}$ (notation of §13).

Remark 18.1. The construction just described is a special case of Example 9.5, as one sees restricting g, τ to $V \subset M$ and using Remark 16.3. Thus, by Remark 9.6, τ obtained here has two critical manifolds, which realize case 2) of (ii) in §22 below.

Conversely, every triple $\mathbf{C}\mathrm{P}^m, g, \tau$ of Example 9.5 can also be obtained as described in this section. This fact, which will not be used, is an immediate consequence of Theorem 29.2 in §29.

§19. Dimensions of critical manifolds

Suppose that τ is a special Kähler-Ricci potential on a Kähler manifold (M, g) of complex dimension m (see (0.1)) and N is a critical manifold of τ, cf. Remark 5.3(ii), while $Q = g(\nabla\tau, \nabla\tau)$, $Y = \Delta\tau$, and $\phi, \psi : M' \to \mathbf{R}$ are as in (8.3), M' denoting the open set on which $d\tau \neq 0$ (i.e., $Q > 0$). Thus, either $\phi = 0$ identically on M' or $\phi \neq 0$ everywhere in M' (see Lemma 8.2). To refer to the former case, we will just write $\phi = 0$, while the latter one will be tacitly assumed whenever we mention the constant c defined (only when $\phi \neq 0$) in Lemma 8.2.

We have $dY = 0$ wherever $d\tau = 0$, as $dY = -2\,\mathrm{r}(\nabla\tau, \cdot\,)$ (cf. the two lines preceding Remark 8.1), so that $Y = \Delta\tau = \langle g, \nabla d\tau\rangle$ is constant on N. Letting τ_0 be the constant value of τ on N, we define a real constant a, depending on N, by

$$(19.1) \qquad 2a = Y \text{ on } N \text{ if } \phi = 0 \text{ or } \tau_0 \neq c, \text{ and } 2a = Y/m \text{ on } N \text{ if } \tau_0 = c.$$

Then $\psi(x) \to a$ as $x \to y \in N$, where x is a variable point of M'. Also,

$$(19.2) \qquad a \neq 0 \text{ and } \nabla_w v = aw \text{ for every vector } w \text{ normal to } N \text{ at any point,}$$

where $v = \nabla\tau$. Furthermore, one of the following two cases must occur:

(19.3)
a) N is a complex submanifold of complex codimension 1 in M, or
b) N consists of a single point.

Finally,

(i) In case a) of (19.3), $\phi = 0$ on M' or $\tau_0 \neq c$. In case b) of (19.3), $\tau_0 = c$.
(ii) If $\phi = 0$ and $m \geq 2$, then no critical manifold of τ is a one-point set.
(iii) In the case where ϕ is not identically zero, a point $y \in M$ has $\tau(y) = c$ if and only if $\{y\}$ is a critical manifold of τ.
(iv) In case a) of (19.3) with $m \geq 2$, the limit relation $\mathcal{H}_x \to T_y N$ as $x \to y$, for any $y \in N$, holds in an appropriate Grassmannian bundle, with $\mathcal{H}$ as in (8.1), x being a variable point of M'.

In fact, let us fix $y \in N$. Since M' is dense in M (Remark 4.4), we may choose a sequence of points in M' converging to y and, at each point x of the sequence, an orthonormal basis of T_xM formed by eigenvectors of $(\nabla d\tau)(x)$, the last two of which correspond to the eigenvalue $\psi(x)$, and the others to $\phi(x)$, cf. (8.3). A subsequence of this sequence of bases converges, in a suitable frame bundle, to an orthonormal basis of T_xM that has all the properties just listed for $x = y$, with some eigenvalues ψ_0, ϕ_0 that are limits of the $\psi(x)$ and $\phi(x)$.

If $\phi = 0$ or $\tau(y) \neq c$, then $\phi_0 = 0$, which is obvious when $\phi = 0$ and, if $\tau(y) \neq c$, follows if we let $x \to y$ in $Q = 2(\tau - c)\phi$ (see Lemma 8.2).

If $\tau(y) = c$, we must have $\psi_0 = \phi_0$. In fact, let us choose a curve $s \mapsto x(s)$ as in Remark 8.4(ii), so that $\dot\tau \neq 0$ for all $s \neq 0$ close to 0. Also, $\psi(x(s)) \to \psi_0$ as $s \to 0$, and similarly for ϕ, since ψ_0, ϕ_0 are the limits of all convergent sequences of such values $\psi(x(s)), \phi(x(s))$ (to see this, consider two separate cases, $\psi_0 = \phi_0$ and $\psi_0 \neq \phi_0$). Hence, by l'Hospital's rule, $Q/(\tau - c)$ evaluated at $x(s)$ tends, as $s \to 0$, to the limit of $\dot Q/(\tau - c)^{\cdot} = 2\psi\dot\tau/\dot\tau$ (see Remark 8.4(i)), that is, to $2\psi_0$, while, by Lemma 8.2, $Q/(\tau - c) \to 2\phi_0$ as $s \to 0$.

Our ψ_0, ϕ_0 are the eigenvalues of $(\nabla d\tau)(y)$, with the multiplicities $2(m-1)$ and 2. Hence the constant value of $Y = \Delta\tau$ on N equals $2\psi_0 + 2(m-1)\phi_0$, that is, its value at $y \in N$. As $\phi_0 = 0$ in one case discussed above and $\phi_0 = \psi_0$ in the other, we obtain $\psi_0 = a$ for a given by (19.1). Also, $a \neq 0$, or else we would have $\phi_0 = \psi_0 = 0$ in both cases, contradicting the relation $(\nabla d\tau)(y) \neq 0$ in Remark 4.4. According to (1.3), the complex space T_yN thus is the orthogonal direct sum of two subspaces, of which one is the eigenspace of $(\nabla v)(y)$ for the unique nonzero eigenvalue $\psi_0 = a$, and the other is the kernel of $(\nabla v)(y)$ (trivial when $\phi_0 = \psi_0$, of complex codimension one when $\phi_0 = 0$), so that Remark 5.3(iii) implies (19.2), (19.3) and (i). Next, (ii) follows since its premise precludes the case $\phi_0 = \psi_0 \neq 0$, and, as we saw, the remaining case $\phi_0 = 0$ yields (19.3.a) for *every* critical manifold N, and, similarly, (iii) is obvious from (i) and the fact that, by Lemma 8.2, we have $\tau \neq c$ at all points with $d\tau \neq 0$. Finally, the limit relations $\psi(x) \to a$ and (iv) follow since the convergence involving the $\psi(x)$ and ψ_0, as well as that for orhonormal bases, was established for some subsequence of *any* given sequence of points $x \in M'$ tending to $y \in N$.

Lemma 19.1. *Suppose that N is a critical manifold of a function τ satisfying* (0.1) *on a Kähler manifold (M, g), cf.* Remark 5.3(ii)*, and $[0, \ell) \ni s \mapsto x(s) \in M$ is a unit-speed geodesic with $\dot x(0)$ normal to N at $x(0) = y \in N$, where $\dot x = dx/ds$, and such that $d\tau \neq 0$ at $x(s)$ for all $s \in (0, \ell)$. If we set $v = \nabla\tau$ and let* $\operatorname{sgn} a = \pm 1$ *stand for the sign of a in* (19.1) – (19.2)*, then $\dot x = (\operatorname{sgn} a)v/|v|$ at $x(s)$, for every $s \in (0, \ell)$, and, for $s \in [0, \ell)$,*

$$d\tau/ds \,=\, (\operatorname{sgn} a)\sqrt{Q}\,, \quad \text{with the initial value } \tau \,=\, \tau_0 \text{ at } s = 0, \tag{19.4}$$

where $d\tau/ds = d[\tau(x(s))]/ds$ and τ_0 is the constant value of τ on N.

In fact, $\nabla_v v = \psi v$ by (8.4) with $w = v$, and $\nabla_w v = aw$ for $w = \dot x(0)$ by (19.2), so that Lemma 6.4(b) applied to the Levi-Civita connection ∇ of g, the geodesic segment $X \subset M$ which is the image of $s \mapsto x(s)$, and our y, v, a yields

$\dot{x} = \pm v/|v|$ for some sign $\pm$ and all $s \in (0, \ell)$. Remark 8.4(i) now gives $\dot{\tau} = g(v, \dot{x})$ and $\dot{Q} = 2\psi\dot{\tau}$. At any $s > 0$ close to 0 we thus have $\pm\dot{\tau} > 0$ and $a\psi > 0$ (due to the relation $\psi(x) \to a$ preceding (19.2)), so that $\pm a\dot{Q} > 0$, which implies $\pm a > 0$, since $\dot{Q} > 0$. (Note that $Q(x(s)) > 0$ for such s, as $Q = g(\nabla\tau, \nabla\tau)$, while $Q(x(0)) = 0$.) Finally, (19.4) now follows since $d\tau/ds = d_{\dot{x}}\tau$ and, by (8.2), $|\nabla\tau| = \sqrt{Q}$. ■

§20. A consequence of Gauss's Lemma

The *normal exponential mapping* of a submanifold N of a Riemannian manifold (M, g) is the restriction of $\mathrm{Exp} : U^{\mathrm{Exp}} \to M$ to the set $U^{\mathrm{Exp}} \cap \mathcal{L}$, with $\mathcal{L}$ denoting the total space of the normal bundle of N (see (2.2)) and $U^{\mathrm{Exp}} \subset TM$ defined as in Remark 6.3 for the Levi-Civita connection ∇ of (M, g).

For $M, g, N, \mathcal{L}$ as above and $y \in N$, let s be the norm function (Remark 2.2) of the real fibre metric in $\mathcal{L}$ obtained by restricting g to $\mathcal{L}$. The inverse mapping theorem allows us to choose a connected neighborhood N' of y in N and a number $\ell \in (0, \infty)$ such that, for the open subset U' of $\mathcal{L}'$ given by $0 \le s < \ell$, where $\mathcal{L}'$ is the portion of $\mathcal{L}$ lying over N', we have $U' \subset U^{\mathrm{Exp}}$ and the normal exponential mapping sends U' diffeomorphically onto an open set in M.

The following fact is well-known (and also immediate from (d) in §26):

Gauss's Lemma. *Under these assumptions, all half-open geodesic segments of length ℓ, emanating from N' in directions normal to N, intersect orthogonally the* Exp-*images of all level sets of the norm function restricted to U'.* ■

One of its consequences is

Lemma 20.1. *For a function τ satisfying* (0.1) *on a Kähler manifold (M, g), let $Q = g(\nabla\tau, \nabla\tau)$ and let $\psi : M' \to \mathbf{R}$ be characterized by* (8.3)*, with M' standing for the open set on which $d\tau \neq 0$. Then*

(a) *ψ has a unique extension to a C^∞ function $M \to \mathbf{R}$, also denoted ψ.*

(b) *Every point of M has a neighborhood U on which Q is a C^∞ function of τ, that is, a composite consisting of τ followed by a C^∞ function $\tau \mapsto Q$ defined on a suitable interval of the variable τ and such that $dQ/d\tau = 2\psi$ for $dQ/d\tau$ and ψ treated as functions on U.*

Proof. At points with $d\tau \neq 0$, (b) is obvious from Remark 8.5(a). Suppose now that $y \in M$ is a point at which $d\tau = 0$, and let N be the critical manifold of τ containing y (cf. Remark 5.3(ii)). We may choose N', ℓ, U' as in the second paragraph of this section and, making N' and ℓ smaller if necessary, also require that $d\tau \neq 0$ at every point of $\mathrm{Exp}(U' \smallsetminus N')$. (Cf. Lemma 5.2(a) for $u = J(\nabla\tau)$.)

The gradients $v = \nabla\tau$ and $\nabla Q = 2\psi v$ (see (8.5.i)), which, by Lemma 19.1, are tangent to the geodesic segments mentioned in Gauss's Lemma, must therefore be normal to the Exp-images of all level sets of the norm function restricted to U'. Any such level set is a bundle of positive-dimensional spheres over N' (cf. the inequality in Lemma 5.2(c) for $u = J(\nabla\tau)$), unless it is the zero section N', i.e., the zero level; therefore, it is connected, and so τ, Q must both be constant along its Exp-image. Thus, both τ and Q, restricted to $\mathrm{Exp}(U')$ and then pulled back to U' via Exp, are functions of the norm function.

Let $(-\ell,\ell)\ni s\mapsto x(s)\in M$ be any unit-speed geodesic such that $x(0)\in N'$ and $\dot x(0)$ is normal to N at $x(0)$, where $\dot x = dx/ds$. As $x(s)=\mathrm{Exp}(x(0),s\dot x(0))$ and the value of the norm function at $(x(0),s\dot x(0))$ is $|s|$, it follows that τ, Q treated as C^∞ functions of the variable $s\in(-\ell,\ell)$ (via the substitutions $\tau(x(s)), Q(x(s)))$ depend just on $|s|$, i.e., are even. Their restrictions to $[0,\ell)$ describe how their Exp-pullbacks depend on the norm function (also denoted s). By (19.4), the dependence of τ on s is homeomorphic, i.e., Q restricted to $\mathrm{Exp}(U')$ is also a function of τ. Finally, $d^2\tau/ds^2\neq 0$ at $s=0$ in view of Remark 8.4(ii), since, by (19.2) and (1.3), $\dot x(0)$ is an eigenvector of $(\nabla d\tau)(y)$ for the eigenvalue $a\neq 0$. Assertion (b) for the point y is therefore immediate from Remark 1.2. Finally, relation $dQ/d\tau = 2\psi$, valid locally in M' (see Remark 8.5(a)) can now be used to define a C^∞ extension of ψ to a suitable neighborhood of any given point in M and, as every such extension is unique (due to denseness of M', cf. Remark 4.4), all such extensions together form a function $\psi: M\to\mathbf{R}$. This completes the proof. ∎

For M, g, τ as in Lemma 20.1 and $\psi,\phi: M'\to\mathbf{R}$ given by (8.3), the unique C^∞ extension of ψ to M provided by Lemma 20.1(a) leads to a similar extension of ϕ. In fact, $\phi: M\to\mathbf{R}$ then is defined either by (8.5.ii) with $m\geq 2$, or by $\phi=0$ when M is of complex dimension 1. Both extensions are constant on every critical manifold N of τ.

