THE ASYMPTOTICS OF SPECTRAL GAPS OF 1D DIRAC OPERATOR WITH COSINE POTENTIAL

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Abstract. The 1D Dirac operator

$$Ly = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + \begin{pmatrix}0 & 2a\cos 2\pi x\\ 2a\cos 2\pi x & 0\end{pmatrix} y, \qquad y = \begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

considered as a selfadjoint operator in $L^2(\mathbb{R}, \mathbb{C}^2)$, has a continuous spectrum that equals the real line $\mathbb{R}$ without countably many open intervals (spectral gaps). There exists an integer $N_* = N_*(a) > 0$ such that for each odd $n \in \mathbb{Z}$ with $|n| > N_*$ there is a unique spectral gap $(\lambda_n^-, \lambda_n^+)$ with length $\gamma_n = \lambda_n^+ - \lambda_n^-$ that satisfies $(|\lambda_n^\pm - n\pi| < \pi/2$, and in this way we obtain all spectral gaps situated outside $[-\pi N_*, \pi N_*]$. Moreover, $\gamma_{-n} = \gamma_n$ and

$$\gamma_n = 2a\left(\frac{a}{4\pi}\right)^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^{-2}\left[1 + 0\left(\frac{\ln n}{n}\right)\right], \quad n > N_*.$$

1. Introduction

Let

$$(1) \quad L = L^0 + V, \qquad L^0 = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx}, \qquad V = \begin{pmatrix}0 & P(x)\\ Q(x) & 0\end{pmatrix},$$

where P and Q are periodic functions of period 1 such that $P, Q \in L^2([0,1])$. If $Q(x) = \overline{P(x)}$ then the operator L gives a rise to a selfadjoint operator in $L^2(\mathbb{R}, \mathbb{C}^2)$ which spectrum is a real line $\mathbb{R}$ with gaps $(\lambda_n^-, \lambda_n^+)$, may be empty; for large $|n|$ these spectral gaps are close to πn.

As in the case of Schrödinger operator [3], [9], [11], [15] these points $\{\lambda_n^-, \lambda_n^+\}$ are eigenvalues of the operator L considered on the interval $[0,1]$ with periodic (for n even) and antiperiodic (for n odd) boundary conditions (see [3], [12], [13]).

In this note we find a sharp formula for the asymptotics of spectral gaps of L in the case where $P(x) = Q(x) = 2a\cos 2\pi x$, a real. Our interest in this question was inspired by the results of E. Harrell [6]

Key words and phrases. Dirac operator, spectral gaps, cosine potential.

The first author acknowledges the hospitality of The Mathematics Department of The Ohio State University during academic year 2002/2003.

and J. Avron and B. Simon [1] on the asymptotics of the spectral gaps of the Mathieu operator $d^2/dx^2 + 2a\cos 2x$. It seems there has been no explicit evaluation of gaps of specific 1D periodic Dirac operators so far. Our note is trying to fill this gap.

2. Preliminaries

The Dirac operator L^0, considered on the interval $[0,1]$ with periodic $(y(0) = y(1))$ or antiperiodic $(y(0) = -y(1))$ boundary conditions, has a discrete spectrum, respectively, $\{2k\pi, k \in \mathbb{Z}\}$ and $\{(2k+1)\pi, k \in \mathbb{Z}\}$. Each eigenvalue $n\pi$, both for periodic (if n is even), or antiperiodic (if n is odd) boundary conditions has multiplicity 2, and

$$e_n^1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-in\pi x}, \quad e_n^2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{in\pi x} \tag{2}$$

are eigenfunctions corresponding to the eigenvalue $n\pi$. Moreover, if the Hilbert space $\mathbb{H} = L^2[0,1] \times L^2[0,1]$ is equipped with the scalar product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \int_0^1 (f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)})dx, \tag{3}$$

then each of the systems

$$\{e_{2k}^1, e_{2k}^2,\ k \in \mathbb{Z}\}, \qquad \{e_{2k+1}^1, e_{2k+1}^2,\ k \in \mathbb{Z}\} \tag{4}$$

is an orthonormal basis in $\mathbb{H}$.

