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POTENTIAL TECHNIQUES AND REGULARITY OF BOUNDARY VALUE PROBLEMS IN EXTERIOR NON-SMOOTH DOMAINS

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Abstract. Via Potential Theory, we obtain optimal solvability results in weighted Sobolev spaces for the Poisson's Problem for the Laplacian, with Dirichlet and Neumann Boundary Conditions in the exterior Ω of a bounded Lipschitz domain. As a consequence we present suitable Helmholtz decompositions of vector fields defined on Ω. As a further application of our methods we study similar regularity issues for the 3-dimensional Stokes System in Ω.

1. Introduction

In this paper we initiate the study of regularity of Boundary Value Problems on the complement Ω of a bounded Lipschitz domain (commonly referred to as an exterior domain). In particular, we obtain optimal results for Poisson's equation for the Laplacian with both, Dirichlet and Neumann boundary conditions. As a further application we also present some regularity results for the Poisson's problem associated with the 3-dimensional stationary Stokes System. More specifically (see Section 1 for the notation), we consider the problems

$$\begin{cases} \Delta u = f \quad \text{in } \Omega \\ \mathbf{B}u = g, \end{cases}$$

were $\mathbf{B}$ stands for either the Dirichlet or the Neumann boundary operator and the three-dimensional Stoke's system

$$\begin{cases} \Delta \mathbf{u} = \nabla \pi + \mathbf{f} \quad \text{in } \Omega \\ \mathbf{div}\, \mathbf{u} = 0 \\ \mathbf{u}|_{\Omega} = 0 \end{cases}.$$

for an unknown vector-valued function $\mathbf{u}$. We refer the reader to Section 9 for the specifics. The dimensional restriction is due to the estimates in [3], to which we refer in Section 9. The standard Sobolev spaces used for regularity in the case of bounded domains are in general not suitable for the treatment of boundary value problems in exterior domains. Instead, weighted Sobolev spaces are the natural alternative. Several authors have succesfully used weighted Sobolev spaces in the study of boundary value problems in exterior smooth domains, in particular Amrouché et al (see [1] and [2]). Their methods, however are heavily dependent on smoothness (except for $p = 2$) and cannot be carried over to domains with rough boundaries. We present a Potential-Theoretic approach, which, in addition to being specially fitted for the non-smooth situation, has the advantage of producing

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an explicit solution to each of the above problems, expressed in terms of the Newtonian Potential of the domain and a boundary Layer Potential (see Theorems 5.1, 5.7 and 8.4.)

2. Function Spaces

In this section we introduce the spaces of functions which are suitable for our study of regularity in the exterior of a bounded Lipschitz domain.

2.1. **Definitions and Notation.** $\mathbb{R}$ In what follows, unless otherwise stated, Ω will stand for $\mathbb{R}^n \setminus \overline{\tilde{\Omega}}$, where $\tilde{\Omega} \subset \mathbb{R}^n$ is a bounded Lipschitz domain with a connected boundary, in turn denoted by $\partial\Omega$. The ball of radius r centered at x will be denoted by $B_r(x)$. As usual, $\mathcal{S}$ will stand for the space of rapidly decreasing functions and (for any domain $\Sigma \subseteq \mathbb{R}^n$), $C_0^\infty(\Sigma)$ will denote the space of smooth compactly supported functions in Σ. The dual of a topological vector space X and the transpose of an operator T will be indicated by X^* and T^* respectively.
Let I be big enough so that $\mathbb{R}^n \setminus \Omega \subset B_{2^{I-1}}(0)$. Let $(\psi_i)_i$ be the partition of the unity given in [23], subordinated to the open cover $(\Phi_i)_i$ of $\mathbb{R}^n$ defined by

$$\Phi_0 = \{x : x \in \mathbb{R}^n\,,\, |x| < 4\}\,,$$

and

$$\Phi_i = \left\{x \in \mathbb{R}^n\,,\, 2^{i-1} < |x| < 2^{i+2}\right\},$$

and such that $\psi_i \geq 0$ for all i and that for any multi-index α there is a positive constant $c(\alpha)$ for which

$$|D^\alpha \psi_i(x)| \leq c(\alpha) 2^{-i|\alpha|} \text{ for all } x \in \mathbb{R}^n. \tag{2.1}$$

We refer the reader to [23] for the definition of the potential (Sobolev) and Besov spaces, $H^p_s(\mathbb{R}^n)$ (for $1 \leq p < \infty$, $s \in \mathbb{R}$) and $B^{p,q}_s(\mathbb{R}^n)$) respectively, with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Given $f \in \mathcal{S}^*$, the distribution $f \circ (2^j x)$ acts on $\phi \in \mathcal{S}$ as

$$\langle f \circ (2^j x), \phi \rangle = 2^{-jn} \langle f, \phi(2^{-j} \cdot) \rangle.$$

For $\mu \in \mathbb{R}$, $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq q \leq \infty$, the weighted Besov spaces are defined as follows:

$$B^{p,q}_{s,\mu} = \left\{ f : f \in \mathcal{S}^*\left(\mathbb{R}^n\right), \|f\|_{B^{p,q}_{s,\mu}} = \left(\sum_0^\infty 2^{j(\mu - sp + n)} \left\|(f\psi_j)(2^j x)\right\|^p_{B^{p,q}_s} \right)^{\frac{1}{p}} < \infty \right\}. \tag{2.2}$$

Similarly, for $1 < p < \infty$, $s, \mu \in \mathbb{R}$, the weighted potential spaces $H^p_{s,\mu}$ are defined as:

$$H^p_{s,\mu} = \left\{ f : f \in \mathcal{S}^*\left(\mathbb{R}^n\right), \|f\|_{H^p_{s,\mu}} = \left(\sum_0^\infty 2^{j(\mu - sp + n)} \left\|(f\psi_j)(2^j x)\right\|^p_{H^p_s} \right)^{\frac{1}{p}} < \infty \right\} \tag{2.3}$$

Furnished with the natural norms, $B^{p,q}_{s,\mu}$ and $H^p_{s,\mu}$ become Banach spaces and have been extensively studied in [25] and [26]. In particular, it should be pointed out that the above spaces are independent of the choice of the partition of unity $(\psi_j)_j$ (see [26]). We mention for future reference that for $0 \leq s = [s] + \{s\}$, $0 \leq \{s\} < 1$, the spaces $B^{p,q}_{s,\mu}$ ($H^p_{s,\mu}$) (for p,q and μ as above), can alternatively be characterized as follows:

$$B^{p,q}_{s,\mu} = \left\{ f : f \in \mathcal{S}^* \left(\mathbb{R}^n\right), \left(\sum_0^\infty 2^{j\mu} \left\| f\psi_j \right\|^p_{\dot{B}^{p,q}_s} \right)^{\frac{1}{p}} < \infty \right\}. \tag{2.4}$$

and

$$H^p_{s,\mu} = \left\{ f : f \in \mathcal{S}^* \left(\mathbb{R}^n\right), \left(\sum_0^\infty 2^{j\mu} \left\| f\psi_j \right\|^p_{\dot{H}^p_s} \right)^{\frac{1}{p}} < \infty \right\}, \tag{2.5}$$

where

$$\|f\|_{\dot{B}^{p,p}_s} = \left(\iint \frac{|\nabla^{[s]} f(x) - \nabla^{[s]} f(y)|^p}{|x-y|^{n+\{s\}p}} \right)^{\frac{1}{p}}$$

and

$$\|f\|_{\dot{H}^p_s} = \left\| \mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f)) \right\|_{L^p},$$

$\mathcal{F}$ standing for the Fourier transform. We now describe other norms, that generate the topology of the above spaces and which will prove useful in due time. (see [25], [26] [27]). For $x \in \mathbb{R}^n$, let $\varrho(x) = (1+|x|^2)^{\frac{1}{2}}$.

Theorem 2.1. *If s is a non-negative integer and $\mu \in \mathbb{R}$, the norm $\|\cdot\|^*$ defined as*

$$\|u\|^*_{H^p_{s,\mu}} = \sum_{\alpha=0}^{s} \left\| \varrho^{\frac{\mu}{p} - (s-|\alpha|)} \left| D^\alpha u \right| \right\|_{L^p} \tag{2.6}$$

is equivalent to $\|\cdot\|_{H^p_{s,\mu}}$. If $0 \le s = [s] + \{s\}$ with $[s] \in \mathbb{Z}$ and $0 < \{s\} < 1$, then

$$\|u\|^*_{B^{p,p}_{s,\mu}} = \|u\|_{H^p_{[s],\mu-\{s\}p}} + \left(\iint \sum_{|\alpha|=[s]} \frac{\left| \varrho(x)^{\frac{\mu}{p}} D^\alpha u(x) - \varrho(y)^{\frac{\mu}{p}} D^\alpha u(y) \right|^p}{|x-y|^{n+\{s\}p}} \, dxdy \right)^{\frac{1}{p}} \tag{2.7}$$

is a norm that generates the topology of $B^{p,p}_{s,\mu}$.

For the proof, we refer to [25] and [26]. In the sequel, unless otherwise stated, we will use $\|\cdot\|_X$ for any norm that generates the topology of the space X.
Next, for any domain $\Sigma \subset \mathbb{R}^n$, let

$$\mathcal{E}_\Sigma : C^\infty_0(\Sigma) \to C^\infty_0$$

be the "extension by zero outside Σ". For $s, \mu \in \mathbb{R}$, $1 < p, q < \infty$, we define

$$B^{p,q}_{s,\mu}(\Sigma) = \{ f : f \in (C^\infty_0)^*(\Sigma) \text{ and } f = \mathcal{E}^*_\Sigma(F), F \in B^{p,q}_{s,\mu} \} \tag{2.8}$$

and

$$H^p_{s,\mu}(\Sigma) = \{ f : f \in (C^\infty_0)^*(\Sigma) \text{ and } f = \mathcal{E}^*_\Sigma(F), F \in H^p_{s,\mu} \}, \tag{2.9}$$

and furnish each space with the corresponding quotient norm, namely

$$\|f\|_{B^{p,q}_{s,\mu}(\Sigma)} := \inf\{ \|F\|_{B^{p,q}_{s,\mu}} : \mathcal{E}^*_\Sigma F = f \}$$

and

$$\|f\|_{H^p_{s,\mu}(\Sigma)} := \inf\{ \|F\|_{H^p_{s,\mu}} : \mathcal{E}^*_\Sigma F = f \}.$$

A straightforward application of the Hahn-Banach Theorem shows that for $s<0$, the above space $B^{p,q}_{s,\mu}(\Omega)$ coincide with the space of distributions $f\in C_0^\infty(\Omega)$ for which the norm

$$\|f\|_{B^{p,q}_{s,\mu}(\Omega)}=\sup\left\{|\langle f,\phi\rangle|\ ,\ \phi\in C_0^\infty(\Omega)\,,\ \|\phi\|_{B^{p',q'}_{s,-\frac{\mu p'}{p}}}=1\right\}<\infty,$$

for p' and q' the Hölder conjugates of p and q respectively. A similar statement holds for the space $H^p_{s,\mu}(\Omega)$. We notice in passing that on locally integrable functions, $\mathcal{E}^*_\Sigma$ coincides with the restriction to Σ. Observe that the obvious modifications in the norms (2.6) and (2.7) provide norms on $H^p_{s,\mu}(\Omega)$ (for integer $s\geq 0$) and on $B^{p,q}_{s,\mu}(\Omega)$ (for fractional $s\geq 0$) respectively, that are equivalent to the infimum norms just defined. To the effect of simplifying the notation, in the sequel, we will write $B^{p,p}_{s,\mu}$ ($B^{p,p}_{s,\mu}(\Omega)$) as $B^p_{s,\mu}$ ($B^p_{s,\mu}(\Omega)$) and adopt the convention that $A^p_{s,\mu}$ ($A^p_{s,\mu}(\Omega)$) represents either $B^p_{s,\mu}$ ($B^p_{s,\mu}(\Omega)$) or $H^p_{s,\mu}$ ($H^p_{s,\mu}(\Omega)$), with the obvious modification to denote the usual spaces with weight 1.