Furthermore, $\psi=a$ on N, for the constant $a\neq 0$ depending on N as in (19.1) – (19.2) (due to the relation $\psi(x)\to a$ preceding (19.2)). Therefore, $\phi=a$ on N when N consists of a single point, and $\phi=0$ on N otherwise; this is clear from (8.5.ii) restricted to N (so that $\psi=a$) along with (19.1) and (i) in §19.

§21. Isometric actions of the circle

For a C^2 function τ on a Riemannian manifold (M,g), let $\mathrm{Crit}^1(\tau)$ be the set of those critical points y of τ at which the Hessian $\mathrm{Hess}_y\tau$ has exactly one nonzero eigenvalue (of any multiplicity). Thus,

$$\text{(21.1)}\qquad \mathrm{Hess}_y\tau \quad\text{is semidefinite for every}\quad y\in\mathrm{Crit}^1(\tau).$$

If $y\in\mathrm{Crit}^1(\tau)$ and a is the nonzero eigenvalue of $\mathrm{Hess}_y\tau$, while N is the critical manifold of τ containing y, and $u=Jv$ with $v=\nabla\tau$, then, for any $z\in T_yM$,

$$\text{(21.2)}\qquad \nabla_z u = 0 \quad\text{if}\quad z\in T_yN, \quad\text{and}\quad \nabla_z u = aJz \quad\text{if}\quad z\in(T_yN)^\perp.$$

In fact, as ∇v commutes with J (see Lemma 4.2(ii) and (4.1.a)), so does $\nabla u = J\circ(\nabla v)$, by (4.1.b). Thus, $(\nabla u)(y)$ is complex-linear has the same eigenvectors as $(\nabla v)(y)$ (or $(\nabla d\tau)(y)$, cf. (1.3)), its eigenvalues being i times those of $(\nabla d\tau)(y)$ Now (21.2) is obvious from Remark 5.3(iii) and (19.2). Next, for the set $\mathrm{Crit}(\tau)$ of all critical points of τ, Remark 5.3(iii), (19.2) and (1.3) give

$$\text{(21.3)}\qquad \mathrm{Crit}(\tau) = \mathrm{Crit}^1(\tau) \quad\text{if } \tau \text{ satisfies (0.1) on a Kähler manifold.}$$

Lemma 21.1. *Let τ be a Killing potential on a Kähler manifold (M,g), cf. §1, and let a point $y\in M$ lie in the set* $\mathrm{Crit}^1(\tau)$ *defined above, so that $u=J(\nabla\tau)$ is a Killing field and $u(y)=0$. If U,U' are chosen as in* Lemma 5.1*, for these u and y, then the flow of u restricted to U is periodic, i.e., represents an isometric action on U of the circle group S^1. The minimum period of the flow of u on U equals $2\pi/|a|$, where a is the nonzero eigenvalue of $\nabla d\tau$ at y.*

In fact, according to Lemma 6.5, u restricted to U is the $\exp_y$-image of the linear vector field on U' given by the skew-adjoint (and hence diagonalizable) operator $z\mapsto\nabla_z u$ with the eigenvalues ai and 0, or just ai (see (21.2)). ■

Corollary 21.2. *Let $\tau:M\to\mathbf{R}$ be a Killing potential on a complete Kähler manifold (M,g) such that the set* $\mathrm{Crit}^1(\tau)$ *defined above is nonempty. Then*

(i) *The flow of the Killing vector field $u=J(\nabla\tau)$ is periodic, i.e., constitutes an isometric S^1 action on (M,g).*
(ii) *The absolute value of the nonzero eigenvalue of $\nabla d\tau$ is the same at all points of* $\mathrm{Crit}^1(\tau)$.

In fact, (i), (ii) are both obvious from Lemma 21.1 and the unique continuation property for isometries (Remark 4.1): in (ii), $2\pi/|\psi(y)|$ is, by Lemma 21.1, the minimum period of the flow of u, and so it is the same for all $y\in\mathrm{Crit}^1(\tau)$. ■

Corollary 21.3. *Let τ satisfy* (0.1) *on a complete Kähler manifold (M,g), and let $\psi:M\to\mathbf{R}$ be the continuous extension to M, described in* Lemma 20.1*, of the eigenvalue function ψ in* (8.3)*. Then the restriction of the function $|\psi|$ to the set* $\mathrm{Crit}(\tau)$ *of critical points of τ is constant and positive.*

This is clear from Corollary 21.2(ii) and (21.3). (Constancy of ψ on each connected component N of $\mathrm{Crit}(\tau)$ has also been shown at the end of §20.) ■

§22. Boundary conditions

Let τ be a special Kähler-Ricci potential on a compact Kähler manifold (M,g) of complex dimension $m\ge 1$, cf. (0.1). Then

(i) τ has exactly two critical manifolds, defined as in Remark 5.3(ii), and they are the τ-preimages of its extremum values $\tau_{\max}$ and $\tau_{\min}$.
(ii) One of the following two cases must occur:
    1) Both critical manifolds of τ are of complex codimension one;
    2) One critical manifold of τ is of complex codimension 1, and the other consists of a single point.
(iii) $Q=g(\nabla\tau,\nabla\tau)$ is a C^∞ function of τ, that is, a composite consisting of τ followed by a C^∞ function $[\tau_{\min},\tau_{\max}]\ni\tau\mapsto Q\in\mathbf{R}$.
(iv) The function $[\tau_{\min},\tau_{\max}]\ni\tau\mapsto Q\in\mathbf{R}$ in (iii) satisfies (17.1).

In fact, by Example 7.1, (21.3), (21.1) and the inequality in Lemma 5.2(c) with $u=J(\nabla\tau)$, our M and τ satisfy the assumptions, and hence the conclusions, of Corollary 7.5. This gives (i). Now (ii) easily follows from (19.3). In fact, unless $m=1$, the critical manifolds of τ cannot both consist of single points, for if they did, we would have $\tau_{\max}=\tau_{\min}=c$ (by (ii), (iii) in §19), contradicting nonconstancy of τ in (0.1). (Also, if both critical manifolds were single points, with $m \geq 2$, Reeb's theorem [13] would imply that M is a topological n-sphere, $n \geq 4$, admitting no Kähler metric.)

By (i), the open set $M' \subset M$ on which $d\tau \neq 0$ is the disjoint union of the τ-preimages of all values in $(\tau_{\min}, \tau_{\max})$. Each of those τ-preimages is connected, due to the assertion of Corollary 7.5 (which, as we saw, hold in our case); therefore, $Q = g(\nabla\tau, \nabla\tau)$ is constant on it (Remark 8.5(b)). This gives (iii), the C^∞-differentiability property of the assignment $[\tau_{\min}, \tau_{\max}] \to \mathbf{R}$ being now obvious from the analogous local conclusion in Lemma 20.1.

Lemma 20.1 also gives $2\psi = dQ/d\tau$ on the interval $[\tau_{\min}, \tau_{\max}]$. The assertion about $|\psi|$ in Corollary 21.3 thus shows that $|dQ/d\tau|$ has the same positive value at both endpoints $\tau_{\min}, \tau_{\max}$. Finally, since $Q = g(\nabla\tau, \nabla\tau)$, the function $\tau \mapsto Q$ is positive on the open interval $(\tau_{\min}, \tau_{\max})$ (formed by non-critical values of τ, cf. (i)), and vanishes at its endpoints $\tau_{\min}, \tau_{\max}$. Hence $dQ/d\tau > 0$ at $\tau_{\min}$ and $dQ/d\tau < 0$ at $\tau_{\max}$, which yields (iv).

§23. The distance between the critical manifolds

For $[\tau_{\min}, \tau_{\max}]$ with a function $\tau \mapsto Q$ satisfying (17.1), let us set

$$L = \int_{\tau_{\min}}^{\tau_{\max}} \frac{d\tau}{\sqrt{Q}} \in (0, \infty). \tag{23.1}$$

To see that $L < \infty$, use Q as the variable of integration near either endpoint.

Lemma 23.1. *Let N, N^* be the two critical manifolds of a function τ satisfying* (0.1) *on a compact Kähler manifold (M, g), and let L be the invariant given by* (23.1)*, with $Q = g(\nabla\tau, \nabla\tau)$ treated as a function of τ, cf.* §22.

(a) *L is the minimum distance between N and any given point $y' \in N^*$.*
(b) *Every point $x \in M$ at which $d\tau \neq 0$ can be joined to N by a geodesic, normal to N, of some length $\ell \in (0, L)$.*
(c) *If $X \subset M$ is a geodesic of length L with endpoints y, y' such that $y \in N$ and X is normal to N at y, then $y' \in N^*$ and $Q > 0$ on $X \smallsetminus \{y, y'\}$.*

Proof. For X, y, y' as in (c), let X' be the maximal half-open geodesic segment containing y as an endpoint along with all points of X sufficiently close to y and such that $d\tau \neq 0$ everywhere in $X' \smallsetminus \{y\}$, and let $[0, \ell) \ni s \mapsto x(s)$ be an arc-length parameterization of X'. By (19.4), $ds = \pm Q^{-1/2} d\tau$ on X', so that $\ell = \int_0^\ell ds = |\int_{\tau_0}^{\tau'} Q^{-1/2} d\tau|$, where $\tau_0 = \tau(y) \in \{\tau_{\min}, \tau_{\max}\}$ is the value of τ on N, and $\tau' = \tau(x(\ell))$ with $x(\ell) = \lim_{s \to \ell} x(s)$. (Note that $\ell \leq L < \infty$ by (23.1), and $\ell < L$ unless $\tau' \in \{\tau_{\min}, \tau_{\max}\}$.) However, maximality of X' now gives $(d\tau)(x(\ell)) = 0$, and so, as $\tau(x(\ell)) = \tau'$, (i) in §22 shows that $\{\tau_0, \tau'\} = \{\tau_{\min}, \tau_{\max}\}$, i.e., $\ell = L$. Consequently, $X' = X$ and (c) follows.

Given $y' \in N^*$, let y be the point of N nearest to y', and let X' be a minimizing geodesic segment of some length L', joining y' to y. As (c) implies that every point in a given critical manifold lies at the distance L from some point in the other critical manifold, we have $L' \le L$. On the other hand, $L' \ge L$. In fact, if we had $L' < L$, by extending X' beyond y' so as to obtain a geodesic segment X of length L we would conclude, from the final clause of (c), that y' is not a critical point of τ. (Note that X' is normal to N at y due to our distance-minimizing choice of y and X'.) Hence $L' = L$, which gives (a).

To prove (b), let us connect any $x \in M' = M \smallsetminus (N \cup N^*)$ with the point y nearest to it in $N \cup N^*$ by a minimizing geodesic segment X' of some length $\ell > 0$. Thus, $\ell < L$, or else some point of X' would lie at the distance L from y, and so, by (c), it would be a point of $N \cup N^*$, closer to x than y is. Extending X' beyond x, we obtain a geodesic segment X of length L and, by (c), one of the endpoints of X lies in N. Moreover, X must be normal to N at that endpoint, since, by (a), X is a minimum-length curve joining N to N^*. This completes the proof. ■

§24. The normal horizontal distribution

The normal bundle $\mathcal{L}$ of any submanifold N of a Riemannian manifold (M, g) carries the usual *normal connection* ∇^{nrm}, characterized by $\nabla^{\mathrm{nrm}}_{\dot y} w = [\nabla_{\dot y} w]^{\mathrm{nrm}}$ whenever $t \mapsto w(t) \in T_{y(t)}M$ is a C^1 vector field normal to N along a C^1 curve $t \mapsto y(t) \in N$. Here $\dot y = dy/dt$, while ∇ is the Levi-Civita connection of (M, g), and $...^{\mathrm{nrm}}$ stands for the component normal to N.

Let $\mathcal{L}$ now denote both the normal bundle of N, and the total space thereof (see (2.2)), where N is a critical manifold of a function τ with (0.1) on a Kähler manifold (M, g), cf. Remark 5.3(ii). By (19.3), two cases are possible:

(a) N is a complex submanifold of complex codimension 1 in M, so that $\mathcal{L}$ is a complex line bundle over N, or
(b) $N = \{y\}$ for some point $y \in M$, and so $\mathcal{L} = \{y\} \times T_yM$.

The *normal horizontal distribution* of N is a distribution $\mathcal{H}^N$ on $\mathcal{L} \smallsetminus N$, defined as follows. In case (a), $\mathcal{H}^N$ is the restriction to $\mathcal{L} \smallsetminus N$ of the horizontal distribution of the normal connection in $\mathcal{L}$ (see above), while, in case (b), $\mathcal{H}^N$ coincides with the distribution $\mathcal{H}$ of Remark 3.1 for $V = T_yM$ with the Hermitian inner product $\langle\,,\rangle$ whose real part is $g(y)$, provided that one identifies $\mathcal{L} = \{y\} \times T_yM$ with T_yM (as we will always do). Note that $\mathcal{H}^N$ is not only a real vector subbundle of the tangent bundle $T(\mathcal{L} \smallsetminus N)$, but also a *complex* vector bundle, with the complex structure in each fibre $\mathcal{H}_* = \mathcal{H}^N_{(y,z)}$ inherited, in case (b), from the ambient space T_yN (in which $\mathcal{H}_*$ is contained as a complex subspace), or provided, in case (a), by requiring the differential at (y, z) of the bundle projection $\mathcal{L} \to N$ to be a complex linear operator $\mathcal{H}_* \to T_yN$.