The operator

$$L = L^0 + V, \qquad V = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \tag{5}$$

where $P(x)$ and $Q(x)$ are 1-periodic functions, may be considered as a perturbation of L^0. Further we always assume that $P(x), Q(x) \in L^2[0,1]$. Then the operator L, considered with periodic or antiperiodic boundary conditions, has also a discrete spectrum. Moreover, if $N_0 = N_0(V)$ is large enough, then for $|n| \geq N_0$ there are exactly two (counted with multiplicity) periodic (if n is even), and antiperiodic (if n is odd) eigenvalues $\{\lambda_n^-, \lambda_n^+\}$ of L such that

$$|\lambda_n^\pm - \pi n| < \pi/2, \quad |n| > N_0 \tag{6}$$

(see, for example [10], [13], [14]). If $\lambda_n^- \neq \lambda_n^+$ then the interval $(\lambda_n^-, \lambda_n^+)$ is a spectral gap of L (considered in $L^2(\mathbb{R})$); moreover every spectral gap of L appears in that way. For convenience, we consider the interval $(\lambda_n^-, \lambda_n^+)$ as spectral gap (empty) even if $\lambda_n^- = \lambda_n^+$.

3. Basic formulas

We use a pure Fourier method to set up a framework of our analysis in the same way as it was done in [7], [8], [2] in the case of Hill operators, or in [4], [5] in the case of Dirac operators.

1. Suppose that $\lambda = n\pi + z$, $|n| > N_0$, is a periodic (or antiperiodic) eigenvalue of L with $|z| < \pi/2$, and that $y \neq 0$ is a corresponding eigenvector. Let $E_n^0 = [e_n^1, e_n^2]$ be the eigenspace of L^0 that corresponds to $n\pi$, and let $\mathbb{H}(n)$ be its orthogonal complement. We denote by P_n^0 and Q_n^0, respectively, the orthogonal projectors on E_n^0 and $\mathbb{H}(n)$. Then the equation $(n\pi + z - L)y = 0$ is equivalent to the following system of two equations:

$$Q_n^0(n\pi + z - L^0 - V)Q_n^0 y + Q_n^0(n\pi + z - L^0 - V)P_n^0 y = 0 \tag{7}$$

$$P_n^0(n\pi + z - L^0 - V)Q_n^0 y + P_n^0(n\pi + z - L^0 - V)P_n^0 y = 0 \tag{8}$$

Taking into account that $P_n^0 Q_n^0 = Q_n^0 P_n^0 = 0$ and $P_n^0 L^0 Q_n^0 = Q_n^0 L^0 P_n^0 = 0$ we obtain that (7) and (8) can be written as

$$Q_n^0(n\pi + z - L^0 - V)Q_n^0 y - Q_n^0 V P_n^0 y = 0 \tag{9}$$

$$-P_n^0 V Q_n^0 y - P_n^0 V P_n^0 y + z P_n^0 y = 0 \tag{10}$$

The operator

$$A = A(n,z) := Q_n^0(n\pi + z - L^0 - V)Q_n^0 : \mathbb{H}(n) \to \mathbb{H}(n) \tag{11}$$

is invertible for large $|n|$ (see below (24)). So, solving (9) for $Q_n^0 y$, we obtain $Q_n^0 y = A^{-1} Q_n^0 V P_n^0 y$, where $P_n^0 y \neq 0$ (otherwise $Q_n^0 y = 0$ which implies $y = P_n^0 y + Q_n^0 y = 0$). Now (10) implies (after plugging the above expression for $Q_n^0 y$ in it) that $(S - z)P_n^0 y = 0$, where the operator $S : E_n^0 \to E_n^0$ is defined by

$$S := P_n^0 V A^{-1} Q_n^0 V P_n^0 + P_n^0 V P_n^0. \tag{12}$$

Hence we obtain (since $P_n^0 y \neq 0$)

$$\det \begin{vmatrix} S^{11} - z & S^{12} \\ S^{21} & S^{22} - z \end{vmatrix} = 0 \tag{13}$$

where

$$S^{ij} = S^{ij}(n,z) = \langle e_n^i, S e_n^j \rangle, \quad i,j \in \{1,2\}. \tag{14}$$

Equation (13) and the expressions (14) play a crucial role throughout the paper.