Notice that if Σ is a bounded domain the weighted spaces defined above coincide with the usual Besov and Sobolev spaces $B^{p,q}_s(\Sigma)$ and $H^p_s(\Sigma)$ respectively. We point out that the trace operator maps both $B^{p,q}_{s,\mu}(\Omega)$ and $H^p_{s,\mu}(\Omega)$ into $B^{p,q}_{s-\frac{1}{p}}(\partial\Omega)$ for $\frac{1}{p}<s<1+\frac{1}{p}$ (see [8], [12] and [13]).

A distribution $\varphi\in(C_0^\infty)^*(\Omega)$ is said to belong to $A^p_{s,\mathrm{loc}}(\Omega)$ if it coincides with some distribution $g\in A^p_s(K)$ on any bounded domain $K\subset\overline{\Omega}$, meaning $(\mathcal{E}^\Omega_K)^*(\varphi)\in A^p_s(K)$, where $(\mathcal{E}^\Omega_K)$ is the extension to Ω by 0 acting on $\mathcal{D}(K)$.

Lemma 2.2. *Let $f\in(C_0^\infty)^*(\Omega)$, $s>0$ ($s\geq 0$ if $A^p_{s,\mu}=H^p_{s,\mu}$), $\mu\in\mathbb{R}$ and I as above. Then the following statements are equivalent:*

(i) $f\in A^p_{s,\mu}(\Omega)$

(ii) $f\in A^p_{s,loc}(\Omega)$ *and*

$$\left(\|f(\sum_{j=0}^{I-1}\psi_j)\|^p_{A^p_s(\Omega\cap B_{2^{I+1}}(0))}+\sum_{j=I}^{\infty}2^{\mu j}\,\|f\psi_j\|^p_{\dot{A}^p_s}\right)^{\frac{1}{p}}<\infty. \tag{2.10}$$

(iii) $f\in A^p_{s,loc}(\Omega)$ *and*

$$\left(\|f(\sum_{j=0}^{I-1}\psi_j)\|^p_{A^p_s(\Omega\cap B_{2^{I+1}}(0))}+\sum_{j=I}^{\infty}2^{j(\mu-sp+n)+I(sp-n)}\left\|f\psi_j\circ 2^{j-I}\right\|^p_{A^p_s}\right)^{\frac{1}{p}}<\infty. \tag{2.11}$$

Moreover, the quantities (2.10) and (2.11) are norms, equivalent to the norm $A^p_{s,\mu}(\Omega)$.

In particular,

$$\sum_{i=0}^{\infty}\psi_i f=f$$

in $A^p_{s,\mu}(\Omega)$.

Proof. Let $f=F|_\Omega$ for $F\in A^p_{s,\mu}$. Since for any bounded domain $\Omega_1\subset\overline{\Omega}$, $f|_{\Omega_1}=F|_{\Omega_1}=(F\sum_0^N\psi_i)|_{\Omega_1}\in A^p_s(\Omega_1)$ for big enough N, it is clear that $f\in A^p_{s,loc}(\Omega)$

(with natural estimates), and it easily follows that (ii) is bounded by a positive constant times $\|f\|_{A^p_{s,\mu}(\Omega)}$. Let f satisfy (ii). Put

$$F = G\sum_{j=0}^{I-1}\psi_j + f\sum_{j=I}^{\infty}\psi_j, \tag{2.12}$$

where $G \in A^p_s$, $G = f$ on $\Omega \cap B_{2^{I+1}}(0)$ and

$$\|G\|_{A^p_s} \le C\|f\|_{A^p_s(\Omega\cap B_{2^{I+1}}(0))}.$$

Then $\mathcal{E}^*_\Omega(F) = f$ and one has the estimate

$$\sum_{j=0}^{\infty} 2^{\mu j}\,\|F\psi_j\|^p_{\dot{A}^p_s} = \sum_{j=0}^{\infty} 2^{\mu j}\left\|G\sum_{i=max(0,j-3)}^{min(I-1,j+3)}\psi_i\psi_j + f\sum_{k=max(I,j-3)}^{j+3}\psi_k\psi_j\right\|^p_{\dot{A}^p_s}$$

$$\le 2^{p-1}\sum_{j=0}^{\infty}2^{\mu j}\left(\left\|G\sum_{i=0}^{j+3}\psi_i\psi_j\right\|^p_{\dot{A}^p_s} + \left\|\sum_{max(j-3,I)}^{j+3}\psi_j\psi_k f\right\|^p_{\dot{A}^p_s}\right)$$

$$\le C(\|G\|^p_{A^p_s} + \sum_{j=0}^{\infty}2^{\mu j}\,\|f\psi_j\|^p_{\dot{A}^p_s}), \tag{2.13}$$

for a positive constant C independent of j (Lemma (3) in [25]). Thus, $f \in A^p_{s,\mu}(\Omega)$ with norm bounded by (2.13).

The equivalence $(ii) \Leftrightarrow (iii)$ follows from Theorem 1 in [25]. □

For $s \in \mathbb{R}$, $1 < p, q < \infty$, yet two other scales of Banach spaces is introduced, whose relevance will be apparent later:

$$\tilde{B}^{p,q}_{s,\mu}(\Omega) = \left\{f : f \in B^{p,q}_{s,\mu} \text{ and } \operatorname{supp} f \subset \overline{\Omega}\right\},$$

and

$$\tilde{H}^p_{s,\mu}(\Omega) = \left\{f : f \in H^p_{s,\mu} \text{ and } \operatorname{supp} f \subset \overline{\Omega}\right\},$$

furnished (respectively) with the norms

$$\|f\|_{\tilde{B}^{p,q}_{s,\mu}(\Omega)} = \|f\|_{B^{p,q}_{s,\mu}}$$

and

$$\|f\|_{\tilde{H}^p_{s,\mu}(\Omega)} = \|f\|_{H^p_{s,\mu}}.$$

It is easy to check that $B^{p,q}_{s,\mu}(\Omega)$ and $H^p_{s,\mu}(\Omega)$ are quasi-Banach (Banach) spaces for $0 < p, q \le \infty$ $(1 \le p, q \le \infty)$.

Lemma 2.3. *$C^\infty_0(\Omega)$ is dense in $\tilde{B}^{p,q}_{s,\mu}(\Omega)$ and in $\tilde{H}^p_{s,\mu}(\Omega)$*

Proof. The translation operator is continuous on $H^p_{k,\mu}$ for $k \in \mathbb{N}$ (see [17]), whence by interpolation (see [26]) it is also bounded on $B^{p,q}_{s,\mu}$ and $H^p_{s,\mu}$ for $s, \mu \in \mathbb{R}$. It follows from a straightforward calculation that $\tau_h f \to f$ in $B^{p,q}_{s,\mu}$ and in $H^p_{s,\mu}$ for smooth f. The proof of the lemma follows then as in the bounded case in [12]. □

Theorem 2.4. *Let $s, \mu \in \mathbb{R}$ and $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then*

$$\left(\tilde{B}^{p,q}_{s,\mu}(\Omega)\right)^* = B^{p',q'}_{-s,-\frac{p'}{p}\mu}(\Omega) \tag{2.14}$$

and

$$\left(B^{p,q}_{s,\mu}(\Omega)\right)^* = \tilde{B}^{p',q'}_{-s,-\frac{p'}{p}\mu}(\Omega). \tag{2.15}$$

Similarly,

$$\left(\tilde{H}^{p}_{s,\mu}(\Omega)\right)^* = H^{p'}_{-s,-\frac{p'}{p}\mu}(\Omega) \tag{2.16}$$

and

$$\left(H^{p}_{s,\mu}(\Omega)\right)^* = \tilde{H}^{p'}_{-s,-\frac{p'}{p}\mu}(\Omega). \tag{2.17}$$

Proof. We present the proof of the first two statements, since the assertions corresponding to the potential spaces can be dealt with along the same lines. For $\Lambda \in \left(\tilde{B}^{p,q}_{s,\mu}(\Omega)\right)^*$, denote by τ the map $\tau(\Lambda) = \mathcal{E}^*_\Omega(\tilde{\Lambda})$, where $\tilde{\Lambda}$ stands for the Hahn-Banach extension of Λ to $B^{p,q}_{s,\mu}$. It is not hard to check that τ is an isometric isomorphism from $\left(\tilde{B}^{p,q}_{s,\mu}(\Omega)\right)^*$ onto $B^{p',q'}_{-s,-\frac{p'}{p}\mu}(\Omega)$, by virtue of the duality theorems in [26]. This proves (2.14). The identity (2.15) follows by observing that the map

$$\Lambda \to \Lambda\mathcal{E}^*_\Omega$$

is an isometric isomorphism from $(B^{p,q}_{s,\mu}(\Omega))^*$ onto $\tilde{B}^{p'q'}_{-s,-\mu\frac{p'}{p}}(\Omega)$. Indeed, for $\Lambda \in (B^{p,q}_{s,\mu}(\Omega))^*$, $\Lambda\mathcal{E}^*_\Omega \in B^{p',q'}_{-s,-\frac{p'}{p}\mu}$ via the usual identification, moreover

$$\|\Lambda\mathcal{E}^*_\Omega\|_{B^{p',q'}_{-s,-\frac{p'}{p}\mu}} \leq \|\Lambda\|_{(B^{p,q}_{s,\mu}(\Omega))^*}$$

and it can be readily checked that the support of the latter is contained in $\overline{\Omega}$. Conversely, taking $\tilde{\Lambda} \in \tilde{B}^{p',q'}_{-s,-\frac{p'}{p}\mu}(\Omega)$, the linear functional $\Lambda \in (B^{p,q}_{s,\mu}(\Omega))^*$ given by

$$\langle \Lambda, \phi \rangle = \langle \tilde{\Lambda}, g \rangle$$

for $\phi = \mathcal{E}^*_\Omega(g)$ is well defined and its norm is controlled by the norm of Λ.