We also define vector fields v^N, u^N on $\mathcal{L}$ to be v, u in (2.3) (in case (a)), or v, u in Remark 3.1 (in case (b), with $V = T_yM$ and $\langle\,,\rangle$ as above), where a is the constant determined by N via (19.1).

Remark 24.1. As noted in §3, $\mathcal{H}^N$ is, also in case (b), the horizontal distribution of a connection. Therefore, every vector in $\mathcal{H}^N$ at any given point of $\mathcal{L} \smallsetminus N$ is tangent to a curve in $\mathcal{L} \smallsetminus N$ which is *horizontal*, i.e., tangent to $\mathcal{H}^N$ at every point.

Remark 24.2. Given a totally geodesic submanifold N of a Riemannian manifold (M,g), a point $y \in N$, and vectors $w, w' \in T_yN$, let ∇, R be the Levi-Civita connection and curvature tensor of (M,g), and let a Riemannian metric h on N be a constant multiple of its submanifold metric. Then, for any vector ξ tangent (or, normal) to N at y, the value $R(w,w')\xi$ coincides with the one obtained by replacing R with the curvature tensor of (N,h) (or, respectively, of the normal connection in the normal bundle $\mathcal{L}$ of N).

In fact, extending w, w', ξ to C^∞ vector fields on a neighborhood U of y in M tangent/normal to N along $N \cap U$, we see that $\nabla_w \xi$, restricted to $N \cap U$, is the covariant derivative relative to the Levi-Civita connection of (N,h) (or, respectively, the normal connection in $\mathcal{L}$), and our claim is obvious from (2.1).

§25. Critical manifolds and curvature

Lemma 25.1. *Let τ be a special Kähler-Ricci potential on a Kähler manifold (M,g), cf.* (0.1)*, and let $v, u, \mathcal{V}, \mathcal{H}, Q, \phi, \psi$ be given by* (8.1) – (8.3)*, so that ϕ, ψ are C^∞ functions on the open set M' on which $d\tau \neq 0$. For any two C^∞ vector fields w, w' defined on an open subset of M' and orthogonal to v and u at every point, we then have, with R denoting the curvature tensor,*

(i) $QR(w,w')v = 2(\phi - \psi)\phi g(Jw,w')u$.

(ii) *Either of $g(R(w,v)w',v)$ and $g(R(w,u)w',u)$ equals $g(w,w')$ times a function which does not depend on the choice of w and w'.*

Proof. Let $...^{\mathrm{vrt}}$, $...^{\mathrm{hrz}}$ be the $\mathcal{V}$ and $\mathcal{H}$ components relative to the decomposition $TM' = \mathcal{H} \oplus \mathcal{V}$. Then $Q[\nabla_w w']^{\mathrm{vrt}} = -\phi[g(w,w')v + g(Jw,w')u]$ and

$$Q[w,w']^{\mathrm{vrt}} = -2\phi\, g(Jw,w')u \quad \text{for any local } C^\infty \text{ sections } w, w' \text{ of } \mathcal{H}. \tag{25.1}$$

(See [8], formula (13.1).) Since, by (8.5.i), ϕ is constant in the direction of w, (8.4) gives $\nabla_w \nabla_{w'} v = \phi \nabla_w w'$. Also, from (8.4), $\nabla_{[w,w']} v = \phi[w,w']^{\mathrm{hrz}} + \psi[w,w']^{\mathrm{vrt}} = (\psi - \phi)[w,w']^{\mathrm{vrt}} + \phi[w,w']$. As ∇ is torsion-free, (i) now easily follows from (2.1) and (25.1). Next, writing $\langle\,,\rangle$ for $g(\ ,\ )$ we have $\langle \nabla_v \nabla_w w', v\rangle = d_v \langle \nabla_w w', v\rangle - \langle \nabla_w w', \nabla_v v\rangle = (\phi\psi - d_v\phi)\langle w,w'\rangle - \phi d_v \langle w,w'\rangle$, since $\nabla_v v = \psi v$ (by (8.4) for $w = v$) and $\langle \nabla_w w', v\rangle = \langle [\nabla_w w']^{\mathrm{vrt}}, v\rangle = -\phi\langle w,w'\rangle$ (cf. the above formula for $Q[\nabla_w w']^{\mathrm{vrt}}$ and (8.2)), while $\langle \nabla_v w', v\rangle = -\langle w', \nabla_v v\rangle = 0$ (as $\nabla_v v = \psi v$), and so $\langle \nabla_w \nabla_v w', v\rangle = -\langle \nabla_v w', \nabla_w v\rangle = -\phi\langle w, \nabla_v w'\rangle$ (since (8.4) gives $\nabla_w v = \phi w$). Next, the local flow of v leaves $\mathcal{H}$ invariant (see [8], Remark 17.3, discussion of condition (a)), so that $[w,v]$ is a section of $\mathcal{H}$ and (8.4) gives $\nabla_w v = \phi w$, $\nabla_{[w,v]} v = \phi[w,v]$. Hence, as ∇ is torsion-free, $\langle \nabla_{[w,v]} w', v\rangle = -\langle w', \nabla_{[w,v]} v\rangle = \phi\langle w', [v,w]\rangle = \phi\langle w', \nabla_v w\rangle - \phi^2\langle w,w'\rangle$. Now (2.1) with $d_v\langle w,w'\rangle = \langle \nabla_v w, w'\rangle + \langle w, \nabla_v w'\rangle$ yields assertion (ii) for $\langle R(w,v)w',v\rangle$. However, $\langle R(w,u)w',u\rangle = \langle R(Jw,v)Jw',v\rangle$ (and so (ii) for $\langle R(w,u)w',u\rangle$ follows from $\langle Jw, Jw'\rangle = \langle w,w'\rangle$). Namely, in each tangent space, the operator $w' \mapsto R(w,v)w'$ is complex-linear, i.e., commutes with J, since relation $\nabla J = 0$ means that ∇ is a connection in TM treated as a *complex* vector bundle. Due to skew-adjointness of J, this gives $\langle R(w,u)w',u\rangle = \langle R(w,u)w', Jv\rangle = -\langle R(w,u)Jw', v\rangle$, which in turn equals $-\langle R(Jw',v)w,u\rangle = -\langle R(Jw',v)w, Jv\rangle = \langle R(Jw',v)Jw, v\rangle = \langle R(Jw,v)Jw', v\rangle$. This completes the proof. ■

Suppose that τ satisfies (0.1) on a Kähler manifold (M, g) and a given critical manifold N of τ is of complex codimension 1 in M, cf. (19.3), while ε, a are the constants described in Remark 8.3 and (19.1), and $f : M \to \mathbf{R}$ is given by (10.4) with c as in Lemma 8.2. (The cases $\phi = 0$ and $\phi \neq 0$ in Lemma 8.2 correspond to $\varepsilon = 0$ and $\varepsilon = \pm 1$.)

We then define a Kähler metric h on the complex manifold N to be $1/f_0$ times the restriction of g to TN, where f_0 is the constant value of f on N. Note that $f_0 > 0$ as $\tau \neq c$ on N when $\varepsilon = \pm 1$, cf. (i) in §19.

Lemma 25.2. *Let N be a critical manifold of a special Kähler-Ricci potential τ on a Kähler manifold (M, g) such that $\dim_{\mathbf{C}} M = m \geq 2$ and $\dim_{\mathbf{C}} N = m - 1$, and let $\mathcal{L}$ be the normal bundle of N. Then, for ε, a, h described above, and with the normal connection and curvature form as in* §24 *and* Remark 2.1*,*

(a) *The Kähler manifold (N, h) is Einstein unless $m = 2$.*
(b) *The curvature form of the normal connection in $\mathcal{L}$ equals $-2\varepsilon a$ times the Kähler form of (N, h), defined as in* (1.5)*.*

Proof. Let $v, u, \mathcal{H}$ be as in (8.1). For any $x \in M'$, i.e., a point $x \in M$ at which $v(x) \neq 0$, we define symmetric bilinear forms $h(x)$ and $\mathrm{r}^{(h)}(x)$ on the space $\mathcal{H}_x$ by declaring $h(x)$ to be the restriction of $g(x)/f(x)$ to $\mathcal{H}_x$, with f as in (10.4) (so that $f > 0$ on M', as $\tau \neq c$ on M' by Lemma 8.2), and letting $\mathrm{r}^{(h)}(x)$ assign to vectors $w, w' \in \mathcal{H}_x$ the value $\mathrm{r}^{(h)}(w, w')$ equal to $\sum_j g(R(w, e_j)w', e_j)$, where R is the curvature tensor of (M, g) and the e_j run through any $g(x)$-orthonormal basis of $\mathcal{H}_x$. Since such e_j along with $Q^{-1/2}v$ and $Q^{-1/2}u$ (at x) then form a $g(x)$-orthonormal basis of T_xM, cf. (8.2), our $\mathrm{r}^{(h)}(x)$ and the Ricci tensor $\mathrm{r}(x)$ of g at x are related by $\mathrm{r}^{(h)}(w, w') = \mathrm{r}(w, w') - [g(R(w, v)w', v) + g(R(w, u)w', u)]/Q$, for $w, w' \in \mathcal{H}_x$, with v, u, Q standing for their values at x. As $\mathrm{r} = \lambda g$ on $\mathcal{H}$ (see (8.3)), Lemma 25.1(ii) shows that $\mathrm{r}^{(h)}(x)$ equals a scalar times $h(x)$.

As M' is dense in M (Remark 4.4), choosing a sequence of points $x \in M'$ converging to any given $y \in N$ and using the limit relation $\mathcal{H}_x \to T_yN$ (see (iv) in §19) along with Remark 24.2, we see that the the Ricci tensor of h at y is a scalar multiple of $h(y)$, which proves (a).

Lemma 25.1(i) and (8.1) give $R(w, w')\xi = i\Omega(w, w')\xi$ for any point $x \in M'$ and vectors $w, w' \in \mathcal{H}_x$, $\xi \in \mathcal{H}_x^\perp$, where $\Omega(w, w') \in \mathbf{R}$ equals $2(\phi - \psi)\phi/Q$ at x times $g(Jw, w')$. Let a variable point $x \in M'$ now tend to any fixed $y \in N$. Relation $\mathcal{H}_x \to T_yN$ (cf. (iv) in §19) then gives the same for y and T_yN instead of x and $\mathcal{H}_x$, while $\phi(x) \to 0$, $\psi(x) \to a$ (see end of §20) and, unless $\phi = 0$ identically, Lemma 8.2 yields $Q(x)/\phi(x) \to 2[\tau(y) - c] = \varepsilon f(y)$. Now (b) is immediate from Remark 24.2 along with the definitions of the curvature form and Kähler form (see Remark 2.1 and (1.5)), which completes the proof. ■

§26. Variations and partial covariant derivatives

Let $(s, t) \mapsto x(s, t) \in M$ be a fixed C^∞ *variation of curves* in a manifold M, that is, a C^∞ mapping with real variables s, t ranging independently over some intervals. By (s,t)-*dependent* functions φ or vector fields w we then mean assignments sending each (s,t) to $\varphi(s,t)\in\mathbf{R}$ or $w(s,t)\in T_{x(s,t)}M$. Differentiability of such objects is also well-defined, since they clearly are just sections of appropriate pullback bundles. In particular, the velocities of the curves $s\mapsto x(s,t)$ and $t\mapsto x(s,t)$, with t or s fixed, form (s,t)-dependent vector fields, here denoted x_s and x_t, and having, in any local coordinates, the components $x_s^j=\partial x^j/\partial s$ and $x_t^j=\partial x^j/\partial t$, where $x^j(s,t)$ are the components of $x(s,t)$. Ordinary vector fields u on M or functions $f:M\to\mathbf{R}$ are treated as (s,t)-dependent ones that assign $u(x(s,t))$ or $f(x(s,t))$ to any (s,t).

We use the subscript notation φ_s,φ_t for the partial derivatives of (s,t)-dependent C^1 functions φ, including ordinary C^1 functions on M. If, in addition, there is a fixed connection ∇ in the tangent bundle TM, we may differentiate (s,t)-dependent C^1 vector fields w covariantly with respect to either parameter s or t (i.e., along the curves mentioned above), obtaining (s,t)-dependent fields w_s,w_t equal to $\nabla_{\dot x}w$ for $\dot x=x_s$ (or, $\dot x=x_t$), with the local-coordinate expressions $w_s^j=\partial w^j/\partial s+\mathit{\Gamma}_{kl}^j x_s^k w^l$ and $w_t^j=\partial w^j/\partial t+\mathit{\Gamma}_{kl}^j x_t^k w^l$. Here $\mathit{\Gamma}_{kl}^j$ are the component functions of ∇, evaluated at $x(s,t)$.

Applied to x_s and x_t, this leads to the (s,t)-dependent fields $x_{ss}=(x_s)_s$, $x_{st}=(x_s)_t$, etc. Thus, $x_{ss}=0$ identically if and only if all the curves $s\mapsto x(s,t)$ are uniform-parameter geodesics. If ∇ is torsion-free, then $\mathit{\Gamma}_{kl}^j=\mathit{\Gamma}_{lk}^j$, and so

$(*)$ $x_{st}=x_{ts}$.