2. Let $\mathbb{H}^1$ and $\mathbb{H}^2$ be the subspaces of $\mathbb{H}$ generated, respectively, by $\{e_m^1, m \in \mathbb{Z}\}$ and $\{e_m^2, m \in \mathbb{Z}\}$, and let $\mathbb{H}^1(n)$ and $\mathbb{H}^2(n)$ be, respectively, the intersections of these spaces with $\mathbb{H}(n)$. Then $\mathbb{H} = \mathbb{H}^1 \oplus \mathbb{H}^2$, so

each operator $B : \mathbb{H} \to \mathbb{H}$ may be identified with a 2×2 operator matrix (B^{ij}), where $B^{ij} : \mathbb{H}^j \to \mathbb{H}^i, \quad i,j = 1,2$. If we consider the matrix representation of B in the basis $\{e^1_{2k}, e^2_{2k}, \ k \in \mathbb{Z}\}$ (or $\{e^1_{2k+1}, e^2_{2k+1}, \ k \in \mathbb{Z}\}$) then this matrix itself combines the matrix representations of B^{ij}. Of course, similar remark holds for operators acting in $\mathbb{H}(n)$.

Futher we always work with one of the bases (4) (respectively, using the first basis in the case of periodic boundary conditions, and the second one in the case of antiperiodic boundary conditions). However, we don't specify below which basis is used because the formulas for the matrix representations in these bases are formally the same (with running indices being even in the first case, and odd in the second case).

Let

$$P(x) = \sum_{m\in\mathbb{Z}} p(m) e^{im\pi x}, \qquad Q(x) = \sum_{m\in\mathbb{Z}} q(m) e^{im\pi x} \tag{15}$$

be the Fourier expansions of the functions $P(x)$ and $Q(x)$. It is easy to see that the operator V has the following matrix representation

$$V = \begin{pmatrix} 0 & V^{12} \\ V^{21} & 0 \end{pmatrix}, \qquad V^{12}_{km} = p(-k-m), \quad V^{21}_{km} = q(k+m). \tag{16}$$

The operator $Q^0_n(n\pi + z - L^0)Q^0_n : \mathbb{H}(n) \to \mathbb{H}(n)$ is invertible in $\mathbb{H}(n)$ for any z with $|z| < \pi/2$. Let D_n denote its inverse operator; then the matrix representation of D_n is

$$D_n = \begin{pmatrix} D^{11}_n & 0 \\ 0 & D^{22}_n \end{pmatrix}, \qquad \left(D^{11}_n\right)_{km} = \left(D^{22}_n\right)_{km} = \frac{\delta_{km}}{\pi(n-k)+z}. \tag{17}$$

The operator A defined in (11) can be written as

$$A = Q^0_n(n\pi + z - L^0)Q^0_n(1 - T_n)Q^0_n, \tag{18}$$

where

$$T_n = D_n Q^0_n V Q^0_n. \tag{19}$$

Thus $A = A(n,z)$ is invertible if and only if $1 - T_n$ is invertible in $\mathbb{H}(n)$. By (16) and (17) one can easily see that the operator (19) has a matrix representaion

$$T_n = \begin{pmatrix} 0 & T^{12}_n \\ T^{21}_n & 0 \end{pmatrix}, \tag{20}$$

where

$$(T^{12}_n)_{km} = \frac{p(-k-m)}{\pi(n-k)+z}, \quad (T^{21}_n)_{km} = \frac{q(k+m)}{\pi(n-k)+z}. \tag{21}$$

We need also the matrix representation of its square T_n^2. From (20) and (21) it follows that

$$T_n^2 = \begin{pmatrix} T_n^{12}T_n^{21} & 0 \\ 0 & T_n^{21}T_n^{12} \end{pmatrix}, \tag{22}$$

where

$$(T_n^{12}T_n^{21})_{km} = \sum_{j\neq n} \frac{p(-k-j)q(j+m)}{[\pi(n-k)+z][\pi(n-j)+z]} ,$$

(23)

$$(T_n^{21}T_n^{12})_{km} = \sum_{j\neq n} \frac{q(k+j)p(-j-m)}{[\pi(n-k)+z][\pi(n-j)+z]} .$$