Definition 2.5. *Let X_i be a sequence of Banach spaces. Then $l^p((X_i)_i)$ denotes the Banach space of sequences $(a_i)_i$ with $a_i \in X_i$ such that*

$$\|(a_i)_i\|_{l^p((X_i)_i)} = \left(\sum \|a_i\|^p_{X_i}\right)^{\frac{1}{p}} < \infty,$$

furnished with the obvious norm.

For fixed j consider the homeomorphism Υ_j on $\mathcal{S}(\mathbb{R}^n)$ given by composition with 2^{I-j}, i.e.,

$$\Upsilon_j(\phi)(x) = 2^{(I-j)n}\phi(2^{I-j}x).$$

Put $A^p_s(j) = (\Upsilon^*_j)^{-1}(A^p_s)$ furnished with the norm

$$\|\cdot\|_{A^p_s(j)} = 2^{\frac{j}{p}(\mu-sp+n)+I(s-\frac{n}{p})}\|\Upsilon^*_j(\cdot)\|_{A^p_s}.$$

Then, we have the following lemma:

Lemma 2.6. *For $s \geq 0$, $1 < p < \infty$, $\mu \in \mathbb{R}$, the space $A^p_{s,\mu}(\Omega)$ is a retract of $l^p((X_i)_i)$, where $X_0 = A^p_s(\Omega \cap B_{2^{I+1}}(0))$ and $X_i = A^p_s(I + i - 1)$ for $i \geq 1$.*

Proof. It follows as in [26] : let $(\chi_i)_i$ and $(\psi_i)_i$ be two systems of functions as in (2.1), with $\psi_i = 1$ on the support of χ_i. Then the map

$$R : A^p_{s,\mu}(\Omega) \longrightarrow l^p(A^p_s(i))$$
$$R(f) = (f_i)_i$$

where $f_0 = f\sum_{j\le I-1}\chi_j$ and $f_i = f\chi_{I+i-1}$ for $i \ge 1$, is a retraction. In fact, it is not hard to see that $\sum_{i=1}^{\infty}\psi_{I+i-1}a_i$ converges in $A^p_{s,\mu}$, and that for

$$T : l^p((X_i)_i) \longrightarrow A^p_{s,\mu}(\Omega)$$
$$T((a_i)_i) = a_0 + \mathcal{E}^*_{\Omega}\left(\sum_{i=1}^{\infty}\psi_{I+i-1}a_i\right)$$

TR is the identity on $A^p_{s,\mu}(\Omega)$. □

It follows now (see [27]) that for $1 < p_1, p_2 < \infty$, $\mu_1, \mu_2 \in \mathbb{R}$ and $0 \le s_1, s_2$, for $0 < \theta < 1$,

$$\left[A^{p_1}_{s_1,\mu_1}(\Omega), A^{p_1}_{s_2,\mu_2}(\Omega)\right]_{\theta} = A^p_{s,\mu}(\Omega),$$

and

$$\left[H^{p_1}_{s_1,\mu_1}(\Omega), H^{p_2}_{s_2,\mu_2}(\Omega)\right]_{p,\theta} = B^p_{s,\mu}(\Omega)$$

with $s = (1-\theta)s_1 + \theta s_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{\mu}{p} = \frac{(1-\theta)\mu_1}{p_1} + \frac{\theta\mu_2}{p_2}$.

Corollary 2.7. *Let $s \ge 0$,$1 < p < \infty$ and $\mu \in \mathbb{R}$, the following interpolation results hold:*

$$\left[A^p_{0,\mu p}(\Omega), \tilde{A}^p_{-2,-\mu p}(\Omega)\right]_{\theta} = \tilde{A}^p_{-2\theta,(1-2\theta)\mu p}(\Omega)$$

$$\left[H^p_{0,\mu p}(\Omega), \tilde{H}^p_{-2,-\mu p}(\Omega)\right]_{\theta,p} = \tilde{B}^p_{-2\theta,(1-2\theta)\mu p}(\Omega),$$

$$\left[H^p_{2,\mu p}(\Omega), H^p_{0,-\mu p}(\Omega)\right]_{\theta,p} = B^p_{2(1-\theta),(1-2\theta)\mu p}(\Omega),$$

$$\left[A^p_{2,\mu p}(\Omega), A^p_{0,-\mu p}(\Omega)\right]_{\theta} = A^p_{2(1-\theta),(1-2\theta)\mu p}(\Omega).$$

Proof. The proof of the Lemma follows from Lemma 2.6 and the interpolation results (Theorem 3) in [26]. The corollary is a direct consequence of the Lemma and the duality theorems for the real and complex interpolation methods. This Theorem can be adapted to other situations. In particular, a vector valued version that will be tacitly employed in Section 9 can be obtained by carrying out minor changes in the proof presented above. □

3. The Newtonian Potential on Unbounded Domains

Definition 3.1. *Let $\Lambda \in \mathcal{E}'$, where $\mathcal{E}'$ is the space of distributions with compact support. The Newtonian potential of $\Lambda \in \mathcal{E}'$,$\mathbf{L}(\Lambda)$ is defined to be the convolution*

$$\mathcal{N}(\Lambda) = \mathcal{K} * \Lambda \tag{3.1}$$

where $\mathcal{K}$ is the distribution defined by the locally integrable function ($n \ge 3$):

$$\frac{1}{n(2-n)\omega_n|X|^{n-2}}, \tag{3.2}$$

and ω_n is the surface measure of the unit ball in $\mathbb{R}^n$.

Let $\mathcal{R}_\Omega$ be the restriction map to Ω, so that

$$\mathcal{R}_\Omega : H^p_{s,\mu p} \longrightarrow H^p_{s,\mu p}(\Omega).$$

Notice that $\mathcal{R}_\Omega = \mathcal{E}^*_\Omega$ on locally integrable functions. The Newtonian potential of $f \in H^p_{0,(\delta+2)p}(\Omega)$ is defined as

$$\mathcal{N}_\Omega(f) = \mathcal{E}^*_\Omega \mathcal{N} \mathcal{R}^*_\Omega(f). \tag{3.3}$$

The next theorem is the key to the mapping properties of $\mathcal{N}_\Omega$:

Theorem 3.2. *Let $n \geq 3\,, 1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $-\frac{n}{p} < \delta_1 < -2 + \frac{n}{q}$ $-\frac{n}{q} < \delta_2 < -2 + \frac{n}{p}$ and $0 < s < 2$. Then the Newtonian Potential is a bounded linear map from $\tilde{A}^p_{s-2,(\frac{s}{2}(\delta_1+\delta_2)-\delta_2+2(s-1))p}(\Omega)$ into $A^p_{s,(\frac{s}{2}(\delta_1+\delta_2)-\delta_2+2(s-1))p}(\Omega)$*

Proof. Let A_r denote the Muckenhoupt class for $r \in [1,\infty]$. For $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$ and $-\frac{n}{r} < \delta < -2 + \frac{n}{r'}$, one has $(1+|x|)^{\delta r} \in A_r \subseteq A_\infty$, whereas $(1+|x|)^{-(\delta+2)r'} \in A_{r'}$. Moreover, it can be easily verified by direct calculation that there exists a positive constant C such that for $i = 1,2$ and any ball $B \subset \mathbb{R}^n$,

$$|B|^{r(\frac{i}{n}-1)} \left(\int_B (1+|x|)^{(\delta+2-i)r}\, dx \right) \left(\int_B (1+|x|)^{\frac{-(\delta+2)r}{r-1}}\, dx \right)^{r-1} \leq C.$$

Thus, the boundedness of the Riesz Potentials

$$I_i(f) = \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-i}} : H^r_{0,(\delta+2)r} \to H^p_{0,(\delta+2-i)r} \;\; i = 1,2$$

is guaranteed by the results in [21]. It can be readily verified that

$$\mathcal{R}^*_\Omega : H^p_{0,(\delta+2)p}(\Omega) \to H^p_{0,(\delta+2)p}$$

boundedly (so that in particular (3.3) is well defined) whence $\mathcal{N}_\Omega$ is a bounded linear map from $H^p_{0,(\delta_1+2)p}(\Omega)$ into $H^p_{2,(\delta_1+2)p}(\Omega)$ and from $H^q_{0,(\delta_2+2)q}(\Omega)$ into $H^q_{2,(\delta_2+2)q}(\Omega)$. We claim next that

$$\mathcal{N}^*_\Omega : H^q_{-2,-(\delta_1+2)q} \to H^q_{0,-(\delta_1+2)q}$$

is also given by convolution with $\mathcal{K}$ (in particular, the convolution with $\mathcal{K}$ is well defined on $H^q_{-2,-(\delta_1+2)q}$). Indeed, for $\Lambda \in H^q_{-2,-(\delta+2)q}$, $\psi \in H^p_{0,(\delta_1+2)p}$, one has

$$\langle \mathcal{N}^*_\Omega(\Lambda), \psi \rangle = \langle \Lambda, \int \Gamma(y)\psi(x-y)dy \rangle = \langle \Lambda, \int \Gamma(z-x)\psi(z)dz \rangle$$
$$= \langle \Lambda, \int \Gamma(y)\psi(x+y)dy \rangle = \langle \Lambda, \langle \Gamma, \tau_x \psi \rangle \rangle = \langle \Lambda * \Gamma, \psi \rangle.$$

Theorem 3.2 follows now by duality and interpolation. □

We highlight a particular case, namely

Corollary 3.3. *For $1 < p < \infty$ and δ_1, δ_2 as in theorem 3.2, the map*

$$\mathcal{N}_\Omega : \tilde{A}^{p,p}_{-1,(\delta_1-\delta_2)\frac{p}{2}}(\Omega) \to A^{p,p}_{1,(\delta_1-\delta_2)\frac{p}{2}}(\Omega)$$

is bounded and linear. In particular, for $\frac{n}{n-1} < p < n$,

$$\mathcal{N}_\Omega : \tilde{H}^p_{-1,0}(\Omega) \to H^p_{1,0}(\Omega)$$

is bounded and linear.

4. Mapping Properties of the Boundary Layers

For $n \geq 3$, f a measurable function on $\partial\Omega$ we denote the exterior unit normal of Ω at P by $N(P)$ and define the double layer potential with density f at $X \in \Omega$ as

$$\mathcal{D}f(X) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{(Q - X) \cdot N(Q)}{|X - Q|^n} f(Q)\, d\sigma(Q). \tag{4.1}$$

For $X \in \mathbb{R}^n$ and a distribution $\Lambda \in \mathcal{L}'(\partial\Omega)$ the single layer potential with density Λ is defined as:

$$\mathbf{S}(\Lambda)(X) = \frac{1}{(2-n)\omega_n} \langle \Lambda, \frac{1}{|X - \cdot|^{n-2}} \rangle. \tag{4.2}$$

Theorem 4.1. *Let* $0 < s < 1$, δ_1, δ_2 *and* p *be as in theorem* 3.2 *and*

$$\mu = \frac{1}{2}(s + \frac{1}{p})(\delta_1 + \delta_2) - \delta_2 + 2(s + \frac{1}{p} - 1).$$

Then

$$\mathcal{D} : B^p_s(\partial\Omega) \longrightarrow B^p_{s+\frac{1}{p},\mu p}(\Omega) \cap H^p_{s+\frac{1}{p},\mu p}(\Omega) \tag{4.3}$$

and

$$\mathbf{S} : B^p_{s-1}(\partial\Omega) \longrightarrow B^p_{s+\frac{1}{p},\mu p}(\Omega) \cap H^p_{s+\frac{1}{p},\mu p}(\Omega) \tag{4.4}$$

are bounded linear maps (the intersection spaces are furnished with the usual maximum norm).