Let us now assume that ∇ is the Levi-Civita connection of a fixed Riemannian metric g on M, while N is a submanifold of M and $t\mapsto\zeta(t)$ is a C^∞ unit vector field normal to N along some given C^∞ curve $t\mapsto y(t)\in N$, where t ranges over some interval. Let us set $x(s,t)=\mathrm{Exp}(y(t),s\zeta(t))$ for all s in some interval of the form $[0,\ell]$ with $\ell>0$, where $\mathrm{Exp}:U^{\mathrm{Exp}}\to M$ is defined as in Remark 6.3. (Such ℓ exists, i.e., $(y(t),s\zeta(t))\in U^{\mathrm{Exp}}$ for all s,t, provided that one replaces the original interval of t with a suitable subinterval.) Then

(a) $|x_s|=1$ and $x_{ss}=0$ for all s,t,
(b) $x_s(0,t)=\zeta(t)$ is unit and normal to N, and $x_t(0,t)=\dot y(t)$ is tangent to N, with $\dot y=dy/dt$,
(c) $x_{st}=\nabla_{\dot y}\zeta$ at $s=0$ and any t,
(d) $\langle x_s,x_t\rangle=0$ for all s,t, which is known as *Gauss's Lemma* (cf. §20),

with $\langle\,,\rangle$ standing for $g(\;,\;)$. In fact, the formula for $x(s,t)$ implies (a) – (c). The Leibniz rule for (s,t)-dependent functions such as $\langle x_s,x_t\rangle$ yields $\langle x_s,x_t\rangle_s=\langle x_{ss},x_t\rangle+\langle x_s,x_{ts}\rangle$, and, from $(*)$, $2\langle x_s,x_{ts}\rangle=2\langle x_s,x_{st}\rangle=\langle x_s,x_s\rangle_t$. Hence (a) gives $\langle x_s,x_t\rangle_s=0$, i.e., $\langle x_s,x_t\rangle$ does not depend on s, and (d) follows since, by (b), $\langle x_s,x_t\rangle=0$ when $s=0$.

Still making all the assumptions listed in the paragraph following $(*)$, let us also suppose that (M,g) is a Kähler manifold with a special Kähler-Ricci potential τ (cf. (0.1)), while N is a critical manifold of τ (Remark 5.3(ii)), v,u,Q,ϕ,ψ are given by (8.1) – (8.3) (so that ϕ,ψ are C^∞ functions on the open set M' defined by $d\tau\neq0$), and $x(s,t)\in M'$ for all $s>0$ and all t. Then, for all s,t with $s>0$,

(e) $|v|=|u|=Q^{1/2}$,

(f) $v = \pm Q^{1/2} x_s$,
(g) $\langle u, x_t\rangle_s = \pm 2\langle u, x_t\rangle \psi Q^{-1/2} = 2\langle u, x_{st}\rangle$,
(h) $Q_s = \pm 2\psi Q^{1/2}$ and $\phi_s = \pm 2(\psi - \phi)\phi Q^{-1/2}$,

with a specific fixed sign $\pm$, namely, the sign of a in (19.1) - (19.2). In fact, (8.2) implies (e), while Lemma 19.1 and (a) give $v = \pm|v|x_s$ with the required sign $\pm$, so that (f) follows from (e). Also, $f_s = \langle x_s, \nabla f\rangle$ for $f = Q$ and $f = \phi$, and so (8.5.i) combined with (f), (a) yields (h). Next, $\langle u, x_{ts}\rangle = -\langle u_t, x_s\rangle = \langle u_s, x_t\rangle$. Namely, the first relation follows from the Leibniz rule and $(*)$, as $\langle u, x_s\rangle = 0$ (by (f), since $\langle u, v\rangle = 0$, cf. (8.2)), while the second is clear from skew-symmetry of ∇u (§4), as $u_s = (\nabla u)x_s$ (see above). The Leibniz rule now yields $\langle u, x_t\rangle_s = \langle u_s, x_t\rangle + \langle u, x_{ts}\rangle = 2\langle u_s, x_t\rangle$. This in turn implies both $\langle u, x_t\rangle_s = 2\langle u, x_{ts}\rangle$ and $\langle u, x_t\rangle_s = \pm 2\langle u, x_t\rangle\psi Q^{-1/2}$ (since (8.4) gives $\nabla_v u = \nabla_v(Jv) = J\nabla_v v = \psi u$, so that, by (e) – (f), $u_s = \pm\psi Q^{-1/2} u$), which proves (g).

§27. The normal exponential mapping

Suppose that N is a critical manifold of a special Kähler-Ricci potential on a Kähler manifold (M, g) of complex dimension $m \geq 2$ (see (0.1) and Remark 5.3(ii)), while $\mathcal{H}^N, v^N, u^N$ (or, $\mathcal{H}, v, u$) are the complex vector bundles and vector fields defined in §24 (or, (8.1)). Letting $\mathcal{L}$ stand, again, for the total space of the normal bundle of N, we also assume that the image of the set $U' = (U^{\mathrm{Exp}} \cap \mathcal{L}) \smallsetminus N$ under the normal exponential mapping of N (see §20) is contained in the open set $M' \subset M$ on which $d\tau \neq 0$. Recall that, by Lemma 8.2, ϕ in (8.3) is either identically zero, or nonzero everywhere in M'. If $\phi \neq 0$ on M', there is a constant c such that $Q/\phi = 2(\tau - c)$ on M' and, by (i) in §19, in case (a) of §24 we have $\tau \neq c$ everywhere in N, while, in case (b) of §24, $N = \{y\}$ and $\tau(y) = c$.

In addition, let $\Theta : T_{(y,z)}\mathcal{L} \to T_x M$ denote the differential of Exp at some fixed $(y, z) \in U'$, with $x = \mathrm{Exp}(y, z)$, and let the symbol $\mathcal{H}_*$ stand for $\mathcal{H}^N_{(y,z)}$.

Lemma 27.1. *Under these assumptions, $\Theta(\mathcal{H}_*) \subset \mathcal{H}_x$ and $\Theta : \mathcal{H}_* \to \mathcal{H}_x$ is complex-linear. Also, letting $w, w' \in T_y N$ be the images of any given $\xi, \xi' \in \mathcal{H}_*$ under the differential at (y, z) of the bundle projection $\mathcal{L} \to N$, we have*

(i) $g(\Theta\xi, \Theta\xi') = g(w, w')$ *if* $\phi = 0$ *identically on* M'.
(ii) $[\tau(y) - c]\, g(\Theta\xi, \Theta\xi') = [\tau(x) - c]\, g(w, w')$ *if* $\phi \neq 0$ *on* M' *and case* (a) *of* §24 *occurs.*
(iii) $a\, g(z, z)\, g(\Theta\xi, \Theta\xi') = 2\,[\tau(x) - c]\, g(\xi, \xi')$ *with* a *as in* (19.1), *if* $\phi \neq 0$ *on* M' *and case* (b) *of* §24 *occurs.*

Finally, the Θ-images of $v^N(y, z)$ and $u^N(y, z)$ are $|az|v(x)/|v(x)|$ and, respectively, $u(x)$, with a as in (19.1).

Proof. Let $y(t), \zeta(t), x(s, t)$ be as in the paragraph following $(*)$ in §26, and in addition such that, in case (a) of §24, the unit vector field $t \mapsto \zeta(t)$ normal to N along the curve $t \mapsto y(t) \in N$ is parallel relative to the Levi-Civita connection of (M, g), while, in case (b) of §24, $y(t) = y$ for all t and $\dot\zeta = d\zeta/dt \in T_y M$ is $g(y)$-orthogonal to $\zeta(t)$ and $J\zeta(t)$ for every t. These assumptions mean that, for any fixed s, the curve $t \mapsto (y(t), s\zeta(t))$ in U' is horizontal in the sense of Remark 24.1.

(In case (a) we assume that $\nabla_{\dot y}\zeta = 0$, rather than just $[\nabla_{\dot y}\zeta]^{\rm nrm} = 0$ as required by the definition of the normal connection in §24, since N is totally geodesic, cf. Remark 5.3(ii), and so $\nabla_{\dot y}\zeta$ is normal to N whenever ζ is.)

Writing $\langle\,,\rangle$ for $g(\ ,\ )$ we have $\langle v, x_t\rangle = \langle u, x_t\rangle = 0$ for all s, t (notation of §26). First, $\langle v, x_t\rangle = 0$ by (f), (d) in §26. Next, (g) – (h) in §26 yield $[\langle u, x_t\rangle/Q]_s = 0$, i.e., $\langle u, x_t\rangle/Q$ is constant as a function of s with fixed t. To see that its constant value is 0 we may evaluate its limit as $s \to 0$ using l'Hospital's rule and noting that, by (g), (h) in §26, $\langle u, x_t\rangle_s/Q_s = \pm\langle u, x_{st}\rangle Q^{-1/2}/\psi$. In case (a), $\langle u, x_t\rangle_s/Q_s = \pm\langle u/|u|, x_{st}\rangle/\psi \to 0$ as $s = 0$, by (c), (e) in §26, since $\nabla_{\dot y}\zeta = 0$, while $\psi = a \neq 0$ on N (see end of §20). In case (b), $\langle u, x_t\rangle_s/Q_s = \langle Jx_s, x_{st}\rangle/\psi \to \langle J\zeta(t), \dot\zeta\rangle/a = 0$ as $s \to 0$ by (f), (b) in §26 with $u = Jv$, $\psi = a \neq 0$ on N, and our orthogonality assumption for case (b). Thus, $\langle u, x_t\rangle = 0$.

Relations $\langle v, x_t\rangle = \langle u, x_t\rangle = 0$ state that v, u are g-normal to the Exp-image of every horizontal curve in U'. Hence $\varTheta(\mathcal{H}_*) \subset \mathcal{H}_x$ (cf. Remark 24.1 and (8.1)).

To prove (i) – (iii) we may assume, due to symmetry of g, that $\xi = \xi'$. By (1.2) and (8.3), $v_t = \phi x_t$ for every s, t, since $x_t(s,t) \in \mathcal{H}_{x(s,t)}$. (Note that v_t equals $(\nabla v)x_t$, the covariant derivative of v in the direction of x_t, cf. §26 and (1.1).) Also, $Q_t = \langle x_t, \nabla Q\rangle = 0$, as $\nabla Q = 2\psi v$ (see (8.5.i)), while $\langle v, x_t\rangle = 0$, and so, since $x_s = \pm Q^{-1/2}v$ by (f) in §26, we have $x_{st} = \pm Q^{-1/2}\phi x_t$. The Leibniz rule and $(*)$ in §26 now imply $\langle x_t, x_t\rangle_s = 2\langle x_t, x_{st}\rangle = \pm 2\langle x_t, x_t\rangle\phi Q^{-1/2}$. This, along with (b) in §26, yields $\langle x_t, x_t\rangle = g(\dot y, \dot y)$ when $\phi = 0$, thus proving (i); at the same time, combined with (h) in §26, it gives $[\langle x_t, x_t\rangle\phi/Q]_s = 0$, so that $\langle x_t, x_t\rangle\phi/Q$ is constant as a function of s. When $\phi \neq 0$ on M', we have $Q/\phi = 2(\tau - c)$ (by Lemma 8.2), so that $\langle x_t, x_t\rangle/(\tau - c)$ is constant as a function of s, and we can find its constant value by evaluating its limit as $s \to 0$. Specifically, in case (a) of §24, $\tau(y) \neq c$ (see (i) in §19), and (ii) follows as $\langle x_t, x_t\rangle/(\tau - c)$ at $x = x(s,t)$ equals $g(\dot y, \dot y)/(\tau - c)$ at $y = x(0,t)$, cf. (b) in §26. In case (b) of §24, however, we find the limit by applying l'Hospital's rule *twice*, as $\tau(y) = c$ (cf. (i) in §19) and $d\tau = 0$ at y, while $x_t = 0$ at $s = 0$ by (b) in §26; this gives $2\langle x_{ts}, x_{ts}\rangle$ in the numerator (at $s = 0$) and τ_{ss} in the denominator. By Remark 8.4(ii) and $(*)$, (c) in §26, $\tau_{ss} = \psi(y) = a$ and x_{ts} equals $\dot\zeta = d\zeta/dt$ at $s = 0$. (Note that $\psi = a$ on N, cf. end of §20, while $\nabla_{\dot y}\zeta = \dot\zeta$ as $y(t) = y$ is constant.) Now (iii) follows: $\langle x_t, x_t\rangle/(\tau - c)$ at any $x = x(s,t)$ is the same as at $y = x(0,t)$, i.e., $\langle x_t, x_t\rangle/(\tau - c) = 2g(\dot\zeta, \dot\zeta)/a = 2g(\xi, \xi)/(as^2) = 2g(\xi, \xi)/[a g(z, z)]$, where $z = s\zeta(t)$ and $\xi = s\dot\zeta(t)$.

The equality $x_{st} = \pm Q^{-1/2}\phi x_t$ established in the last paragraph means that

$$\nabla_{\dot x} w \,=\, \pm Q^{-1/2}\phi w\,, \qquad \text{with} \quad \dot x = x_s\,, \tag{27.1}$$

where $w = x_t$ stands for the vector field $s \mapsto w(s) = x_t(s,t)$ along the geodesic $s \mapsto x(s,t)$, for any fixed t. As $\nabla J = 0$, (27.1) holds for $\tilde w = Jw$ whenever it does for w. In case (a) of §24, the Exp-preimage of w is a vector field along the curve $s \mapsto (y(t), s\zeta(t)) \in \mathcal{L}$ which arises as the horizontal lift of $w(0)$. (In fact, at any s, t, the preimage is the velocity vector of the curve $t \mapsto (y(t), s\zeta(t))$, which we chose to be horizontal, and which has the projection image $t \mapsto y(t)$ with the velocity $w(0)$, cf. (b) in §26.) Replacing $w(0)$ by $Jw(0)$ causes such a horizontal-lift field to become multiplied by i in the complex vector bundle $\mathcal{H}^N$ (see §24), and at the same time results in replacing w with (27.1) by $\tilde{w} = Jw$, since a solution w to (27.1) is determined by the initial value $w(0)$. This proves our complex-linearity assertion in case (a) of §24. In the remaining case (b), under the identification $\mathcal{L} = T_yM$ (§24), the Exp-preimage of w is the vector field $s \mapsto s\dot{\zeta}(t)$ along the line segment $s \mapsto s\zeta(t) \in T_yM$ (with fixed t). Therefore, $w(0) = 0$ and $w(s)/s$ has a limit as $s \to 0$, equal (in view of the local-coordinate formula for $\nabla_{\dot{x}}w$) to the value of $\nabla_{\dot{x}}w$ at $s = 0$. As the Exp-preimage of the limit is $\dot{\zeta}(t)$, it follows that, in case (b) of §24, a solution w to (27.1) is uniquely determined by $(\nabla_{\dot{x}}w)(0) = \dot{\zeta}(t)$ (with fixed t). Thus, replacing w by $\tilde{w} = Jw$ amounts to using $J\dot{\zeta}(t)$ instead of $\dot{\zeta}(t)$, i.e., to multiplying the Exp-preimage of w by i in the complex vector bundle $\mathcal{H}^N$ (cf. §24), which establishes our complex-linearity claim also in case (b).