There exists $N_* = N_*(V)$ such that if $|n| > N_*$ then $1 - T_n$ is an invertible operator (for special V we consider this follows from Lemma 4 below). Therefore $(1-T_n)^{-1} = \sum_{\ell=0}^{\infty} T_n^\ell$, so, in view of (18), we have

$$A^{-1} = \sum_{\ell=0}^{\infty} T_n^\ell D_n. \tag{24}$$

Now from (12) and (19) it follows that

$$S = \sum_{\ell=0}^{\infty} P_n^0 V T_n^\ell D_n Q_n^0 V P_n^0 + P_n^0 V P_n^0 = \sum_{\ell=0}^{\infty} P_n^0 V T_n^\ell P_n^0, \tag{25}$$

so, in view of (14), we have

$$S^{ij} = \left\langle e_n^i, S e_n^j \right\rangle = \sum_{\ell=0}^{\infty} \left\langle e_n^i, V T_n^\ell e_n^j \right\rangle, \quad i,j = 1,2. \tag{26}$$

4. Main result

Throughout this section we are dealing with the case where $P(x) = Q(x) = 2a\cos 2\pi x$. Thus we have

$$p(\pm 2) = q(\pm 2) = a, \quad p(m) = q(m) = 0 \quad \text{for} \quad m \neq \pm 2. \tag{27}$$

Theorem 1. *Let γ_n, $n \in \mathbb{Z}$ be the length of spectral gaps of Dirac operator*

$$Ly = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{dy}{dx} + \begin{pmatrix} 0 & 2a\cos 2\pi x \\ 2a\cos 2\pi x & 0 \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then $\gamma_{-n} = \gamma_n$, and there exists $N_* = N_*(a)$ such that for $|n| > N_*$ $\gamma_n = 0$ for even n and

$$\gamma_n = 2a\left(\frac{a}{4\pi}\right)^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^{-2}\left[1 + 0\left(\frac{\ln n}{n}\right)\right], \quad n \geq N_*(a), \tag{28}$$

for odd $n > 0$.

The proof of the theorem consists of several steps that are given below as separate statements.

The next lemma follows easily from (16) and (21 - 26).

Lemma 2.

$$S^{11}(n,z) = S^{22}(n,z), \quad S^{21}(n,z) = S^{12}(n,z), \quad |n| \geq N_*. \tag{29}$$

Moreover, if z is real, then $S^{ij}(n,z)$ are real-valued.

Further we set for convenience

$$\alpha(n,z) := S^{11}(n,z) = S^{22}(n,z), \quad \beta(n,z) := S^{12}(n,z) = S^{21}(n,z).$$

With these notations the quasi-quadratic equation (13) becomes

$$(\alpha(n,z) - z)^2 = (\beta(n,z))^2. \tag{30}$$

Lemma 3. *The periodic and antiperiodic spectra of L are symmetric with respect to 0, that is if λ is a periodic (or antiperiodic) eigenvalue of L, then $-\lambda$ is respectively, periodic or antiperiodic eigenvalue of L. Hence*

$$\gamma_{-n} = \gamma_n. \tag{31}$$

Indeed, if $\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$ is an eigenvector corresponding to λ then it is easy to see that $\begin{pmatrix} f(-x) \\ -g(-x) \end{pmatrix}$ is an eigenvector corresponding to $-\lambda$ (since $\cos x$ is an even function, so $P(-x) = P(x)$ and $Q(-x) = Q(x)$).

Lemma 4. *If $|z| < \pi/2$ and $|n| > 2$ then*

$$(a) \quad \|T_n^2\| \leq 7a^2/|n|, \qquad (b) \leq \|T_n D_n T_n\| = 7a^2/|n|. \tag{32}$$

Proof. (a) The norm of T_n^2 does not exceed its Hilbert-Schmidt norm which by (22) is less than the sum of the Hilbert-Schmidt norms of the operators $T_n^{12}T_n^{21}$ and $T_n^{21}T_n^{12}$.