Proof. Recall I is large enough so that $\mathbb{R}^n \setminus \Omega$ is contained in ball of radius 2^{I-1} centered at 0. To start with, we point out that from the trace theorem and the results in [8],

$$\|\mathcal{D}f\|_{B^p_{s+\frac{1}{p}}(\Omega \cap B_{2^I}(0))} \leq C \|f\|_{B^p_s(\partial\Omega)}.$$

It is easy to see that for $i \geq I$,

$$\|\mathcal{D}f\|^p_{L^p(\Phi_i)} \leq 2^{(n-1)(1-p)+1} \|f\|^p_{B^p_s(\partial\Omega)}.$$

On the other hand, for $i \geq I$, it follows from (2.1) that for $0 < s + \frac{1}{p} < 1$,

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus \Phi_i} \int_{\Phi_i} \frac{|\mathcal{D}f(X)\psi_i(X)|^p}{|X - Y|^{n+(s+\frac{1}{p})p}} \\ &\leq \|f\|_{L^p(\partial\Omega)} \int_{\Phi_i} \frac{dX}{|X|^{(n-1)p}} \left(\int_{|Y| \leq 2^{i-1}} + \int_{|Y| \geq 2^{i+2}} \right) \frac{|\psi_i(X)|^p}{|X - Y|^{n+(s+\frac{1}{p})p}}. \end{aligned} \tag{4.5}$$

The first of the two integrals above is dominated by

$$\begin{aligned} &\|f\|_{L^p(\partial\Omega)} \int_{\Phi_i} \frac{dX}{|X|^{(n-1)p}} \int_{|X|-2^{i-1} < |X-Y| < |X|+2^{i-1}} \frac{dY}{|X - Y|^{n+(s+\frac{1}{p})p}} \\ &\leq 2^{i(n-np-(s+\frac{1}{p}-1)p)} \|f\|_{L^p(\partial\Omega)}. \end{aligned}$$

As for the second integral, it can be easily seen to be bounded above by

$$\|f\|_{L^p(\partial\Omega)} \int_{\Phi_i} \frac{dX}{|X|^{(n-1)p}} \left(\int_{|X-Y|<2^i} + \int_{|X-Y|>2^i} \right) \frac{|\psi_i(X)|^p dY}{|X-Y|^{n+(s+\frac{1}{p})p}} \leq$$
$$\|f\|_{L^p(\partial\Omega)} \int_{\Phi_i} \frac{dX}{|X|^{(n-1)p}} \left(\int_{|X-Y|<2^i} \frac{2^{-ip} dY}{|X-Y|^{n+(s+\frac{1}{p}-1)p}} + \int_{|X-Y|\geq 2^i} \frac{dY}{|X-Y|^{n+(s+\frac{1}{p})p}} \right) \leq$$
$$2^{i(n-np-(s+\frac{1}{p}-1)p)} \|f\|_{L^p(\partial\Omega)}. \tag{4.6}$$

Similarly, for $1 < s + \frac{1}{p} < 2$,

$$\int_{\mathbb{R}^n \setminus \Phi_i} \int_{\Phi_i} \frac{|\nabla \mathcal{D} f(X) \psi_i(X)|^p}{|X-Y|^{n+(s+\frac{1}{p}-1)p}} \leq 2^{i(n-np-(s+\frac{1}{p}-1)p)} \|f\|_{L^p(\partial\Omega)}.$$

On the other hand,

$$\begin{aligned} &|\mathcal{D}f(X) - \mathcal{D}f(Y)| \\ &\leq \int_{\partial\Omega} \frac{|X-Y| \left(|X-Q| \, |\xi - Q|^{n-1} + |X-Q|^n \right)}{|X-Q|^n \, |Y-Q|^n} |f(Q)| \, d\sigma(Q), \end{aligned} \tag{4.7}$$

for some ξ in the segment joining X and Y, so that for $0 < s + \frac{1}{p} < 1$,

$$\begin{aligned} &2^{i\mu p} \int_{\Phi_i} \int_{\Phi_i} \frac{|\mathcal{D}f(X) - \mathcal{D}f(Y)|^p}{|X-Y|^{n+(s+\frac{1}{p})p}} dX dY \\ &\leq C 2^{i(\mu p - np)} \|f\|^p_{B^{p,p}_s(\partial\Omega)} \int_{\Phi_i} \int_{\Phi_i} \frac{dY dX}{|X-Y|^{n+(s+\frac{1}{p})p-p}} \\ &\leq C 2^{i(\mu p - np + n - (s+\frac{1}{p}-1)p)} \|f\|^p_{B^{p,p}_s(\partial\Omega)}. \end{aligned} \tag{4.8}$$

Analogously,

$$|\nabla \mathcal{D} f(X) - \nabla \mathcal{D} f(Y)| \leq C \|f\|_{L^p(\Omega)} |X-Y| \, 2^{-i(n+1)}, \tag{4.9}$$

whence, for $1 < s + \frac{1}{p} < 2$,

$$\begin{aligned} &2^{i\mu p} \int_{\Phi_i} \int_{\Phi_i} \frac{|\nabla\mathcal{D}f(X) - \nabla\mathcal{D}f(Y)|^p}{|X-Y|^{n+(s+\frac{1}{p}-1)p}} dX dY \\ &\leq C 2^{i(\mu p - (n+1)p)} \|f\|^p_{B^{p,p}_s(\partial\Omega)} \int_{\Phi_i} \int_{\Phi_i} \frac{dY dX}{|X-Y|^{n+(s+\frac{1}{p})p-p-p}} \\ &\leq C 2^{i(\mu p - np + n - (s+\frac{1}{p}-1)p)} \|f\|^p_{B^p_s(\partial\Omega)}. \end{aligned} \tag{4.10}$$

It remains to tackle the case $s + \frac{1}{p} = 1$. Let $f \in B^p_{1-\frac{1}{p}}(\partial\Omega)$, and choose $\chi \in C_0^\infty(B_{2^I(0)})$, with $\chi = 1$ on $B_{2^{I-1}}(0)$. Take $G \in H^p_1$ be such that $\mathcal{E}^*_\Omega(G) = \mathcal{E}^\Omega_{\Omega \cap B_{2^{I-1}(0)}} \mathcal{D}(f)$ and put

$$F = G\chi + \mathcal{R}^*_\Omega \left((1-\chi)\mathcal{D}(f) \right).$$

Then one has $\mathcal{E}^*_\Omega F = \mathcal{D} f$ and $F \in H^p_{1,\mu p}$, which can be easily verified, since

$$\|\nabla(\psi_i \mathcal{D}(f)\|_{L^p} \leq 2^{i(n-np)} \|f\|_{B^p_s(\partial\Omega)}.$$

Moreover, let $F_i \in B_1^p$ be an extension of $\mathcal{D}(f)|_{\Phi_i}$ with (see [8])

$$\|F_i\|_{B_1^p} \le \|\mathcal{D}(f)|_{\Phi_i}\|_{B_1^p(\Phi_i)} = \|\mathcal{D}(f)|_{\Phi_i}\|_{H_1^p(\Phi_i)}.$$

The estimate

$$\|F_i\psi_i\|_{B_1^p} = \|\mathcal{D}(f)\psi_i\|_{B_1^p} \le C\|F_i\|_{B_1^p}$$

shows that $\mathcal{D}(f) \in B^p_{1,\mu p}(\Omega)$.
Let now $0 < s < 1$ and $G \in H^p_{s+\frac{1}{p}}$ be such that $\mathcal{D}(f)|_{\Omega\cap B_{2^{I-1}}(0)} = G|_{\Omega\cap B_{2^{I-1}}(0)}$ and

$$\|G\|_{H^p_{s+\frac{1}{p}}} \le C\|\mathcal{D}(f)\|_{H^p_{s+\frac{1}{p}}(\Omega\cap B_{2^{I-1}}(0))}.$$

Furthermore, for $j \ge I$, $G_j \in H^p_{s+\frac{1}{p}}$ choose G_j coinciding with $\mathcal{D}f$ on Φ_j and

$$\|G_j\|_{H^p_{s+\frac{1}{p}}} \le C\,\|\mathcal{D}(f)\|_{H^p_{s+\frac{1}{p}}(\Phi_j)} = C\,\|\mathcal{D}(f)\|_{B^{p,p}_{s+\frac{1}{p}}(\Phi_j)}\,.$$

Define

$$H = G\sum_{0}^{I-1}\psi_i + \sum_{j=I}^{\infty} G_j\psi_j.$$

Then $H = \mathcal{D}(f)$ in Ω and $H \in H^p_{s+\frac{1}{p}}$. Indeed, for $i \ge I+2$,

$$\begin{aligned} \|H\psi_i\|_{H^p_{s+\frac{1}{p}}} &= \left\|\mathcal{D}(f)\psi_i\sum_{j=i-3}^{i+3}\psi_j\right\|_{H^p_{s+\frac{1}{p}}} \\ &\le C\|\mathcal{D}(f)\psi_i\|_{H^p_{s+\frac{1}{p}}} = C\|\mathcal{D}(f)\psi_i\|_{B^p_{s+\frac{1}{p}}} \le C2^{\frac{n}{p}-n-(s+\frac{1}{p}-1)}, \end{aligned} \tag{4.11}$$

since $\mathcal{D}(f)$ is harmonic in the bounded domain Φ_i and a simple calculation shows that for every muti-index γ there is a positive constant $c(\gamma)$ such that for all $x \in \mathbb{R}^n$, the inequality

$$|D^\gamma \sum_{i-3}^{i+3}\psi_j(x)| \le c(\gamma)|x|^{-|\gamma|}$$

holds (see Lemma 3 in [25]). In conclusion, since

$$\mu p + n - np - (s + \frac{1}{p} - 1)p < 0$$

for all values of the parameters involved, one has (for $0 < C = C(\Omega, p, s)$),

$$\left(\|\mathcal{D}(f)\sum_{i=0}^{I-1}\phi_i\|^p_{B^p_{s+\frac{1}{p}}(\Omega\cap B_{2^{I-1}}(0))} + \sum_{i=I}^{\infty}2^{\mu i p}\|\mathcal{D}(f)\psi_i\|^p_{\dot{B}^p_{s+\frac{1}{p}}}\right)^{\frac{1}{p}} \le C\|f\|_{B^p_s(\partial\Omega)}$$

and

$$\left(\|\mathcal{D}(f)\sum_{i=0}^{I-1}\phi_i\|^p_{H^p_{s+\frac{1}{p}}(\Omega\cap B_{2^{I-1}}(0))} + \sum_{i=I}^{\infty}2^{\mu i p}\|\mathcal{D}(f)\psi_i\|^p_{\dot{H}^p_{s+\frac{1}{p}}}\right)^{\frac{1}{p}} \le C\|f\|_{B^p_{1-\frac{1}{p}}(\partial\Omega)}.$$