Finally, relation $\dot{x} = (\operatorname{sgn} a)v/|v|$ in Lemma 19.1, for $x(s) = \operatorname{Exp}(y, sz/|z|)$ at $s = |z|$, shows that Θ sends $z/|z|$, treated as a vertical vector in $T_{(y,z)}\mathcal{L}$ (cf. §2), onto $(\operatorname{sgn} a)v(x)/|v(x)|$. Multiplying both vectors by $a|z|$, we obtain our assertion about the Θ-image of $v^N(y,z)$. Also, since $z \in (T_yN)^\perp$, (19.2) and (21.2) – (21.3) give $\nabla_z u = iaz$. Now $\Theta(u^N(y,z)) = u(x)$ by Lemma 6.5, since on the normal space $\mathcal{L}_y \subset T_yM$ (identified, as usual, with $\{y\} \times \mathcal{L}_y \subset \mathcal{L}$), the normal exponential mapping of N coincides with $\exp_y$. This completes the proof. ■

§28. A normal exponential diffeomorphism

The normal exponential mapping of N, used below, was introduced in §20; this time it is defined on the whole total space of the normal bundle of N, since, due to compactness of M, we have $U^{\mathrm{Exp}} = TM$ (notation of Remark 6.3).

Lemma 28.1. *Suppose that L is given by* (23.1) *for $\tau_{\min}, \tau_{\max}$ and $\tau \mapsto Q$ determined as in* (iii) *of* §22 *by a function τ satisfying* (0.1) *on a compact Kähler manifold (M,g), and N is a critical manifold of τ, cf.* Remark 5.3(ii). *If $\mathcal{L}$ denotes the total space of the normal bundle of N, while $M' \subset M$ and $\mathcal{L}' \subset \mathcal{L}$ are the open sets defined by $d\tau \neq 0$ and, respectively, $0 < s < L$, where s is the norm function $\mathcal{L} \to [0,\infty)$ corresponding as in* Remark 2.2 *to the fibre metric obtained by restricting g to $\mathcal{L}$, then the restriction to $\mathcal{L}'$ of the normal exponential mapping of N is a C^∞ diffeomorphism* $\operatorname{Exp} : \mathcal{L}' \to M'$.

Proof. The Exp-image of any open line segment of length L emanating from 0 in any fibre $\mathcal{L}'_y$ of the punctured-disk bundle $\mathcal{L}'$ has the form $X \smallsetminus \{y, y'\}$, with X and y, y' as in Lemma 23.1(c), so that, by Lemma 23.1(c), $X \smallsetminus \{y, y'\} \subset M'$. Hence Exp actually sends $\mathcal{L}'$ into M'.

Surjectivity of $\operatorname{Exp} : \mathcal{L}' \to M'$ is obvious from Lemma 23.1(b). To prove its injectivity, suppose that $(y,z) \in \mathcal{L}'$ and $x = \operatorname{Exp}(y,z) \in M'$. Since $0 < |z| < L$, we can express (y,z) in terms of x by travelling backwards along the unit-speed geodesic $t \mapsto x(t) = \operatorname{Exp}(y, tz/|z|)$, which has $x(0) = y$, $\dot{x}(0) = z/|z|$, $x(s) = x$ (where $\dot{x} = dx/dt$ and $s = |z| \in (0,L)$) and, by Lemma 19.1, $\dot{x}(s) = w(x)$ for the vector field $w = (\operatorname{sgn} a)v/|v|$ on M' (with v, a defined as in Lemma 19.1). In fact, the re-parameterized geodesic $t \mapsto y(t) = \operatorname{Exp}(x, -tw(x))$ clearly has $y(0) = x$,

$\dot{y}(0) = -w(x)$, $y(s) = y$ and $\dot{y}(s) = -z/|z|$, so that $(y,z) = (y(s), -s\dot{y}(s))$. Moreover, s is uniquely determined by x and depends C^∞-differentiably on x (via $\tau(x)$), since the assignment $s \mapsto \tau$ with (19.4) is a C^∞ diffeomorphism $(0,L) \to (\tau_{\min}, \tau_{\max})$ (cf. (23.1)). The last formula for (y,z) thus shows that (y,z) is determined by x, i.e., $\mathrm{Exp} : \mathcal{L}' \to M'$ is injective, and its inverse $M' \to \mathcal{L}'$ is of class C^∞. This completes the proof. ∎

Lemma 28.2. *Let $\mathcal{L}$ be the total space, with* (2.2)*, of a real/complex vector bundle with a Riemannian/Hermitian fibre metric $\langle\,,\rangle$ over a manifold N, and let $\mathcal{L}' \subset \mathcal{L}$ denote the open set given by $0 < s < L$, for some $L \in (0,\infty)$, where s is the norm function $\mathcal{L} \to [0,\infty)$, cf.* Remark 2.2. *Then, setting $\varPhi(y,z) = (y, sz/|z|)$ for $y \in N$ and $z \in \mathcal{L}_y \smallsetminus \{0\}$, where $s \in (0,L)$ depends on $r = |z| \in (0,\infty)$ via a fixed C^∞ diffeomorphism $(0,\infty) \to (0,L)$, we obtain a C^∞ diffeomorphism $\varPhi : \mathcal{L} \smallsetminus N \to \mathcal{L}'$.*

In fact, if $(y,w) = \varPhi(y,z)$ and $r = |z| > 0$, then $w = sz/r$ and $z = rw/s$ with r obtained from $s = |w|$ via the inverse diffeomorphism $(0,L) \to (0,\infty)$. ∎

Remark 28.3. With the notations and under the assumptions of Lemma 28.2, let $\mathcal{H}$ be the horizontal distribution of a fixed connection in $\mathcal{L}$ making $\langle\,,\rangle$ parallel, and let $\varXi : T_{(y,z)}\mathcal{L} \to T_{\varPhi(y,z)}\mathcal{L}$ denote the differential of $\varPhi$ at any given point $(y,z) \in \mathcal{L} \smallsetminus N$. Then, for $\mathcal{H}' = \mathcal{H}_{(y,z)}$ and $\mathcal{H}'' = \mathcal{H}_{\varPhi(y,z)}$,

(a) $\varXi$ maps $\mathcal{H}'$ onto $\mathcal{H}''$.

(b) In the case where $\mathcal{L}$ is a *complex* vector bundle, $\varXi$ sends the vectors $v(y,z)$ and $u(y,z)$, defined by (2.3) for any fixed $a \in \mathbf{R} \smallsetminus \{0\}$, onto r/s times ds/dr times $v(\varPhi(y,z))$ and, respectively, onto $u(\varPhi(y,z))$, with s and ds/dr evaluated at $r = |z|$.

In fact, (a) is immediate: the norm function r is constant along any horizontal curve in $\mathcal{L}$, and so $\varPhi$ multiplies such a curve by a constant factor. Also, (b) for u is obvious as the flow of u consists of multiplications by scalars of modulus one, each of which commutes with $\varPhi$ (treated as a mapping $\mathcal{L} \smallsetminus N \to \mathcal{L} \smallsetminus N$). Finally, (c) for v follows since an integral curve $t \mapsto (y, e^{at}z)$ of v, for a fixed $z \in \mathcal{L}_y$ with $|z| = 1$, has the $\varPhi$-image $t \mapsto (y, w(t))$ such that $w(t) = sz$, with s depending on t via the given diffeomorphism $r \mapsto s$, where $r = e^{at}$. Hence dw/dt equals r/s times ds/dr times $aw(t)$, as required.

§29. A global classification of special Kähler-Ricci potentials

We will now show that every triple M, g, τ formed by a special Kähler-Ricci potential τ on a compact Kähler manifold (M,g) is biholomorphically isometric to one of the examples constructed in §17 and §18. Since we already know that, conversely, each of those examples constitutes a compact Kähler manifold with a a special Kähler-Ricci potential, the result of this section amounts to a complete classification theorem for such triples M, g, τ.

Note that, in contrast with its use elsewhere in the text, the symbol (S,γ) stands in this section for a Riemannian (or Kähler) manifold *of any dimension*.

Lemma 29.1. *Let (S,γ), (M,g) be complete Riemannian manifolds with open subsets $S'\subset S$, $M'\subset M$ such that both $S\smallsetminus S'$ and $M\smallsetminus M'$ are unions of finitely many compact submanifolds of codimensions greater than one. Any isometry Ψ of (S',γ) onto (M',g) then can be uniquely extended to an isometry of (S,γ) onto (M,g). If, in addition, (S,γ) and (M,g) are Kähler manifolds and the isometry $\Psi: S'\to M'$ is a biholomorphism, then so is the extension $S\to M$.*

In fact, by the codimension hypothesis S' (or, M') is connected and dense in S (or, in M), and the inclusion mappings $S'\to S$, $M'\to M$ are distance-preserving. Thus, as metric spaces, S and M are the completions of S' and M'. Our assertion now follows since distance-preserving mappings are isometries [11], p. 169, the Kähler case being obvious from a continuity argument. ■

Theorem 29.2. *Let τ be a special Kähler-Ricci potential on a compact Kähler manifold (M,g) of complex dimension $m\ge 2$, cf.* (0.1). *Then either*

(i) *M,g,τ are, up to a biholomorphic isometry, obtained as in* §17; *or,*
(ii) *M can be biholomorphically identified with $\mathbf{C}\mathrm{P}^m$ in such a way that g,τ become the objects constructed in* §18.

Proof. We denote N,N^* the two critical manifolds of τ, ordered so that either

(a) both N,N^* are of complex codimension one, or
(b) $N=\{y\}$ for some $y\in M$ (cf. (i), (ii) in §22).

In case (a) (or, (b)), we define the data (10.3) (or, (11.1)) as follows. First, in both cases, m is the complex dimension of M, while $\mathcal{I}'=(\tau_{\min},\tau_{\max})$, the variable $\tau\in\mathcal{I}'$ and the function Q of τ are chosen to be the ones appearing in the assignment $[\tau_{\min},\tau_{\max}]\ni\tau\mapsto Q$ determined by our M,g,τ as in (iii) of §22; r is a fixed function with (10.1), ε is the invariant defined in Remark 8.3, $a\in\mathbf{R}$ satisfies (19.1) with our N, and c is the constant with $Q/\phi=2(\tau-c)$ on M' (notation of Lemma 8.2). Thus, c remains undefined when $\phi=0$ identically on M', cf. Lemma 8.2, which, by (ii) in §19, may happen only in case (a).

Next, in case (a), N is the critical manifold chosen above, $\mathcal{L}$ is its normal bundle, $\mathcal{H}$ in (10.3) is the horizontal distribution of the normal connection in $\mathcal{L}$ (see §24), while $\langle\,,\rangle$ is the Hermitian fibre metric in $\mathcal{L}$ whose real part is the restriction of g to $\mathcal{L}$, and h is the metric on N defined in the paragraph preceding Lemma 25.2. On the other hand, in case (b), $V=T_yM$ and $\langle\,,\rangle$ is the Hermitian inner product with the real part $g(y)$.

The data (10.3) (or, (11.1)) just defined in case (a) (or, (b)) satisfy the conditions listed in the paragraph following (10.3) (or, (11.1); see Lemma 25.2 and Remark 16.2). They also fulfill the additional requirements in §17 (or, §18): namely, (iv) in §22 gives (17.1) in both cases, while, by (iii) in §19, $c\notin[\tau_{\min},\tau_{\max}]$ in case (a) and $\tau(y)=c$ in case (b). The construction of §17 (or, §18) now yields a compact Kähler manifold of complex dimension m, which we will denote (S,γ), rather than (M,g), and a special Kähler-Ricci potential on (S,γ), still denoted τ.

We define the set S' to be $\mathcal{L}\smallsetminus N$ in case (a) and $T_yM\smallsetminus\{0\}$ in case (b), so that S' may be treated as an open subset of S (cf. §17 or §18). Then, in both cases,

$S' = \mathcal{L} \smallsetminus N$, since $\mathcal{L}$ is, also in case (b), the total space of the normal bundle of N, provided that we identify $\{y\} \times T_yM$ with T_yM as in §24.

The inverse of our function r of the variable τ, satisfying (10.1), is a diffeomorphism $(0,\infty) \ni r \mapsto \tau \in \mathcal{I}'$. Another diffeomorphism is $\mathcal{I}' \ni \tau \mapsto s \in (0,L)$, for L as in (23.1), with the inverse characterized by (19.4). The composite assignment $r \mapsto \tau \mapsto s$ is a C^∞ diffeomorphism $(0,\infty) \to (0,L)$, leading to a diffeomorphism $\varPhi : \mathcal{L} \smallsetminus N \to \mathcal{L}'$ defined as in Lemma 28.2. Then, by Lemmas 28.1 and 28.2, the composite $\varPsi = \mathrm{Exp} \circ \varPhi$ is a C^∞ diffeomorphism $\mathcal{L} \smallsetminus N \to M'$, that is, $S' \to M'$.