A symmetry argument shows that it is enough to estimate the Hilbert-Schmidt norm of $T_n^{12}T_n^{21}$. By (23), since $p(-k-j)q(j+m) \neq 0$ if and only if $|-k-j| = |j+m| = 2$, we have $(T_n^{12}T_n^{21})_{km} = 0$ if $m \neq k, \ k \pm 4$, and

$$\left(T_n^{12}T_n^{21}\right)_{kk} = t_k + s_k, \quad \left(T_n^{12}T_n^{21}\right)_{k,k-4} = t_k, \quad \left(T_n^{12}T_n^{21}\right)_{k,k+4} = s_k,$$

where

$$t_k = \frac{a^2}{[\pi(n-k)+z][\pi(n+k-2)+z]}, \quad k \neq n, \ -n+2, \ \ t_{-n+2} = 0$$

$$s_k = \frac{a^2}{[\pi(n-k)+z][\pi(n+k+2)+z]}\ , \quad k \neq n,\ -n-2,\ \ s_{-n-2} = 0.$$

Taking into account that $|\pi(n-j)+z|^{-1} < (2/\pi)|n-j|^{-1}$ for $j \neq n$, $|z| < \pi/2$, and

$$\sum_{k \neq n, -n \pm 2} \frac{1}{|n-k|^2 |n+k \pm 2|^2} < \frac{\pi^2}{|2n \pm 2|^2} < \frac{a^2}{n^2}$$

we obtain that

$$\|T_n^{12} T_n^{21}\|_{HS}^2 = \sum_{k \neq n} [|t_k + s_k|^2 + |t_k|^2 + |s_k|^2] \leq 24a^4/n^2.$$

The same estimate holds for the Hilbert-Schmidt norm of $T_n^{12} T_n^{21}$ which proves (a).

(b) The norm of the operator $T_n D_n T_n$ can be estimated in an analogous way. The corresponding matrix representation satisfies

$$(TDT)^{11} = (TDT)^{22} = 0, \qquad (TDT)^{12} = (TDT)^{21}.$$

Moreover, $(TDT)^{12}_{mk} = 0$ if $m \neq k,\ k \pm 4$, and

$$(TDT)^{12}_{kk} = \tilde{t}_k + \tilde{s}_k, \quad (TDT)^{12}_{k,k-4} = \tilde{t}_k, \quad (TDT)^{12}_{k,k+4} = \tilde{s}_k,$$

with $\tilde{t}_k = t_k/[\pi(n+k-2)+z], \quad \tilde{s}_k = s_k/[\pi(n+k+2)+z]$, where t_k and s_k are the same as in the proof of (a). Thus we have

$$\|TDT\| \leq \|TDT\|_{HS} \leq \|T^2\|_{HS}.$$

Lemma 5. *The following estimates hold for large* $|n|$:

$$(a)\ \alpha(n,z) = O(\frac{1}{|n|}), \ \ (b)\ \frac{d\alpha}{dz}(n,z) = O(\frac{1}{|n|}), \ \ (c)\ \frac{d\beta}{dz}(n,z) = O(\frac{1}{|n|}).$$

Proof. (a) By (25) we have

$$S = P_n^0 V P_n^0 + P_n^0 V T_n P_n^0 + R, \tag{33}$$

where $R = \sum_{k \geq 2} P_n^0 V T_n^k P_n^0$. Since by Lemma 4 we have $\|T_n^2\| = O(1/|n|)$ we obtain that $\|R\| = O(1/|n|)$. Therefore (33) implies that

$$\alpha(n,z) = \langle e_n^1, P_n^0 V e_n^1 \rangle + \langle e_n^1, P_n^0 V T_n e_n^1 \rangle + O(1/|n|).$$

By (16) we have $\langle e_n^1, P_n^0 V e_n^1 \rangle = 0$. From (16) - (21) it follows that

$$\langle e_n^1, P_n^0 V T_n e_n^1 \rangle = \sum_{k \neq n} \frac{p(-n-k)q(k+n)}{\pi(n-k)+z}.$$

By (27) the latter sum has only two nonzero terms, namely

$$\langle e_n^1, P_n^0 V T_n e_n^1 \rangle = \frac{a^2}{\pi(2n-2)+z} + \frac{a^2}{\pi(2n+2)+z} = O(1/|n|), \tag{34}$$

which proves (a).