This finishes the proof of the statement concerning $\mathcal{D}$. The corresponding argument for $\mathbf{S}$ is similar and will be omitted. $\square$

4.1. **Boundary Operators.** Recall that for $P \in \partial\Omega$, the exterior unit normal to Ω at P is denoted by $N(P)$, and σ stands for the Hausdorff measure on $\partial\Omega$. For $f \in L^1(\partial\Omega)$, we define the operators

$$Kf(P) = \frac{1}{\omega_n}\int_{\partial\Omega}\frac{(Q-P)\cdot N(Q)}{|P-Q|^n}f(Q)d\sigma(Q),$$

its formal transpose,

$$K^*f(P) = \frac{1}{\omega_n}\int_{\partial\Omega}\frac{(P-Q)\cdot N(P)}{|P-Q|^n}f(Q)d\sigma(Q),$$

and

$$Sf(Q) = \frac{1}{(2-n)\omega_n}\int_{\partial\Omega}\frac{f(Q)}{|P-Q|^n}d\sigma(Q).$$

The boundary behavior of the layer potentials is given by the identities

$$\mathbf{Tr}\mathcal{D}(f) = (-\frac{1}{2}I + K)(f)$$

and

$$\frac{\partial}{\partial N}S(f) = (\frac{1}{2}I - K^*)(f).$$

For $\epsilon > 0$ let $\mathcal{R}_\epsilon$ be the region in the $(s, \frac{1}{p})$ plane inside the hexagon with vertices $A = (1-\epsilon, 1)$, $B = (1,1)$ $C = (1, \frac{1}{2}-\epsilon)$, $D = (\epsilon, 0)$, $E = (0,0)$ and $F = (0, \frac{1}{2}+\epsilon)$. Also, let $< 1 >$ be the subspace generated by constant functions and let $\dot{B}^{q,q}_r(\partial\Omega)$ stand for the distributions in $B^{q,q}_r(\partial\Omega)$ that vanish on constants. The first five statements of following theorem have been proved in [20]. The proof of statement (6) can be found in [15] :

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, with a connected boundary. Then there exists $\epsilon = \epsilon(\Omega) \in (0, 1]$, such that for $(s, \frac{1}{p}) \in \mathcal{R}_\epsilon$, $\frac{1}{p} + \frac{1}{q} = 1$, the operators listed below (which are well defined and bounded) are isomorphisms:*

(1) $\frac{1}{2}I - K : B^p_s(\partial\Omega) \to B^p_s(\partial\Omega)$
(2) $\frac{1}{2}I - K^* : B^q_{-s}(\partial\Omega) \to B^q_{-s}(\partial\Omega)$
(3) $S : B^q_{-s}(\partial\Omega) \to B^q_{1-s}(\partial\Omega)$
(4) $\frac{1}{2}I + K : B^p_s(\partial\Omega)/ < 1 > \longrightarrow B^p_s(\partial\Omega)/ < 1 >$
(5) $\frac{1}{2}I + K^* : \dot{B}^q_{-s}(\partial\Omega) \to \dot{B}^q_{-s}(\partial\Omega)$.
(6) $\frac{1}{2}I + K + S : B^p_s(\partial\Omega) \to B^p_s(\partial\Omega)$.

5. Boundary Value Problems

5.1. The Dirichlet Problem.

Theorem 5.1. *Let Ω be the complement in $\mathbb{R}^n$ of a bounded Lipschitz domain with a connected boundary. Let $1 < p < \infty$ and δ_i, $i = 1, 2$ be as in Theorem 3.2 and.*

$$\mu = \frac{-t}{2}(\delta_1 + \delta_2) + \delta_1 + 2(1-t),$$

Then there exists a positive number ϵ depending only on Ω such that whenever $(t, \frac{1}{p})$ belongs to the hexagon with vertices $(0, 1)$,$(\epsilon, 1)$,$(\frac{3}{2}-\epsilon, \frac{1}{2}+\epsilon)$, $(2, 0)$, $(2-\epsilon, 0)$ and $(\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon)$, $f \in H^p_{-t,\mu p}(\Omega)$ and $g \in B^{p,p}_{2-\frac{1}{p}-t}(\partial\Omega)$, the problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \mathbf{Tr}u = g \end{cases} \tag{5.1}$$

has a unique solution $u \in H^p_{2-t,\mu p}(\Omega)$. Moreover, $u \in B^{p,p}_{2-t,\mu}(\Omega)$ and there exists a positive constant C depending only on Ω such that

$$\max\{\|u\|_{H^p_{2-t,\mu p}(\Omega)}, \|u\|_{B^{p,p}_{2-t,\mu p}(\Omega)}\} \leq C(\|f\|_{H^p_{-t,\mu p}(\Omega)} + \|g\|_{B^{p,p}_{2-\frac{1}{p}-t}(\partial\Omega)}). \tag{5.2}$$

If Ω is C^1, the previous statement holds for $\epsilon = 1$. Notice that the region described above when $\epsilon = 0$ is common to all Lipschitz domains.

Proof. The estimates of the last section coupled with the invertibility result cited in part (6) of Theorem 4.2 yield the existence part of the theorem. Uniqueness follows from Lemma 2.3 as in [12]. The (unique) solution to the problem is thus given by

$$u(X) = \mathcal{D}(T^{-1}(h))(X) - \mathbf{S}(T^{-1}(h))(X) + \mathcal{N}_\Omega(f)(X),$$

where

$$T = -(\frac{1}{2}I + K + \mathbf{S})$$

and the density is given by

$$h = g - \mathcal{N}_\Omega(f)|_\Omega.$$

5.2. **The Neumann Problem.** With the following few lemmas we prepare the ground for the formulation of the Neumann problem.

Lemma 5.2. *Let $1 < p < \infty$, $\mu \in \mathbb{R}$ and $s \geq 0$. Then for $1 \leq k \leq n$ the Partial Derivative Operator*

$$\partial_k : H^p_{s,\mu}(\Omega) \to H^p_{s-1,\mu}(\Omega)$$

is bounded.

Proof. For $1 < p < \infty$, $0 \leq s \leq 1$, ∂_k

$$\partial_k : H^p_{s,\mu} \to H^p_{s-1,\mu}$$

is bounded (the statement is obvious for $s = 1$, hence the full range of s follows by duality and interpolation).
Now, considering $f \in H^p_{s,\mu}(\Omega)$ and $F \in H^p_{s,\mu}$ with $\mathcal{E}^*_\Omega F = f$, one can easily verify that

$$\mathcal{E}^*_\Omega \partial_i F = \partial_i F|_\Omega = \partial_i f,$$

which completes the proof □

Lemma 5.3. *Let $0 < s < 1$, $1 < p < \infty$, $\mu \in \mathbb{R}$ and $\phi \in B^p_s(\partial\Omega)$. Then there exists an extension (in the trace sense) $\tilde{\phi} \in H^p_{s+\frac{1}{p},\mu}(\Omega)$ of ϕ, so that for some positive constant $C = C(\Omega)$,*

$$\|\tilde{\phi}\|_{H^p_{s+\frac{1}{p},\mu}(\Omega)} \leq C\|\phi\|_{B^p_s(\partial\Omega)}.$$

Proof. For arbitrary $\epsilon > 0$, let $\varphi \in C^\infty_0(B_{2^I}(0))$ be equal to 1 on $B_{2^I-\epsilon}(0)$ and $\Phi \in H^p_{s+\frac{1}{p}}(\mathbb{R}^n)$ be such that $\Phi|_{\partial\Omega} = \phi$. Then $\varphi\Phi \in H^p_{s+\frac{1}{p},\mu}(\Omega)$ for any $\mu \in \mathbb{R}$ and its restriction to $\partial\Omega$ is ϕ. □

Lemma 5.4. *For $1 < p < \infty$, $\mu \in \mathbb{R}$ and $\frac{1}{p} < s \leq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ the following holds:*

$$H^p_{-s+\frac{1}{p},\mu p}(\Omega) = \left(H^q_{s-1+\frac{1}{q},-\mu q}(\Omega)\right)^*.$$

For μ and p as above, and $s \leq \frac{1}{p}$, one has

$$\left(H^p_{-s+\frac{1}{p},\mu p}(\Omega)\right)^* = H^q_{s-1+\frac{1}{q},-\mu q}(\Omega).$$

Proof. In fact, it is well known (see [11]) that for bounded Ω and $\frac{1}{p} < s \leq 1$, $C_0^\infty(\Omega)$ is dense in $H^q_{s-1+\frac{1}{q}}(\Omega)$. From this, it follows that test functions are also dense in the corresponding weighted space $H^q_{s-1+\frac{1}{q},-\mu q}(\Omega)$ for any $\mu \in \mathbb{R}$. Indeed, let $f = \mathcal{E}^*_\Omega F = F|_\Omega$ for $F \in H^q_{s-1+\frac{1}{q},-\mu q}$. Let ζ be a smooth cut off function supported in $B_{\frac{3}{2}2^{I-1}(0)} \cap (\mathbb{R}^n \backslash \mathrm{supp}\psi_I)$ with $\zeta = 1$ in $\mathbb{R}^n \backslash \Omega$. Put $\Omega_0 = \Omega \cap B_{\frac{3}{2}2^{I-1}(0)} \cap (\mathbb{R}^n \setminus \mathrm{supp}\psi_I)$. Then, $f\zeta = (\zeta F)|_{\Omega_0} \in H^q_{s-1+\frac{1}{q}}(\Omega_0)$ and can be approximated in the $H^q_{s-1+\frac{1}{q}}(\Omega_0)$-norm by $g \in C_0^\infty(\Omega_0)$. Also, F can be approximated in $H^q_{s-1+\frac{1}{q},-\mu q}$ by $h \in C_0^\infty$. Notice that for some positive constant C,

$$\|(\zeta F - g)|_{\Omega \cap B_{2^{I+1}(0)}}\|_{H^q_{s-1+\frac{1}{q}}(\Omega \cap B_{2^{I+1}}(0))} \leq C \|(\zeta F - g)|_{\Omega_0}\|_{H^q_{s-1+\frac{1}{q}}(\Omega_0)},$$

as it is easy to see by taking $\varphi \in C_0^\infty(B_{\frac{3}{2}2^{I-1}}(0) \cap (\mathbb{R}^n \setminus \mathrm{supp}\psi_I))$ identically equal to 1 on $\mathrm{supp}(\zeta F - g)$ and observing that $\varphi G|_{\Omega \cap B_{2^{I+1}}(0)} = (\zeta F - g)|_{\Omega \cap B_{2^{I+1}}(0)}$ for every $G \in H^q_{s-1+\frac{1}{q}}$ such that $G|_{\Omega_0} = (\zeta F - g)|_{\Omega_0}$. It follows that

$$\begin{aligned}&\|f - g - (1-\zeta)h\|_{H^q_{s-1+\frac{1}{q},-\mu q}(\Omega)} \leq \\ &\|(F\zeta - g)|_\Omega\|_{H^q_{s-1+\frac{1}{q},-\mu q}(\Omega)} + \|(1-\zeta)(f-h)|_\Omega\|_{H^q_{s-1+\frac{1}{q},-\mu q}(\Omega)}.\end{aligned}$$