Just as we did for τ, let us use the symbol Q both for a function on M' (with $Q = g(\nabla\tau, \nabla\tau)$) and a function on S', obtained from τ on S' via our assignment $\tau \mapsto Q$ (cf. (iii) in §22). The diffeomorphism $\varPsi : S' \to M'$ then makes τ and Q on S' correspond to τ and Q on M'. In fact, τ becomes a function on $S' = \mathcal{L} \smallsetminus N$ by being treated as the composite of the diffeomorphism $r \mapsto \tau$ (see the last paragraph) with r which now stands for the norm function $\mathcal{L} \to \mathbf{R}$ (cf. §17, §18). On the other hand, the norm function restricted to $\mathcal{L}'$, which we denote s, is mapped by Exp onto the arc-length parameter (with the initial value 0) for normal geodesics emanating from N, so that our claim for τ follows from relation (19.4), established in §23 for $\tau : M \to \mathbf{R}$ and the arc length s, but also used in the present proof to define τ as a function on S'. The claim for Q now is obvious since Q on S' is the same function of τ as Q on M'.

The diffeomorphism $\varPsi : S' \to M'$ sends the objects $\mathcal{H}^N, v^N, u^N$ in $S' = \mathcal{L} \smallsetminus N$, introduced in §24, onto $\mathcal{H}, v, u$ in M', defined by (8.1). This is clear from the initial and final clauses of Lemma 27.1 combined with Remark 28.3, where, in the case of v^N and v, we use the fact that $|a|r\,ds/dr = Q^{1/2}$ by (10.1) and (19.4), while the factor $|az|/|v(x)|$ in Lemma 27.1 equals $|a|sQ^{-1/2}$, cf. (8.2).

Furthermore, $\varPsi$ is a biholomorphism, i.e., has a complex-linear differential at every point. Namely, its complex-linearity holds separately on $\mathcal{H}^N$ and on the distribution in $\mathcal{L} \smallsetminus N$ spanned by v^N and u^N, the former conclusion being immediate from Lemma 27.1 and Remark 2.3, the latter obvious as $u^N = Jv^N$ in $\mathcal{L} \smallsetminus N$ ((2.3), or Remark 3.1) and $u = Jv$ in M (see (8.1)).

Next, $\varPsi$ is an isometry of (S', γ) onto (M', g). In fact, by (i) – (iii) in Lemma 27.1 combined with Remark 2.3, the differential of $\varPsi$ at any point is isometric when restricted to $\mathcal{H}^N$. Also, as we just saw, $\varPsi$ sends $\mathcal{H}^N$ and the vector fields v^N, u^N (which span the γ-orthogonal complement of $\mathcal{H}^N$ in S') onto $\mathcal{H}$ and v, u (which span the g-orthogonal complement of $\mathcal{H}$ in M'). Finally, due to the last line in Remark 2.2, relations (8.2) remain valid if one replaces Q (in M') and g, v, u with Q (in S') and γ, v^N, u^N.

Our (S, γ), (M, g) and S', M' clearly satisfy the assumptions of Lemma 29.1, as $M \smallsetminus M' = N \cup N^*$ ((i) in §22) and $S \smallsetminus S' = N \cup N^*$ (Remarks 17.2 and 18.1). Combined with the two preceding paragraphs, this completes the proof. ■

§30. Compact conformally-Einstein Kähler manifolds

The simplest examples of quadruples M, g, m, τ with (0.2) for which M is compact are provided by certain well-known Riemannian products having S^2 as a factor; see [8], §25. In this section we describe their immediate generalization, in which g is a locally reducible metric on the total space M of an S^2 bundle with a flat connection.

Suppose that we are given an integer $m \geq 2$, a real number $K > 0$, a compact Kähler-Einstein manifold (N, h) of complex dimension $m-1$ with the Ricci tensor $\mathrm{r}^{(h)} = (3-2m)Kh$, and a C^∞ complex line bundle $\mathcal{L}$ over N with a Hermitian fibre metric and a fixed flat connection making the metric parallel (i.e., a flat $\mathrm{U}(1)$ connection). The simplest choice of such $\mathcal{L}$ is the product bundle $\mathcal{L} = N \times \mathbf{C}$.

Let $\mathcal{E} = N \times \mathbf{R}$ now denote the product real-line bundle over N, with the obvious ("constant") Riemannian fibre metric, and let M be the unit-sphere bundle of the direct sum $\mathcal{L} \oplus \mathcal{E}$. Thus, M is a 2-sphere bundle over N, with $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is the vertical distribution (tangent to the fibres), and $\mathcal{H}$ is the restriction to M of the horizontal distribution of the direct-sum connection in $\mathcal{L} \oplus \mathcal{E}$. Since the latter connection is flat, the distributions $\mathcal{V}, \mathcal{H}$ are both integrable. We now define a metric g on M by choosing g on $\mathcal{V}$ to be $1/K$ times the standard unit-sphere metric of each fibre, declaring $\mathcal{V}$ and $\mathcal{H}$ to be g-orthogonal, and letting g on $\mathcal{H}$ to be the pullback of h under the bundle projection $M \to N$. Finally, we define $\tau : M \to \mathbf{R}$ to be any nonzero constant times the restriction to M of the composite $\mathcal{L} \oplus \mathcal{E} \to \mathcal{E} \to \mathbf{R}$ consisting of the direct-sum projection morphism $\mathcal{L} \oplus \mathcal{E} \to \mathcal{E}$ followed by the Cartesian-product projection $\mathcal{E} = N \times \mathbf{R} \to \mathbf{R}$.

Let U be any open submanifold of N which admits a system of parallel, orthonormal, trivializing sections for both $\mathcal{L}$ and $\mathcal{E}$. We may use such sections to trivialize the portion M_U of M lying over U, i.e., identify it with $U \times S^2$, where S^2 is the unit sphere centered at 0 in a Euclidean 3-space V. Since $\mathcal{H}, \mathcal{V}$ are the factor distributions of such a product decomposition, it is clear that (M_U, g) is a Riemannian-product manifold with the factors (U, h) and S^2, the latter carrying a metric of constant Gaussian curvature K (namely, $1/K$ times the submanifold metric it inherits from V); in fact, g on M_U is the sum of the pullbacks of h and the S^2 metric via the factor projections, which, for the S^2 factor, follows from invariance of the fibre metric in $\mathcal{E}$ under parallel transports. This local-product structure makes (M, g) a Kähler manifold: the bundle $\mathcal{V}$ (and the 2-sphere fibres of M) are naturally oriented, since so are $\mathcal{L}$ and $\mathcal{E}$. Finally, in terms of such a local-product decomposition, our τ is a function on M, constant in the direction of the N factor, while, as a function $S^2 \to \mathbf{R}$, it is the restriction to S^2 of a nonzero linear homogeneous function $V \to \mathbf{R}$, with V as above.

All quadruples M, g, m, τ obtained from the construction just described satisfy (0.3), along with condition (a) in Proposition 33.1 below. The last claim is immediate from (8.3): namely, $\phi = 0$, i.e., $\nabla d\tau = 0$ on $\mathcal{H}$, since $\mathcal{V}, \mathcal{H}$ defined above are obviously the same as in (8.1), and τ is constant along the factor distribution $\mathcal{H}$. As for (0.3), it is easily verified through direct local calculations using the Riemannian-product description of g in the preceding paragraph; such calculations can also be found in [8], section 25.

§31. Some metrics on the Riemann sphere

The *Riemann sphere* S of a complex vector space V of complex dimension 1 is obtained, as usual, from the disjoint union of V and its dual V^* by identifying the open sets $V \smallsetminus \{0\}$ and $V^* \smallsetminus \{0\}$ via the inversion biholomorphism $z \mapsto z^{-1}$ (see §12). In this way both V, V^* become identified with open subsets of S. Namely, $V = S \smallsetminus \{\infty\}$ and $V^* = S \smallsetminus \{0\}$, where $0 \in S$ stands for $0 \in V$ and ∞ is the zero functional $0 \in V^*$ treated as an element of S.

A given Hermitian inner product $\langle\,,\rangle$ in V and a function $\tau \mapsto Q$ satisfying (17.1) on an interval $[\tau_{\min}, \tau_{\max}]$, along with a selected endpoint $\tau_0 \in \{\tau_{\min}, \tau_{\max}\}$, then give rise to a septuple $\mathcal{I}', \tau, Q, r, a, V, \langle\,,\rangle$ with (12.1), which consisting, besides $V, \langle\,,\rangle$, of $\mathcal{I}' = (\tau_{\min}, \tau_{\max})$, the variable $\tau \in \mathcal{I}'$, our function Q of τ, the constant a such that $dQ/d\tau = 2a$ at $\tau = \tau_0$, and a fixed solution r to (10.1).

Formula $\gamma = (ar)^{-2}Q\,\mathrm{Re}\langle\,,\rangle$ now defines, as in §12, a Riemannian metric on $V \smallsetminus \{0\}$, the annulus U of §12 being $V \smallsetminus \{0\}$ since Remark 17.1 gives $(r_-, r_+) = (0, \infty)$ in (10.2). Here r stands, as usual, also for the norm function of $\langle\,,\rangle$.

The metric γ then has a C^∞ extension to a metric, also denoted γ, on the Riemann sphere S, in which $V \smallsetminus \{0\}$ is contained as the open subset $S \smallsetminus \{0, \infty\}$.

In fact, γ has an extension from $V \smallsetminus \{0\}$ to a C^∞ metric on V, since §14 allows us to treat Q/r^2 as a positive C^∞ function of $r^2 \in [0, \infty)$. The same applies to the metric γ^* on $V^* \smallsetminus \{0\}$ obtained from the corresponding "dual" data as in §12, while (a) in §12 states precisely that the two metrics agree on the intersection $S \smallsetminus \{0, \infty\} = V \cap V^*$.

Example 31.1. Formula $Q = K(\tau_0^2 - \tau^2)$ with any constants $K > 0$ and $\tau_0 \neq 0$ defines a function $\tau \mapsto Q$ which satisfies (17.1) with $\tau_{\max} = -\tau_{\min} = |\tau_0|$.

Remark 31.2. The above construction for $\tau_{\min}, \tau_{\max}$ and $\tau \mapsto Q$ chosen as in Example 31.1 with any given $K > 0$ and $\tau_0 \neq 0$ yields a metric γ *constant Gaussian curvature* K on the Riemann sphere S, and $\varphi : S \to S_1$ defined below is an isometry between (S, γ) and the unit sphere S_1 about $(0,0)$ in $V \times \mathbf{R}$ with $1/K$ times its submanifold metric.

In fact, let $\varphi(z) = \chi(z)/|\chi(z)| = \chi(z)/|\rho_0|$ for $z \in V$ and $\varphi(\infty) = (0, -1)$, with $\chi : V \to \hat{S}$ given by $\chi(0) = (0, \rho_0)$ and $\chi(z) = (\sqrt{Q}z/|z|, \sqrt{K}\tau)$ for $z \in V \smallsetminus \{0\}$, where $\rho_0 = \sqrt{K}\tau_0$ and $\hat{S} \subset V \times \mathbf{R}$ is the sphere, centered at 0, of radius $|\rho_0|$. Here both τ and $Q = K(\tau_0^2 - \tau^2)$ depend on $r = |z|$ via (10.1) with $\mathcal{I}' = (-|\tau_0|, |\tau_0|)$ and $a = -K\tau_0$ (cf. (14.1)). Since (10.2) is a diffeomorphism, χ is injective and its image misses just the point $(0, -\rho_0) \in \hat{S}$.

The pullback under χ of the submanifold metric of $\hat{S}$ equals Q/r^2 times the Euclidean metric $\mathrm{Re}\langle\,,\rangle$ on V. Namely, χ sends *lines* (through 0 in V) and *circles* (about 0, in V) into *meridians* and, respectively, *parallels* in $\hat{S}$, in the cartographic terminology based on the *poles* $(0, \pm\sqrt{K}\tau_0)$. Our lines are orthogonal to circles, and meridians to parallels; thus, all we need to show is that χ restricted to any line or circle deforms the metric by the conformal factor Q/r^2.

For the circles, this is immediate: the obvious S^1 actions make χ equivariant, while χ sends the circle of any radius $r > 0$ onto a parallel which is a circle of radius $\sqrt{Q}$, where $Q = K(\tau_0^2 - \tau^2)$, with the required ratio $\sqrt{Q/r^2}$ of the radii.

For the lines, let $z(r) = rz_0$ with $\langle z_0, z_0\rangle = 1$. Then $\chi(z(r)) = (\sqrt{Q}z_0, \sqrt{K}\tau)$ (where τ, Q depend on $r \in (0, \infty)$ as before), and so, as $dQ/d\tau = -2K\tau$, we have $|d[\chi(z(r))]/dr|^2 = K(Q + K\tau^2)(d\tau/dr)^2/Q$. This equals Q/r^2, i.e., Q/r^2

times $|d[z(r)]/dr|^2$. (In fact, $Q + K\tau^2 = K\tau_0^2$, $a = -K\tau_0$ and, by (10.1), $d\tau/dr = Q/(ar)$.)

Obviously, φ has an isometric extension $S \to S_1$ (cf. Lemma 29.1), with $\varphi(0) = (0,1)$, $\varphi(\infty) = (0,-1)$.

§32. Special cases of conditions (17.1)

Lemma 32.1. *Let* $f = (k-1)\beta^{k+1} - (k+1)\beta^k + (k+1)\beta - (k-1)$ *for* $\beta \in \mathbf{R}$ *and an integer* $k \geq 2$. *Then* $f \neq 0$ *unless* $\beta = 1$ *or* $\beta = (-1)^k$.