(b) By (11) we have $\frac{d}{dz}A^{-1}(n,z) = -A^{-2}(n,z)$, therefore (12) implies that

$$\frac{dS}{dz}(n,z) = -P_n^0 V A^{-2} Q_n^0 V P_n^0. \tag{35}$$

By Lemma 4, $\|T_n^2\| = O(1/|n|)$, therefore we obtain by (24) that

$$A^{-1} = D_n + T_n D_n + R_1, \qquad A^{-1}Q_n^0 V = \sum_{k=1}^{\infty} T_n^k = T_n + R_2,$$

where $\|R_i\| = O(1/|n|),\ i = 1,2$. Multiplying the above expressions we obtain (since by Lemma 4 we have $\|T_n D_n T_n\| = O(1/|n|)$)

$$A^{-2}Q_n^0 V = D_n T_n + R_3, \quad \|R_3\| = O(1/|n|).$$

Now (35) gives us that

$$\frac{dS}{dz}(n,z) = -P_n^0 V D_n T_n P_n^0 + R_4, \quad \|R_4\| = O(1/|n|).$$

Hence, in view of (34), we have

$$\frac{d\alpha}{dz}(n,z) = -\langle e_n^1, V D_n T_n e_n^1\rangle + O(1/|n|) = O(1/|n|).$$

In an analogous way we obtain

$$\frac{d\beta}{dz}(n,z) = -\langle e_n^1, V D_n T_n e_n^2\rangle + O(1/|n|).$$

Now (16), (20) and (21) imply that $\langle e_n^1, V D_n T_n e_n^2\rangle = 0$, hence $d\beta/dz(n,z) = O(1/|n|)$.

Lemma 6. *For large enough $|n|$ we have*

$$\lambda_n^+ - \lambda_n^- = z_n^+ - z_n^- = |\beta(n,z_n^+) + \beta(n,z_n^-)|[1 + O(1/|n|)]. \tag{36}$$

Proof. The equation (30) splits into two equations:

$$(i)\ \ \alpha(n,z) - z = \beta(n,z), \quad (ii)\ \ \alpha(n,z) - z = -\beta(n,z). \tag{37}$$

For large enough $|n|$ each of the equations (i) and (ii) has at exactly one solution z with $|z| < \pi/2$ because the mapping $z \to \alpha(n,z) - \beta(n,z)$ defines a contraction in the disc $\{z : |z| < \pi/2\}$. Indeed, for z_1, z_2 with $|z_i| < \pi/2$ Lemma 5 implies that

$$|z_1 - z_2| = |[\alpha(n,z_1) - \beta(n,z_1)] - [\alpha(n,z_2) - \beta(n,z_2)]| \le O(\frac{1}{|n|})|z_1 - z_2|.$$

On the other hand by (6) for large $|n|$ equation (30) has exactly two solutions (counted with their multiplicities) $z_n^-, z_n^+ \in (-\pi/2, \pi/2)$.

In the case where $z_n^- = z_n^+$ formula (36) holds because then

$$\beta(n, z_n^-) = \beta(n, z_n^+) = 0. \tag{38}$$

Indeed, otherwise we get a contradiction: one of the equations (i) and (ii) would have a double root, the other would have a different root (at least one), so the equation (30) would have more than two roots (counted with multiplicities)

Assume now that $z_n^- \neq z_n^+$. Observe that z_n^-, z_n^+ are real numbers (because the operator L is selfadjoint). So by Lemma 2 $\alpha(n, z_n^\pm)$ and $\beta(n, z_n^\pm)$ are real-valued.

Let z', z'' be, respectively, the solutions of (i) and (ii). Then we have either $z' = z_n^-, z'' = z_n^+$ or $z' = z_n^+, z'' = z_n^-$, so $|z' - z''| = z_n^+ - z_n^-$.

Now by mean value theorem and Lemma 5 we obtain

$$|(\alpha(n, z') - z') - (\alpha(n, z'') - z'')| = |z' - z''|[1 + O(1/|n|].$$

Hence by (37)

$$(z_n^+ - z_n^-)\,[1 + O(1/|n|] = |\beta(n, z_n^+) + \beta(n, z_n^-)|,$$

which proves Lemma 6.