The first term above is, according to Lemma 2.2, less than or equal to

$$\|\sum_0^{I-1} \psi_i (F\zeta - g)|_{\Omega \cap B_{2^{I+1}}(0)}\|_{H^q_{s-1+\frac{1}{q}}(\Omega \cap B_{2^{I+1}}(0))} \leq C \|(F\zeta - g)|_{\Omega_0}\|_{H^q_{s-1+\frac{1}{q}}(\Omega_0)},$$

while the second one is clearly bounded by a positive constant times

$$\|F - h\|_{H^q_{s-1+\frac{1}{q},-\mu q}}.$$

The duality assertions now follow in a standard manner (see [8]). □

Lemma 5.5. *Let $1 < p < \infty$, $0 < s < 1$, q the Hölder conjugate of p and u be a vector field on Ω whose components are in $H^p_{s+\frac{1}{p}-1,\mu p}(\Omega)$. Then $\mathrm{div}\, u$ can be extended to a distribution (still denoted by* div*) in $\tilde{H}^p_{s-1-\frac{1}{q},\mu p}(\Omega)$. The vector u has a normal component $u \cdot N \in B^p_{s-1}(\partial\Omega)$ (which depends on the particular extension* $\mathrm{div}\, u$*), defined as*

$$\langle u \cdot N, \phi \rangle = \langle u\,, \nabla \tilde{\phi} \rangle + \langle \mathrm{div}\, u, \tilde{\phi} \rangle.$$

for any $\phi \in B^q_{1-s}(\partial\Omega)$ extended to $\tilde{\phi} \in H^q_{1-s+\frac{1}{q},-\mu q}(\Omega)$ via Lemma 5.3. Moreover, $u \cdot N$ is independent of the extension $\tilde{\phi}$ and

$$\|u \cdot N\|_{B^p_{s-1}(\partial\Omega)} \leq \|u\|_{H^p_{s+\frac{1}{p}-1,\mu p}(\Omega)} + \|\mathrm{div}\, u\|_{H^p_{s-\frac{1}{q}-1,\mu p}(\Omega)}.$$

Proof. It follows from Lemma 5.4. □

In combination with Theorem 3.2 and Theorem 4.1, the above Lemma implies that if $1 < p < \infty$, $0 < s < 1$, μ as in Theorem 4.1, $f \in B^p_{s-1}(\partial\Omega)$ and $g \in$

$H^p_{s+\frac{1}{p}-2,\mu}(\Omega)$, then $\mathbf{S}(f)$ and $\mathcal{N}_\Omega(g)$ have normal derivatives $\frac{\partial \mathbf{S}(f)}{\partial N} \in B^p_{s-1}(\partial\Omega)$ and $\frac{\partial \mathcal{N}_\Omega(g)}{\partial N} \in B^p_{s-1}(\partial\Omega)$ respectively.

Lemma 5.6. *For s and f as in the previous paragraph and μ as in Theorem 4.1, one has*

$$\frac{\partial \mathbf{S}(f)}{\partial N} = (\frac{1}{2}I - K^*)(f).$$

Proof. Notice that from the continuity of the gradient operator, for $f \in B^p_{s-1}(\partial\Omega)$, one has

$$\nabla \mathbf{S}(f) = \sum_{i=0}^{I} \nabla(\psi_i \mathcal{S} f) + \sum_{i=I}^{\infty} \nabla(\psi_i \mathcal{S} f), \tag{5.3}$$

the convergence understood in $H^p_{s+\frac{1}{p}-1,\mu p}(\Omega)$. Also notice that

$$\sum_{i=0}^{I} \psi_i \mathbf{S} f = \sum_{i=0}^{I} \psi_i \tilde{\mathbf{S}} \tilde{f},$$

where $\tilde{\mathbf{S}}$ is the single layer on the domain $\Omega \cap B_{2^{(I+2)}}(0)$, and $\tilde{f}$ is f on $\partial\Omega$ and 0 on $\partial B_{2^{I+2}}(0)$. Since $\nabla \tilde{\mathbf{S}} \tilde{f} \in H^p_{\frac{1}{p}+s-1}(\Omega \cap B_{2^{I+2}}(0))$, it follows that for $\phi \in C^\infty(\overline{\Omega})$

$$\langle \sum_{i=0}^{I} \nabla(\psi_i \tilde{\mathbf{S}} \tilde{f}), \nabla\phi \rangle = \langle (\frac{1}{2}I - K^*)(f), \phi|_{\partial\Omega} \rangle - \langle \Delta(\sum_{i=0}^{I} \psi_i \tilde{\mathbf{S}} \tilde{f}), \phi \rangle.$$

On the other hand, since the second term in (5.3) is in $C_0^\infty(\Omega)$,

$$\langle \sum_{i=I}^{\infty} \nabla(\psi_i \mathbf{S} f), \nabla\phi \rangle = -\langle \Delta(\sum_{i=I}^{\infty} \psi_i \mathbf{S} f), \phi \rangle.$$

□

Theorem 5.7. *Let $1 - \frac{1}{p} < t < 2 - \frac{1}{p}$, $1 < p < \infty$, with $(t, \frac{1}{p})$ in the hexagon described in Theorem 5.1, μ as in Theorem 5.1, $f \in \tilde{H}^p_{-t,\mu p}(\Omega)$ and $\Lambda \in B^p_{1-t-\frac{1}{p}}(\partial\Omega)$. Then there is a unique solution $u \in H^p_{2-t,\mu p}(\Omega)$ to the Neumann problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial N} = \Lambda. \end{cases} \tag{5.4}$$

Moreover, $u \in B^{p,p}_{2-t,\mu p}(\Omega)$ and there exists a positive constant C depending only on Ω such that

$$\max\{\|u\|_{H^p_{2-t,\mu p}(\Omega)}, \|u\|_{B^{p,p}_{2-t,\mu p}(\Omega)}\} \le C(\|f\|_{H^p_{-t,\mu p}(\Omega)} + \|g\|_{B^{p,p}_{1-\frac{1}{p}-t}(\partial\Omega)}). \tag{5.5}$$

If Ω is C^1, the previous statement holds for $\epsilon = 1$. Notice that the region described above when $\epsilon = 0$ is common to all Lipschitz domains.

Proof. We begin by pointing out that for the specified values of the parameters and $\frac{1}{p} + \frac{1}{p'} = 1$, constant functions are neither in $H^{p'}_{t,-\mu p'}(\Omega)$ nor in $H^p_{2-t,\mu p}(\Omega)$. The first claim can be readily verified by taking a subsequence $(\psi_{j_k})_k$ of the sequence $(\psi_k)_k$ such that

$$|\psi_{j_k}(x)| \ge \frac{1}{3} \text{ on } \{x : 2^{3(k-1)} \le |x| \le 2^{3k-2}\}$$

(notice that $3k-5 \le j_k \le 3k-1$)and invoking standard embedding theorems to conclude that for $t = n(\frac{1}{p'} - \frac{1}{r})$, one has $-\mu + \frac{n}{r} > 0$ and

$$\sum_{k=0}^{\infty} 2^{-\mu p' j_k} \|\psi_{j_k}\|_{L^r}^{p'} \ge C \sum_{k=0}^{\infty} 2^{-\mu p' j_k + \frac{np'}{r}(j_k+2)}$$

for a positive constant C. The second assertion follows similarly and its proof will be omitted. It follows from these observations that the usual compatibility condition needed in the formulation of the Neumann Problem in a bounded domain, namely

$$\langle f, 1 \rangle = \langle \Lambda, 1 \rangle$$

is not necessary in this context. We now proceed to the proof of the Theorem: The existence part follows from the invertibility results in Theorem 4.2,combined with Theorems 3.2, 4.1 and Lemma 5.6. To prove uniqueness, as in [8], we consider a solution of (5.4) with $f = 0$, $\Lambda = 0$, say $u \in H^p_{s+\frac{1}{p},\mu p}(\Omega)$. From the trace theorem and the existence theorem for 5.4, it follows that the trace of u on $\partial\Omega$ anihilates every $\Lambda \in B^q_s(\partial\Omega)$. Thus, from uniqueness for the Dirichlet problem, $u = 0$ in Ω. $\square$

6. The Helmholtz Decomposition for vector fields

In this section we will be mainly concerned with vector fields whose components are functions in certain weighted spaces. We will denote by $\mathbf{A}^p_{s,\mu}(\Omega)$ the Banach space of vector fields with components in $A^p_{s,\mu}(\Omega)$. Let $1 < t + \frac{1}{p} < 2$, $1 < p < \infty$ and $\mathbf{u} \in \mathbf{H}^p_{1-t,\mu p}(\Omega)$ be a vector field defined on Ω. Then $\operatorname{div} \mathbf{u}$ can be extended to a distribution (still denoted by $\operatorname{div} \mathbf{u}$) in $\tilde{H}^p_{-t,\mu p}(\Omega)$ (Lemma 5.5), where q denotes the Hölder conjugate of p.. Accordingly, the normal component $\mathbf{u} \cdot N \in B^p_{1-\frac{1}{p}-t}(\partial\Omega)$ can be identified with the linear functional (which depends on the extension of $\operatorname{div} \mathbf{u}$) defined by

$$\langle \mathbf{u} \cdot N, \phi \rangle = \langle \operatorname{div} \mathbf{u}, \tilde{\phi} \rangle + \langle \mathbf{u}, \nabla \tilde{\phi} \rangle,$$

for any extension $\tilde{\phi} \in H^q_{t,-\mu q}(\Omega)$ of ϕ. Notice that $\mathbf{u} \cdot N$ does not depend on the particular extension $\tilde{\phi}$, for it is readily verified that any $\xi \in H^p_{1+\frac{1}{q}-s,r}(\Omega)$ ($r \in \mathbb{R}$) with vanishing trace can be approximated by smooth functions with compact support in Ω. The spaces

$$\mathbf{E}^p_{1-t,\mu p}(\Omega) = \{\mathbf{u} : \mathbf{u} \in \mathbf{H}^p_{1-t,\mu p}(\Omega)\,,\ \operatorname{div} \mathbf{u} = 0\,,\ \mathbf{u} \cdot N = 0\}$$

and

$$\mathbf{G}^p_{1-t,\mu p}(\Omega) = \{\nabla \pi : \pi \in H^p_{2-t,\mu p}(\Omega)\}$$

are well known by their connection with the Navier-Stokes equations. As an immediate consequence of Theorem 5.7, we have