In fact, $d^2f/d\beta^2 = k(k+1)(k-1)(\beta-1)\beta^{k-2}$, and so $f' = df/d\beta$ is strictly decreasing (or, increasing) on $(0,1)$ (or, on $(1,\infty)$), while $(-1)^k f'$ is strictly decreasing on $(-\infty,0)$. Evaluating f' at 1, 0 and -1, we now obtain $f' > 0$ on $(0,1) \cup (1,\infty)$, and, if k is even, $f' > 0$ on $(-\infty,0]$ while, if k is odd, $f' < 0$ on $(-\infty,\beta_0)$ and $f' > 0$ on $(\beta_0,0)$, for some $\beta_0 \in (-1,0)$. Therefore, evaluating f at 1, 0 and -1, we see that $f > 0$ on $(-\infty,-1)$ and $f < 0$ on $(-1,1)$ (for odd k), $f < 0$ on $(-\infty,1)$ (for even k), and $f > 0$ on $(1,\infty)$ (for all k). ■

Another, purely algebraic proof of Lemma 32.1 can be obtained by noting that f equals $(\beta-1)^3\mathit{\Pi}(\beta)$ with $\mathit{\Pi}(\beta) = \sum_{j=1}^{k-1} j\,(k-j)\,\beta^{j-1}$, while $\mathit{\Pi}(\beta)/(\beta+1)$ or $\mathit{\Pi}(\beta)$ is a sum of squares, as $\mathit{\Pi}(\beta) = 2^{2-k}\sum_{1\,\leq\,j\,\leq\,k/2} j\binom{k+1}{2j+1}(\beta+1)^{k-2j}(\beta-1)^{2j-2}$.

Lemma 32.2. *Let* (17.1) *hold for* $\tau \mapsto Q$ *given by one of the equations*

$$(32.1)\qquad \begin{aligned} &\text{a)}\quad Q \,=\, -K\tau^2 \,+\, (2m-1)^{-1}[\alpha\tau^{2m-1} \,-\, \eta/m]\,,\\ &\text{b)}\quad Q \,=\, m^{-1}K\tau \,+\, \alpha\tau^{m+1} \,-\, 2(m+1)^{-1}\eta/m \end{aligned}$$

and some $\tau_{\min}, \tau_{\max}$, *with an integer* $m \geq 2$ *and real constants* K, α, η.

(i) *In case* (32.1.a) *we have* $Q = K(\tau_0^2 - \tau^2)$ *and* $\tau_{\max} = -\tau_{\min} = |\tau_0|$, *as in* Example 31.1, *for some* $\tau_0 \neq 0$, *while* $\alpha = 0$, $K > 0$ *and* $\eta < 0$.

(ii) *In case* (32.1.b), $\tau_{\max} = -\tau_{\min} > 0$.

Proof. Since $\tau_{\min} \neq \tau_{\max}$, we have $\{\tau_{\min}, \tau_{\max}\} = \{\tau_0, \tau_1\}$ with $\tau_0 \neq 0$. Let φ_0 and φ_1 be the values at τ_0 and τ_1 of any function φ of the variable τ, such as Q and ψ given by $2\psi = dQ/d\tau$. Also, let $k = 2m-2$ and $k' = 2m-1$ (case (32.1.a)), or $k = m$ and $k' = 1$ (case (32.1.b)). For $\beta = \tau_1/\tau_0$ and $f = f(\beta)$ as in Lemma 32.1, (32.1.a) or (32.1.b) yields $2\tau_0^{-k-1}[Q_0 - Q_1 + (\tau_1 - \tau_0)(\psi_0 + \psi_1)] = \alpha f(\beta)/k'$. This gives $\alpha f(\beta) = 0$, since, by (17.1), $Q_0 = Q_1 = \psi_0 + \psi_1 = 0$. As $\tau_1 \neq \tau_0$ (that is, $\beta \neq 1$), Lemma 32.1 now implies that $\alpha = 0$, or k is odd and $\tau_1 = -\tau_0$.

In case (32.1.a), $k = 2m-2$ is even, and so $\alpha = 0$. Now (32.1.a) with $\alpha = 0$ and $Q_0 = Q_1 = 0$ easily yields (i). (In both cases, $|K| + |\alpha| > 0$ due to the nonzero-derivative requirement in (17.1).) In case (32.1.b), however, $\alpha \neq 0$. In fact, relation $\alpha = 0$ in (32.1.b), along with $Q_0 = Q_1 = 0$ and $\tau_1 \neq \tau_0$, would give $K = \eta = 0$. Hence $\tau_1 = -\tau_0 \neq 0$ and (ii) follows, which completes the proof. ■

§33. The four types (a), (b), (c1), (c2)

For a fixed integer $m \ge 1$ and a real variable t, let us set

$$F(t) \,=\, \frac{(t-2)t^{2m-1}}{(t-1)^m}\,, \qquad E(t) \,=\, (t-1)\sum_{k=1}^{m}\frac{k}{m}\binom{2m-k-1}{m-1}t^{k-1}. \tag{33.1}$$

In [8] we established the following result (see [8], Proposition 22.1):

Proposition 33.1. *Let M, g, m, τ satisfy* (0.2) *with $m \ge 3$, or* (0.3) *with $m = 2$, and let $Q : M \to \mathbf{R}$ be given by $Q = g(\nabla\tau, \nabla\tau)$. Then Q is a rational function of τ. More precisely, the open set $M' \subset M$ on which $d\tau \ne 0$ is connected and dense in M and, for ϕ, c as in* Lemma 8.2*, one of the following three cases occurs*:

(a) $\phi = 0$ *identically on M'.*
(b) $\phi \ne 0$ *everywhere in M' and $c = 0$.*
(c) $\phi \ne 0$ *everywhere in M' and $c \ne 0$.*

In case (a), (b)*, or* (c)*, the functions $\tau, Q : M \to \mathbf{R}$ satisfy* (32.1.a) *or* (32.1.b) *for some constants K, α, η, or, respectively, there exist constants A, B, C with*

$$Q = (t-1)[A + BE(t) + CF(t)] \text{ for } t = \tau/c \text{ and } F, E \text{ as in (33.1).} \tag{33.2}$$

Also, in case (c) *we have $\tau \ne c$ everywhere in M unless $C = 0$ in* (33.2). ■

For M, g, m, τ as in the first line of Proposition 33.1, exactly one of conditions (a), (b), (c) is satisfied. If M is compact, (0.4) and §22 show that case (c) consists of two subcases (c1), (c2), corresponding to 1), 2) in (ii) of §22.

Thus, every quadruple M, g, m, τ satisfying (0.2) with $m \ge 3$, or (0.3) with $m = 2$, and such that M is compact, must belong to exactly one of the four *types* (a), (b), (c1), (c2) just described.

We chose not to define analogous "types" (a1), (a2) and (b1), (b2) in cases (a), (b), as type (b) is empty (see Theorem 33.2 below), and hence so are (b1) and (b2), while (a2) is empty by (ii) in §19, and so type (a) coincides with (a1).

The reason we introduce the four types (a), (b), (c1), (c2) is that they allow us a systematic case-by-case approach to classifying quadruples M, g, m, τ with (0.2) and $m \ge 3$, or (0.3) and $m = 2$, such that M is compact.

First, in this section we show that type (b) is empty and provide a complete classification of type (a). Then, in §34, we establish a structure theorem for type (c1), which reduces its classification to the question of finding all objects satisfying conditions (34.1) – (34.5). The latter question is answered in the forthcoming paper [9], where we also prove that type (c2) is empty (cf. §35). However, type (c1) is not empty, as it contains Bérard Bergery's examples [3] (and more; see [9]).

Theorem 33.2. *Let M, g, m, τ satisfy* (0.2) *with $m \ge 3$ or* (0.3) *with $m = 2$. If M is compact, then M, g, m, τ cannot belong to type* (b) *described above.*

In fact, if M, g, m, τ were of type (b), Lemma 8.2 would give $\tau \ne c$, i.e., $\tau \ne 0$, everywhere in the open set $M' \subset M$ on which $d\tau \ne 0$, so that τ would be either nonnegative everywhere, or nonpositive everywhere in M'. As M' is dense in M (cf. Proposition 33.1), the same would be true with M' replaced by M, contrary to the relation $\tau_{\max} = -\tau_{\min} > 0$ in Lemma 32.2(ii), which holds since $\tau \mapsto Q$ in (iii) of §22 satisfies (17.1), while Proposition 33.1 gives (32.1.b). ■

Theorem 33.3. *For M, g, m, τ constructed in* §30*, M is compact,* (0.3) *holds, and the quadruple M, g, m, τ belongs to type* (a) *introduced above.*

Conversely, every quadruple M, g, m, τ with compact M which satisfies (0.2) *with $m \geq 3$ or* (0.3) *with $m = 2$, and belongs to type* (a) *is, up to a τ-preserving biholomorphic isometry, obtained as in* §30*.*

Proof. The first assertion has already been established at the end of §30.

Conversely, let a quadruple M, g, m, τ with compact M be of type (a) and satisfy (0.2) with $m \geq 3$, or (0.3) with $m = 2$. This implies (0.1) (see (0.4)), and so, by Theorem 29.2(i), we may assume that M, g, τ are obtained as in §17 from some data (10.3) with (17.1). (Case (ii) of Theorem 29.2 is excluded by (ii) in §19 and Remark 18.1.) Due to condition (a) in Proposition 33.1, the distribution $\mathcal{H}$ with (8.1) is integrable by (25.1). The connection with the horizontal distribution $\mathcal{H}$ in (10.3) is therefore flat, i.e., $\varepsilon = 0$ in (10.3). (See §10.)

The assignment $\tau \mapsto Q$ in (10.3) used in the construction coincides with that in (iii) of §22 (see (c) in §10), and so, by Proposition 33.1, it is given by (32.1.a) with some K, α, η. However, that assignment also satisfies (17.1) (see (iv) in §22). Therefore Lemma 32.2(i) gives $Q = K(\tau_0^2 - \tau^2)$ and $\tau_{\max} = -\tau_{\min} = |\tau_0|$ for some constants $K > 0$ and $\tau_0 \neq 0$.

This K and $N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ appearing in our data (10.3) now lead to an S^2 bundle constructed as in §30 with a Kähler metric, a special Kähler-Ricci potential, and two distributions, which we denote $\hat{M}, \hat{g}, \hat{\tau}, \hat{\mathcal{V}}, \hat{\mathcal{H}}$ to keep them apart from our $M, g, \tau, \mathcal{V}, \mathcal{H}$. Note that we are still free to multiply $\hat{\tau}$ by a nonzero constant.

Let $\varPhi$ now be the fibre-preserving C^∞ diffeomorphism of the $\mathbf{C}\mathrm{P}^1$ bundle M over N onto the S^2 bundle $\hat{M}$ over N which operates between the fibres over each $y \in N$ as the canonical isometry φ defined in Remark 31.2. Since φ is also orientation-preserving (for the standard Riemann-sphere orientation and the orientation of S^2 used in §30), it is holomorphic, i.e., $\varPhi$ maps fibres of M biholomorphically onto those of $\hat{M}$.

The formulae for φ and χ in Remark 31.2, in which Q and τ are functions of $r = |z|$, also show that $\varPhi$ makes τ correspond to a constant multiple of $\hat{\tau}$ (as the $\mathbf{R}$-component of χ is τ times the constant $\sqrt{K}$) and that $\varPhi$ preserves horizontality of curves, i.e., sends the distribution $\mathcal{H}$ of Remark 13.1 onto $\hat{\mathcal{H}}$. (Note that r is constant along every horizontal curve in M.)

Therefore, $\varPhi$ is a holomorphic isometry: we just verified that for the restriction of $\varPhi$ to the fibres, while the differential of $\varPhi$ at any point, restricted to $\mathcal{H}$, is both complex-linear and isometric by Remark 2.3 and, in addition, both $\varPhi^*\hat{g}$ and g make $\mathcal{H}$ orthogonal to $\mathcal{V}$. This completes the proof. ■

§34. A structure theorem for type (c1)

Let a sextuple m, I, Q, A, B, C consist of

(34.1) An integer $m \geq 2$, a nontrivial closed interval $I \subset \mathbf{R}$ of the variable t, and a function Q of t defined as in (33.2) for some constants A, B, C.

Given such m, I, Q, A, B, C, we may consider the following conditions:

(34.2)
a) Q is analytic on I, i.e., I does not contain 1 unless $C = 0$.
b) $Q = 0$ at both endpoints of I.
c) $Q > 0$ at all interior points of I.
d) dQ/dt is nonzero at both endpoints of I.
e) The values of dQ/dt at the endpoints of I are mutually opposite.

Lemma 34.1. *Let a quadruple M, g, m, τ with compact M satisfy* (0.2) *with $m \geq 3$ or* (0.3) *with $m = 2$, as well as condition* (c) *in* Proposition 33.1 *with c as in* Lemma 8.2. *Treating $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q \in \mathbf{R}$ in* (iii) *of* §22, *cf.* (0.4), *as a function of the variable $t = \tau/c$, we then have* (34.1) *and* (34.2) *for these m, Q along with $I = [\tau_{\min}/c, \tau_{\max}/c]$ and suitable A, B, C.*

In fact, Proposition 33.1 yields (34.1); hence Q is a rational function of t, and so its C^∞-differentiability on I (cf. (iii) in §22) amounts to analyticity. Now (34.2) is immediate from (iv) in §22. ■

Given m, I, Q, A, B, C for which (34.1) – (34.2) hold and

$$1 \notin I\,, \tag{34.3}$$

let us choose

(34.4) $a, c \in \mathbf{R} \smallsetminus \{0\}$ and $\varepsilon = \pm 1$ such that $\pm ac$ are the values of dQ/dt at the endpoints of I, and $\varepsilon c(t-1) > 0$ for all $t \in I$

(such a, c, ε exist by (34.2.e), (34.2.d) and (34.3)), as well as

(34.5) a compact Kähler-Einstein manifold (N, h) of complex dimension $m-1$ with the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$, where $\kappa = \varepsilon m A/c$, along with a C^∞ complex line bundle $\mathcal{L}$ over N carrying a Hermitian fibre metric $\langle\,,\rangle$ and a C^∞ connection making $\langle\,,\rangle$ parallel, whose curvature form (cf. Remark 2.1) equals $-2\varepsilon a$ times the Kähler form of (N, h) (defined as in (1.5)).