Theorem 6.1. *Let Ω be a Lipschitz domain such that $\mathbb{R}^n \setminus \Omega$ is bounded and $\partial\Omega$ is connected. Then, for $(t, \frac{1}{p})$ in the hexagon described in Theorem 5.7, $-\frac{n}{p} < \delta_1 < -2 + \frac{n}{q}$, $-\frac{n}{q} < \delta_2 < -2 + \frac{n}{p}$ and*

$$\mu = \delta_1 - \frac{1}{2} t(\delta_1 + \delta_2) + 2(1-t),$$

the following (topological) decomposition holds:

$$\mathbf{H}^p_{1-t,\mu p}(\Omega) = \mathbf{E}^p_{1-t,\mu p}(\Omega) \oplus \mathbf{G}^p_{1-t,\mu p}(\Omega), \tag{6.1}$$

that is, the projection operator

$$P : \mathbf{H}^p_{1-t,\mu p}(\Omega) \to \mathbf{E}^p_{1-t,\mu p}(\Omega)$$

is bounded. In particular, for $\max\{\frac{3}{2}, \frac{n}{n-1}\} < p < \min\{3, n\}$*, the following (topological) decomposition holds:*

$$\mathbf{L}^p(\Omega) = \mathbf{E}^p(\Omega) \oplus \mathbf{G}^p(\Omega) \tag{6.2}$$

where

$$\mathbf{E}^p(\Omega) = \mathbf{E}^p_{0,0}(\Omega) = \{\mathbf{u} : \mathbf{u} \in \mathbf{L}^p(\Omega)\,,\, \operatorname{div}\mathbf{u} = 0\,,\, \mathbf{u}\cdot \mathrm{N} = 0\}$$

and

$$\mathbf{G}^p(\Omega) = \mathbf{G}^p_{0,0}(\Omega) = \{\mathbf{v} : \mathbf{v} = \nabla\pi\,,\, \pi \in \mathbf{H}^p_{1,0}(\Omega)\}.$$

Proof. Theorem 5.7 guarantees that the projection onto $\mathbf{E}^p_{1-t,\mu p}(\Omega)$ is well defined and bounded and that the zero function is the only element in $\mathbf{E}^p_{1-t,\mu p}(\Omega) \cap \mathbf{G}^p_{1-t,\mu p}(\Omega)$. The decomposition (6.2) results from the definition of μ and the restrictions imposed by Theorem 3.2.

$\square$.

7. Optimality

The proof of the following lemma follows by properly adapting the examples in [12]:

Lemma 7.1. *If* $p > 3$*, there exists a bounded Lipschitz domain* $\Omega' = \mathbb{R}^n \setminus \Omega$ *and* $f \in C^\infty(\Omega)$ *such that if* u *is the* $L^2_{1,\mu}$*-solution to the problem (*5.4*) with* $\Lambda = 0$*, then* $\nabla u \notin \mathbf{L}^p(\Omega)$*.*

We point out that the nature of the counterexample there constructed forces $\nabla u \notin L^p_\mu(\Omega)$ for any $\mu \in \mathbb{R}$. Suitable minor modifications of the results in [8] yield:

Lemma 7.2. *Let* Ω *be the exterior of a bounded, Lipschitz domain,* $1 < p < \infty$*,* $\mu \in \mathbb{R}$ *and* $\Lambda \in B^p_{-\frac{1}{p}}(\partial\Omega)$*. Then there exists a vector field* $\mathbf{v} \in \mathbf{H}^p_\mu(\Omega)$ *with div* $\mathbf{v} \in H^p_{-1,\mu}(\Omega)$ *and* $\mathbf{v}\cdot N = \Lambda$*. Using Theorem* 3.2 *and the regularity result (which follows from* [5] *and Theorem 4.1)*

$$S : L^p(\partial\Omega) \to H^p_{1,\mu p}(\Omega)$$

one concludes that if the Helmholtz decomposition (6.1) holds for some $1 < p \le 2$*, the Neumann Problem (5.4) with* $t = 1$ *and homogeneous boundary datum (*$\Lambda = 0$*) is uniquely solvable in the class* $H^p_{1,\mu p}(\Omega)$*.*

These observations allow us to adapt the methods of [8] to prove the following:

Theorem 7.3. *The range of validity for the decomposition in Theorem* 6.1 *is sharp, that is, for any* $p \notin [\frac{3}{2}, 3]$ *there exists a Lipschitz domain* Ω_p *such that the Helmholtz decomposition (*6.1*) does not hold for any* μ *as in Theorem* 3.2*.*

8. The Stokes System

As a further application of the function-space theoretic tools developed in Chapter 1, we present here some regularity results for the Poisson's Problem with Dirichlet boundary datum for the three-dimensional Stokes system. In the sequel, we retain the notation of Sections 1 and 2 and assume $n = 3$, with the agreement that in this section, a function space such as $\mathbf{A}^{s,p}_{p,\mu}$ or $\mathbf{A}^{s,p}_{p,\mu}(\Omega)$ will stand for the space of n-dimensional vector fields $\mathbf{u}$ on $\mathbb{R}^n$ (or Ω) with components in $A^{s,p}_{p,\mu}$ ($A^{s,p}_{p,\mu}(\Omega)$) and the corresponding norm is defined as the sum of the norms of the components. The Jacobian matrix of the vector valued function $\mathbf{v}$ will be written as $\nabla\mathbf{v}$. For non-zero $X \in \mathbb{R}^3$ the matrix of fundamental solutions of the Stokes system will be denoted by $\Gamma(X) = (\Gamma_{ij}(X))$, where

$$\Gamma_{ij}(X) = \frac{1}{8\pi}\left(\frac{\delta_{ij}}{|X|} + \frac{X_i X_j}{|X|^3}\right)$$

with corresponding pressure vector $q(X) = (q_{ij}(X))$ given by

$$q_{ij}(X) = \frac{1}{8\pi}\frac{X_j}{|X|^3}.$$

In analogy with the notation in Section 3, we introduce the Stokes Potential of $\mathbf{f} \in H^p_{0,(\delta+2)p}$ as

$$\mathcal{P}_\Omega(\mathbf{f}) = \mathcal{E}^*_\Omega \mathcal{P}\mathbf{R}^*_\Omega(\mathbf{f}), \tag{8.1}$$

where

$$\mathcal{P}(f) = \Gamma * \mathbf{f}.$$

The following result is the counterpart of Theorem 3.2 and its proof is similar:

Theorem 8.1. *Let $n \geq 3\,, 1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $-\frac{n}{p} < \delta_1 < -2 + \frac{n}{q}$ $-\frac{n}{q} < \delta_2 < -2 + \frac{n}{p}$, $\mu = (\frac{s}{2}(\delta_1 + \delta_2) - \delta_2 + 2(s-1))$ and $0 < s < 2$. Then the Stokes Potential is a bounded linear map from $\tilde{\mathbf{A}}^p_{s-2,\mu p}(\Omega)$ into $\mathbf{A}^p_{s,\mu p}(\Omega)$.*

Next, we highlight the following properties of the Single Layer Potential, which as in the scalar case, is defined as

$$\mathbf{S}(f)(X) = \int_{\partial\Omega} \Gamma(X - Q)\mathbf{f}(Q)\, d\sigma(Q),$$

for a vector-valued density $\mathbf{f} : \partial\Omega \to \mathbb{C}$. The corresponding boundary operator will be denoted by $S(\mathbf{f})$.

Theorem 8.2. *For $0 < s < 1$, p and μ be as in Theorem 4.1, the operator S maps $\mathbf{B}^p_{s-1}(\partial\Omega)$ boundedly into $\mathbf{B}^p_{s+\frac{1}{p},\mu p}(\Omega) \cap \mathbf{H}^p_{s+\frac{1}{p},\mu p}(\Omega)$.*

Proof. The proof follows (modulo minor self-explanatory modifications) as that of Theorem 4.1. □

Theorem 8.3. *There exists a positive number ϵ depending only on Ω such that for $(s, \frac{1}{p})$ in the interior of the hexagon with vertices $(0, \frac{1}{2} - \epsilon)$, $(0, \frac{1}{2} + \epsilon)$,$(\frac{1}{3} - \epsilon, \frac{2}{3} - \epsilon)$, $(1, \frac{1}{2} + \epsilon)$ and $(1, \frac{1}{2} - \epsilon)$ and $(\frac{2}{3} + \epsilon, \frac{1}{3} - \epsilon)$, the operator S is an isomorphism from $B^p_{s-1}(\partial\Omega)$ onto $B^p_s(\partial\Omega)$.*

Proof. We commence by observing that for $2 \leq p$, the space $\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)$ is topologically embedded into $\mathbf{B}^2_s(\partial\Omega)$ for $0 < s < 1 - \frac{1}{p}$. For an arbitrary distribution $\Lambda \in B^p_{-\frac{1}{p}}(\partial\Omega)$, the vector valued function $\mathbf{v}(X) = \mathbf{S}(\Lambda)(X)$ is a solution (both, in Ω and $\mathbb{R}^n \setminus \overline{\Omega}$) to the (homogeneous) problem

$$\left\{\begin{array}{l} \Delta\mathbf{u} = \nabla\pi \quad \text{in } \Omega \\ \mathbf{div}\,\mathbf{u} = 0 \\ \mathbf{Tr}\,\mathbf{u} = S(\Lambda) \in B^p_{1-\frac{1}{p}}(\partial\Omega) \end{array}\right. \tag{8.2}$$

for $\pi(X) = \langle\Lambda, \mathbf{q}(X - \cdot)\rangle$. Notice that from the proof of Theorem 8.2, for suitable μ, the estimate

$$\|\pi\|_{L^p_{\mu p}(\Omega)} \leq C\|\Lambda\|_{B^p_{-\frac{1}{p}}(\partial\Omega)} \tag{8.3}$$

holds. By virtue of Theorem 8.2 and the L^p-estimate from [3] (which is valid for $2 \leq p < 3 + \epsilon$, where $0 < \epsilon = \epsilon(\Omega)$), the solution $\mathbf{v}$ is subject to the bound

$$\|\nabla\mathbf{v}\|_{L^p(\mathbb{R}^n\setminus\overline{\Omega})} \leq C\|S(\Lambda)\|_{B^p_{1-\frac{1}{p}}(\partial\Omega)}. \tag{8.4}$$