Remark 34.2. The existence of $\mathcal{L}$ with the connection required in (34.5) is by no means guaranteed for a given choice of data with (34.1) – (34.4) and (N, h) as in (34.5). For instance, m, I, Q, A, B, C then must satisfy the following necessary condition: *either $A = 0$, or the values of $A^{-1} dQ/dt$ at the endpoints of I are rational.* In fact, by (34.4) – (34.5) with $A \neq 0$, those values are $\pm m/2$ times the ratio $c_1(\mathcal{L})/c_1(N)$ of two integral cohomology classes in the real cohomology space $H^2(N, \mathbf{R})$. There are also further necessary conditions, stemming from a theorem of Kobayashi and Ochiai [12]. See [9] for details.

We will now use any given data with (34.1) – (34.5) to construct a quadruple M, g, m, τ with (0.3), belonging to type (c1) of §33, in which M is a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over N and, in particular, M is compact.

First, we choose a positive function r of the variable t restricted to the interior of I, such that $dr/dt = acr/Q$, with Q depending on t as in (34.1). This gives (17.1) and (10.1) for Q, r treated as functions of the variable $\tau = ct$ in the interval $[\tau_{\min}, \tau_{\max}] = cI$ or $\mathcal{I}' = (\tau_{\min}, \tau_{\max})$. By Remark 17.1, r ranges over $(0, \infty)$, and τ, Q restricted to the interior of I become functions of $r \in (0, \infty)$.

We will also use the symbol r for the norm function $\mathcal{L} \to (0,\infty)$ of $\langle\,,\rangle$ (Remark 2.2). Being functions of $r > 0$, both τ and Q thus become functions on $\mathcal{L} \smallsetminus N$ (notation of (2.2)). Let g now be the metric on the complex manifold $\mathcal{L} \smallsetminus N$ such that the vertical subbundle $\mathcal{V}$ of the tangent bundle is g-orthogonal to the horizontal distribution $\mathcal{H}$ of the connection chosen in $\mathcal{L}$, while g on $\mathcal{H}$ equals $2|\tau - c|$ times the pullback of h to $\mathcal{H}$ under the bundle projection $\mathcal{L} \to N$, and g on $\mathcal{V}$ is $Q/(ar)^2$ times the standard Euclidean metric $\mathrm{Re}\,\langle\,,\rangle$.

The data (10.3) thus introduced clearly satisfy all the requirements listed in the paragraph following (10.3) along with $\varepsilon = \pm 1$ and $c \notin [\tau_{\min}, \tau_{\max}]$ (due to (34.3)). Also, g defined here satisfies (10.5) with (10.4). Let M now denote the projective compactification of $\mathcal{L}$ (§13). As shown in §17, both g and $\tau : \mathcal{L} \smallsetminus N \to \mathbf{R}$ have C^∞ extensions to a metric and a function on M (still denoted g, τ) such that (M, g) is a Kähler manifold of complex dimension m and τ is a special Kähler-Ricci potential on (M, g).

Theorem 34.3. *Let M, g, m, τ be obtained via the above construction from some data with* (34.1) – (34.5) *and $m \ge 2$. Then M is compact, while the quadruple M, g, m, τ satisfies* (0.3) *and belongs to type* (c1) *of* §33.

Conversely, let M, g, m, τ, with compact M, satisfy (0.2) *with $m \ge 3$, or* (0.3) *with $m = 2$, and belong to type* (c1)*. Then, up to a τ-preserving biholomorphic isometry, M, g, m, τ are obtained as above from some data with* (34.1) – (34.5).

Proof. According to [8], Proposition 23.3, M, g, m, τ constructed above satisfy (0.3), since our description of g and τ on $\mathcal{L} \smallsetminus N$ is a special case of that in [8], §23, case (iii). In addition, since $\varepsilon = \pm 1$, assertion (d) in §10 states that $\phi \ne 0$ and our constant $c \ne 0$ is the same as in Lemma 8.2, and so, by (iii) in §19 with $c \notin [\tau_{\min}, \tau_{\max}]$ and Remark 17.2, the quadruple M, g, m, τ is of type (c1).

The converse statement is obvious from Theorem 29.2 and Lemma 34.1, since $c \notin [\tau_{\min}, \tau_{\max}]$ (i.e., $1 \notin I$) and the function $\kappa : N \to \mathbf{R}$, such that $\mathrm{r}^{(h)} = \kappa h$ is the Ricci tensor of h, is given by $\kappa = \varepsilon m A/c$. In fact, the first claim is obvious as case (ii) of Theorem 29.2 is excluded by (iii) in §19 and Remark 18.1, and the second follows from [8], Remarks 23.2 and 9.4. This completes the proof. ■

§35. Type (c2)

As it eventually turns out, type (c2) is empty: according to Proposition 13.3 of the forthcoming paper [9], the conclusion of the following corollary cannot occur, since conditions (34.1) – (34.2) with any $m \ge 2$ imply that $1 \notin I$.

Corollary 35.1. *Let M, g, m, τ with compact M satisfy* (0.2) *with $m \ge 3$, or* (0.3) *with $m = 2$, and belong to type* (c2) *of* §33*. Then conditions* (34.1) *and* (34.2) *hold for m and some I, Q, A, B, C such that $1 \in I$.*

In fact, for I, Q, A, B, C chosen as in Lemma 34.1, I contains the point $t = 1$ since, by (iii) in §19, $[\tau_{\min}, \tau_{\max}]$ contains $\tau = c$. ■

§36. Appendix: the local structure

In [8], Theorem 24.1, we proved a local classification result for special Kähler-

Ricci potentials τ on Kähler manifolds (M, g), showing that, up to a biholomorphism, such g and τ are, in a neighborhood of any point with $d\tau \neq 0$, obtained as in §10.

A similar local-structure theorem is true for points y at which $d\tau = 0$, provided that one replaces the construction of §10 by that of §15 or Lemma 16.1.

In fact, let $a \neq 0$ be the constant associated as in (19.1) with the critical manifold N of τ containing y. For any sufficiently small connected neighborhood U of y, the values assumed by τ on U form a half-open interval $\mathcal{I}$ whose only endpoint τ_0 is the constant value of τ on N (by (21.3), (21.2), Example 7.1 and Lemma 7.2), while $d\tau \neq 0$ everywhere in $U \smallsetminus N$ (by (a) in Lemma 5.2 for $u = J(\nabla\tau)$), and $Q = g(\nabla\tau, \nabla\tau)$ restricted to U is a C^∞ function of τ with $dQ/d\tau = 2\psi$ (Lemma 20.1(b)), so that $dQ/d\tau = 2a$ at $\tau = \tau_0$ (as $\psi = a$ on N, cf. end of §20). If we now set $\mathcal{I} = \mathcal{I}' \cup \{\tau_0\}$ and fix a function r of $\tau \in \mathcal{I}'$ with (10.1), the resulting objects $Q, \tau, \mathcal{I}, \mathcal{I}', \tau_0, a, r$ clearly satisfy (14.1). We also choose ε and c as in Remark 8.3 and Lemma 8.2, so that c is defined only when $\varepsilon = \pm 1$.

Depending on whether we have case (a) (or, (b)) in (19.3), we introduce the data (10.3) (or, (11.1)) which consist of the objects chosen above along with the complex dimension m of M, and $h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ (or, $V, \langle\,,\rangle$) defined as in the second paragraph following (a), (b) in the proof of Theorem 29.2. The assumptions listed in §15 (or Lemma 16.1) now hold as a consequence of (i) in §19, thus allowing us to construct the "models" required in our classification.

The biholomorphism in question is $\Psi = \mathrm{Exp} \circ \Phi$, defined as in the proof of Theorem 29.2; this time, however, instead of using Lemmas 28.1 and 28.2, we simply conclude from the inverse mapping theorem that Ψ sends a neighborhood of y in the total space of the normal bundle of N diffeomorphically onto a sufficiently small set U selected as above. The rest of the proof is an exact replica of the argument we used to establish Theorem 29.2.

§37. Appendix: another proof of Lemma 19.1

Let $s \mapsto x(s)$ be an arc-length parameterization of X with $x(0) = y$. Since $u = Jv$ is a Killing field (see (0.1) and §1), $g(u, \dot{x})$ is constant along X, for $\dot{x} = dx/ds$. (In fact, $\nabla_{\dot{x}}\dot{x} = 0$, and so $d[g(u, \dot{x})]/ds = g(\nabla_{\dot{x}}u, \dot{x}) = 0$ due to skew-symmetry of ∇u, cf. §4.) Also, $g(u, \dot{x}) = 0$ at $s = 0$, as $u(y) = Jv(y) = 0$. Thus, $g(u, \dot{x}) = 0$ along X. Let M' be the open set where $d\tau \neq 0$ (i.e., $v \neq 0$), and let $X' = X \cap M'$, that is, $X' = X \smallsetminus \{y\}$. For $\mathcal{V}, \mathcal{H}$ as in (8.1), let $\dot{x}^{\mathrm{vrt}}, \dot{x}^{\mathrm{hrz}}$ be the $\mathcal{V}$ and $\mathcal{H}$ components of $\dot{x}$ (restricted to X') relative to the decomposition $TM' = \mathcal{H} \oplus \mathcal{V}$. As $\dot{x} = \dot{x}^{\mathrm{vrt}} + \dot{x}^{\mathrm{hrz}}$, (8.4) applied to $w = \dot{x}^{\mathrm{vrt}}$ and $w = \dot{x}^{\mathrm{hrz}}$ gives $\nabla_{\dot{x}}v = \psi\dot{x}^{\mathrm{vrt}} + \phi\dot{x}^{\mathrm{hrz}}$. However, $\dot{x}^{\mathrm{vrt}} = \dot{\tau}v/Q$ and $\dot{x}^{\mathrm{hrz}} = \dot{x} - \dot{\tau}v/Q$, as one sees using (8.1), (8.2) and the relations $g(v, \dot{x}) = \dot{\tau}$ (Remark 8.4(i)) and $g(u, \dot{x}) = 0$. Thus, $Q\nabla_{\dot{x}}v = \psi\dot{\tau}v + \phi[Q\dot{x} - \dot{\tau}v]$.

Denoting $w^{\mathrm{nrm}} = w - g(w, \dot{x})\dot{x}$ the component normal to X of any vector field w along X we have $\nabla_{\dot{x}}[w^{\mathrm{nrm}}] = [\nabla_{\dot{x}}w]^{\mathrm{nrm}}$ (as $\nabla_{\dot{x}}\dot{x} = 0$), and so, skipping the brackets, we may write $\nabla_{\dot{x}}w^{\mathrm{nrm}}$. Then, for $w = v$ (restricted to X'), $Q\nabla_{\dot{x}}v^{\mathrm{nrm}} = (\psi - \phi)\dot{\tau}v^{\mathrm{nrm}}$. This is obvious from the above formula for $Q\nabla_{\dot{x}}v$, as $\phi Q\dot{x}^{\mathrm{nrm}} = 0$.

Let w be the vector field along X' defined by $w = Q^{-1/2}v^{\mathrm{nrm}}$, when $\phi = 0$

identically on M', or $w = |\phi|^{-1/2} v^{\mathrm{nrm}}$, when $\phi \neq 0$ everywhere in M'. (By Lemma 8.2, one of the two cases must occur.) Also, $\dot{Q} = 2\psi\dot{\tau}$ (Remark 8.4(i)) and $Q\dot{\phi} = 2(\psi - \phi)\phi\dot{\tau}$ (since $Q\nabla\phi = 2(\psi - \phi)\phi v$ according to (8.5.i)). As $Q\nabla_{\dot{x}} v^{\mathrm{nrm}} = (\psi - \phi)\dot{\tau} v^{\mathrm{nrm}}$ (see above), the last two relations give $\nabla_{\dot{x}} w = 0$.

We will now show that the parallel vector field w along X' is identically zero by proving the limit relation $w \to 0$ as $s \to 0$ (i.e., as the variable point $x \in X'$ approaches y). To this end we assume, in both cases, that $w = Q^{-1/2} v^{\mathrm{nrm}}$. (Since $|\phi|^{-1/2} = \sqrt{2|\tau - c|}\, Q^{-1/2}$ when $\phi \neq 0$, by Lemma 8.2, and $|\tau - c|$ is bounded near y, the same limit relation then will follow for $w = |\phi|^{-1/2} v^{\mathrm{nrm}}$.) First, $|v^{\mathrm{nrm}}|^2 = |v|^2 - \dot{\tau}^2$ as $g(v, \dot{x}) = \dot{\tau}$ (Remark 8.4(i)), and so, by (8.2), $|w|^2 = |v^{\mathrm{nrm}}|^2/Q = 1 - \dot{\tau}^2/Q$. Thus, by l'Hospital's rule, $\dot{\tau}^2/Q \to 1$ as $s \to 0$. In fact, $2\dot{\tau}\ddot{\tau}/\dot{Q} = \ddot{\tau}/\psi$ since $\dot{Q} = 2\psi\dot{\tau}$ (Remark 8.4(i)), and (19.2) with (1.3) yield the assumptions of Remark 8.4(ii) with $a = \psi(y)$ and $|\dot{x}(0)| = 1$, which gives $\dot{\tau} \neq 0$ for all $s \neq 0$ close to 0. Consequently, Remark 8.4(ii) yields $\ddot{\tau}/\psi \to 1$ as $s \to 0$. Hence $|w|^2 \to 0$ as $s \to 0$, and so, in both cases, $w = 0$ along X'. Due to our definition of w, this completes the proof.

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