The estimate from [3] alluded to above, applied to the domain $B_{2^I}(0) \cap \Omega$ together with Theorem 2.2 yield

$$\|\mathbf{v}\|_{\mathbf{H}^p_{1,\mu p}(\Omega)} \leq C\left(\|S(\Lambda)\|_{\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)} + E(\Lambda)\right),$$

for μ as in Theorem 8.1 and some positive constant $C = C(\Omega, p, \mu)$. It is not hard to see that for any bounded sequence $(\Lambda_i)_i$ in $B^p_{-\frac{1}{p}}(\partial\Omega)$, $(E(\Lambda_i))_i$ has a convergent subsequence. Estimates (8.4) in conjunction with Lemma 5.5 show that $\mathbf{v}$ has distributional exterior and interior normal derivatives $\frac{\partial^+\mathbf{v}}{\partial N}$ and $\frac{\partial^-\mathbf{v}}{\partial N}$ respectively, whose action on an arbitrary $\boldsymbol{\phi} \in B^p_{-\frac{1}{p}}(\partial\Omega)$ is given by,

$$\langle\frac{\partial^+\mathbf{v}}{\partial N}, \boldsymbol{\phi}\rangle = \langle\nabla\pi, \tilde{\boldsymbol{\phi}}\rangle + \int_\Omega \nabla\mathbf{v}\nabla\tilde{\boldsymbol{\phi}}$$

and

$$\langle\frac{\partial^-\mathbf{v}}{\partial N}, \boldsymbol{\phi}\rangle = \langle\nabla\pi, \tilde{\boldsymbol{\phi}}\rangle + \int_\Omega \nabla\mathbf{v}\nabla\tilde{\boldsymbol{\phi}}$$

for an arbitrary extension $\tilde{\boldsymbol{\phi}}$ in $\mathbf{H}^p_{1,0}(\Omega)$ in the first equality, or $\tilde{\boldsymbol{\phi}}$ in $\mathbf{H}^p_1(\mathbb{R}^3 \setminus \Omega)$ for the latter. As is apparent from the above equalities (using the appropriate version of Lemma 5.5), one has the estimates for $2 \leq p < 3 + \epsilon$

$$\left\|\frac{\partial^+\mathbf{v}}{\partial N}\right\|_{\mathbf{B}^p_{-\frac{1}{p}}(\partial\Omega)} \leq C\left(\|S(\Lambda)\|_{\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)} + E(\Lambda)\right) \tag{8.5}$$

and

$$\left\|\frac{\partial^-\mathbf{v}}{\partial N}\right\|_{\mathbf{B}^p_{-\frac{1}{p}}(\partial\Omega)} \leq C\|S(\Lambda)\|_{\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)}. \tag{8.6}$$

Since

$$\frac{\partial^\pm\mathbf{v}}{\partial\nu} =: \frac{\partial^\pm\mathbf{v}}{\partial N} - N \cdot \pi = (\mp\frac{1}{2}I + T)(\Lambda),$$

where T is a singular integral operator (see [7]), it follows at once that for a positive $C = C(p,\Omega)$,

$$\|\Lambda\|_{\mathbf{B}^p_{-\frac{1}{p}}(\partial\Omega)} \leq C\left(\|S(\Lambda)\|_{\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)} + E(\Lambda)\right),$$

which shows that as an operator between the given spaces, S has closed range and is injective. Choosing now $2-\frac{3}{p} < t < 1$, standard embedding results yield

$$\mathbf{B}^2_t(\partial\Omega) \subseteq \mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega).$$

On the other hand it is known from the L^2-theory developed in [7] and [9] that S is an isomorphism from $\mathbf{B}^2_{t-1}(\partial\Omega)$ onto $\mathbf{B}^2_t(\partial\Omega)$, whence S has dense range as an operator from $\mathbf{B}^p_{-\frac{1}{p}}(\partial\Omega)$ into $\mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)$. In all,

$$S \simeq \mathbf{B}^p_{-\frac{1}{p}}(\partial\Omega) \to \mathbf{B}^p_{1-\frac{1}{p}}(\partial\Omega)$$

is an isomorphism for $2 \leq p < 3+\epsilon$. Duality and interpolation with the aforementioned L^2 theory yield the complete statement of Theorem 8.3. □

As an immediate consequence we have

Theorem 8.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a domain such that $\mathbb{R}^3 \setminus \Omega$ is a bounded Lipschitz domain with a connected boundary. Then there exists a positive number ϵ depending exclusively on the Lipschitz character of Ω such that whenever the point $(t, \frac{1}{p})$ belongs to the hexagon with vertices $(1+2\epsilon, \frac{2}{3}-\epsilon)$, $(\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon)$ $(\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon)$, $(1, \frac{1}{3}-\epsilon)$ $(\frac{3}{2}+\epsilon, \frac{1}{2}-\epsilon)$ $(\frac{3}{2}-\epsilon, \frac{1}{2}+\epsilon)$ and $\mathbf{f} \in \tilde{\mathbf{H}}^p_{-t,\mu p}(\Omega)$ for p and μ as in Theorem 5.1, there is a unique solution $\mathbf{u} \in H^p_{2-t,\mu p}(\Omega)$ and a unique (up to constants) π satisfying the Poisson's problem*

$$\begin{cases} \Delta \mathbf{u} = \nabla\pi + \mathbf{f} & \text{in } \Omega \\ \mathbf{div}\,\mathbf{u} = 0 \\ \mathbf{Tr u} = 0 \end{cases} \tag{8.7}$$

and, for some positive constant $C = C(p,t,\mu,\Omega)$, the a-priori estimate

$$\|\mathbf{u}\|_{\mathbf{H}^p_{2-t,\mu p}(\Omega)} + \|\pi\|_{\mathbf{H}^p_{1-t,\mu p}(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{H}^p_{-t,\mu p}(\Omega)}$$

holds. In particular, for $\frac{3}{2} < p < 3$, if $\mathbf{f} \in \dot{\mathbf{H}}^p_{-1,0}(\Omega)$, then $\mathbf{u} \in \mathbf{H}^p_{1,0}(\Omega)$, and

$$\|\mathbf{u}\|_{\mathbf{H}^p_{1,0}(\Omega)} \leq C\|\mathbf{f}\|_{\dot{\mathbf{H}}^p_{-1,0}(\Omega)}.$$

Proof. Existence and the estimates follow directly from Theorems 8.2 and 8.3. The uniqueness statement is proved by arguing as in Theorem 5.1. The solution to the Poisson's Problem (8.7) is thus given by

$$\mathbf{u}(X) = \mathcal{P}_\Omega(\mathbf{f})(X) + \mathbf{v}(X)$$

where v is the solution to the homogeneous Dirichlet Problem with boundary datum $-\mathcal{P}_\Omega(\mathbf{f})|_{\partial\Omega}$, namely

$$\mathbf{v}(X) = \mathcal{S}(S^{-1}(\mathcal{P}_\Omega|_\Omega))(X).$$

□

References

[1] C. Amrouche, V Girault and J. Giroire, *Weighted Sobolev Spaces for Laplace's Equation in* $\mathbb{R}^n$*,* J. Math. Pures et Appl., 73, 1994, 579-606.

[2] C. Amrouche, V. Girault and J. Giroire, *Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev Spaces,* J. Math. Pures et Appl., 76 1997, 55-81.

[3] R. Brown, Z. Shen, *Estimates for the Stokes Operator in Lipschitz Domains,* Ind. Univ. Mat. J., 44, 1995, 1183-1206.

[4] M. Cantor and J. Matowsky, *Helmholtz decompositionn of* $W^{s,p}$ *vector fields,* Differential geometry, calculus of variations, and their applications, 139–147 Lecture Notes in Pure and Appl. Math., 100, Dekker, New York, 1985.

[5] B. Dahlberg and C. Kenig, *Hardy spaces and the Neumann problem in* L^p *for Laplace's equation in Lipschitz domains,* Ann. of Math., 125, 1987.

[6] E. Fabes, M. Jodeit, N.Riviere, *Potential techniques for boundary value problems on* C^1 *domains,* Acta Math., 1141, 1978, 165-186.

[7] E. Fabes, C. Kenig, G. Verchota, *The Dirichlet Problem for the Stokes Systems in Lipschitz domains,* Duke Math. J., 57, 3, 1988, 769-793.

[8] E. Fabes, O. Méndez, M. Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains,* J. Funct. Anal., 159, 1998, 323-368.

[9] W. Gao, *Layer Potentials and Boundary Value Problems for Elliptic Systems in Lipschitz Domains*, J. Funct. Anal, 95, 1991, 377-399 .

[10] J. Giroire and J. Nedelec, *Numerical Solution of an Exterior Neumann Problem Using a Double Layer Potential,* Mathematics of Computation, 32, 144 1978, 973-990.

[11] P. Grisvard, *Elliptic problems in non-smooth domains,*, Pittman advanced publishing program, 1985.

[12] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains,* J. Funct. Anal., 130, 1995, 161-219.

[13] A. Jonsson and H. Wallin, *Function Spaces on subsets of* $\mathbb{R}^n$*,* Harwood Ac., New York, 1984.

[14] C. Kenig, *Harmonic Analysis techniques for second order elliptic boundary value problems,* AMS regional series in mathematics, American Mathematical Society, Providence, 83, 1994.

[15] L. Lanzani and O. Méndez, *The Poisson's Problem with Robin Boundary Condition in Lipschitz Domains.,* in preparation.

[16] J. Lions and E. Magenes, *Non-Homogeneous boundary value problems and applications, I, II.,* Springer Verlag, 1972.

[17] J. Löfström, *Interpolation of Weighted Spaces of Differentiable Functions.,* Ann. di Mat. Pura e Applicata, 1982.

[18] R.Mc Owen, *Boundary Value Problems for the Laplacian in an Exterior Domain*, Comm. in Partial Diff. Eq. 6 (7), 1981, 783-798.

[19] O. Méndez and M. Mitrea, *Complex powers of the Laplacian in Lipschitz domains,* Mat. Nachr., 223, 2001, 77-88.

[20] O. Méndez and M. Mitrea, *The Banach Envelopes of the Besov and Triebel-Lizorkin spaces and applications,* J. Fourier Anal. Appl. 6 (2000), no. 5, 503–531.

[21] C. Pérez, *Two Weighted Norm Inequalitiesfor Riesz Potentials and Uniform* L^p*-Weighted SObolev Inequalities,* Indiana Univ. Math. Jour. 39 1 (1990).

[22] C. Simader, H Sohr, *A new approach to the Helmholtz decomposition and the Neumann problem in* L^q*-spaces for bounded and exterior domains*, Mathematical problems relating to the Navier-Stokes equation, 1–35, Ser. Adv. Math. Appl. Sci., 11, World Sci. Publishing, River Edge, NJ, 1992.

[23] H. Triebel, *Theory of function spaces I,* Birkhäuser, Berlin, 1983.

[24] H. Triebel, *Theory of function spaces II,* Monographs in Mathematics Birkhäuser Verlag, Basel, 84, 1992.

[25] H. Triebel, *Spaces of Kudrjavcev Type I: Interpolation, Embedding and Structure,* Jour. Math. Anal. and Appl., 56, 1976, 253-277.

[26] H. Triebel, *Spaces of Kudrjavcev Type II: Spaces of Distributions, Duality and Interpolation,* Jour. Math. Anal. and Appl., 56, 1976, 253-277.

[27] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators,* Johann Ambrosius Barth Verlag, Heidelberg, 1995.

[28] G. Verchota, *Layer Potentials and regularity for the Dirichlet Problem for Laplace's Equation in Lipschitz Domains,* J. Funct. Anal. 59, 1984, 572-611.

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