In view of (36) our next goal is to find the asymptotics of

$$\beta(n, z_n) = q(2n) + \sum_{\nu=1}^{\infty} \beta_\nu(n, z_n)$$

where

$$\beta_\nu(n, z) = \sum_{j_1,\dots,j_{2\nu} \neq n} B(j_1, \dots, j_{2\nu}; n, z)$$

with

$$B(j_1, \dots, j_{2\nu}; n, z) = \frac{q(n + j_1)p(-j_1 - j_2)q(j_2 + j_3)\dots q(j_{2\nu} + n)}{[\pi(n - j_1) + z][\pi(n - j_2) + z]\dots[\pi(n - j_{2\nu}) + z]}$$

Note that $q(2n) = 0$ for $n \neq \pm 2$. Fix a $\nu \in \mathbb{N}$ and consider the terms in the sum $\beta_\nu(n, z)$. By (27) a 2ν-tuple of indices $j = (j_1, \dots, j_{2\nu})$ with $j_1, \dots, j_{2\nu} \neq n$ give a rise to a non-zero term of the sum $\beta_\nu(n, z)$ if and only if the numbers

$$x_1 = n + j_1,\ x_2 = -j_1 - j_2, \dots, x_{2\nu} = -j_{2\nu-1} - j_{2\nu},\ x_{2\nu+1} = j_{2\nu} + n \tag{39}$$

are equal to ± 2. On the other hand

$$x_1 + \dots + x_{2\nu+1} = 2n, \tag{40}$$

so we may regard the finite sequence $(x_1, \dots, x_{2\nu+1})$ as steps of a walk from 0 to $2n$. Obviously the indices $j_1, \dots, j_{2\nu}$ are uniquely determined by (39). Moreover, if

$$h_k = x_1 + \dots + x_k, \quad k = 1, \dots, \nu \tag{41}$$

then we have

$$j_{2k-1} = h_{2k-1} - n, \quad j_{2k} = n - h_{2k}, \qquad k = 1, \dots, \nu. \tag{42}$$

Observe that the condition $j_1, \dots, j_{2\nu} \neq n$ holds if and only if the corresponding walk $h_1, \dots, h_{2\nu}$ is inside the segment $[0, 2n]$, that is if $0 < h_k < 2n$. In this case we say that the walk is *admissible*, and that the corresponding sequence of steps $x = (x_1, \dots, x_{2\nu+1})$ is *admissible.* Further we denote by A_ν the set of all admissible sequences of steps $x = (x_1, \dots, x_{2\nu}+1)$. For $x \in A_\nu$ we write $B(x; n, z) = B(j_1, \dots, j_{2\nu}+1; n, z)$ if the sequence x corresponds to $(j_1, \dots, j_{2\nu}+1)$. Then by (42) we have

$$B(x, n, z) = a^{2\nu+1} \left(\prod_{k=1}^{\nu} [\pi(2n - h_{2k-1} + z][\pi h_{2k} + z] \right)^{-1}. \tag{43}$$

If n is an even number then $\beta_\nu(n, z) = 0$ since the set A_ν is empty (because the sum in (40) has $2\nu+1$ terms that are equal to ± 2). Hence

$$\lambda_n^+ - \lambda_n^- = 0 \quad \text{if } n \text{ is even, large enough.} \tag{44}$$

Lemma 7. *If n is an odd number with large enough $|n| = 2m+1$ and $\pi n + z_n$ is an antiperiodic eigenvalue of L with $|z_n| < \pi/2$, then*

$$\beta(n, z_n) = \frac{a^{2m+1}}{(4\pi)^{2m}(m!)^2}[1 + O(\ln m/m)]. \tag{45}$$

Proof. In view of (40), we have $\beta_\nu(n, z_n) = 0$ for $\nu < m$, and $\beta_m(n, z_n)$ is the first non-zero term in the sum $\beta = \sum \beta_\nu$. Thus we have

$$\beta(n, z_n) = \sum_{\nu=m}^{\infty} \langle e_n^1, V T_n^{2\nu} e_n^2 \rangle = O(1/|n|^m)$$

because $\|T_n^2\| = O(1/|n|)$ by Lemma 4. By (37) we have

$$\alpha(n, z_n) - z_n = \pm \beta(n, z_n),$$

so by Lemma 5 we obtain that

$$z_n = O(1/|n|). \tag{46}$$

The sum β_m has only one non-zero term because there is one and only one walk from 0 to $2n = 2(2m+1)$ with $2m+1$ steps, each of size 2, namely, its steps are $x_k = 2, \; k = 1, \dots, 2m+1$. By (43) we obtain with $h_k = 2k$ and $\tilde{z}_n = z_n/4\pi$ that

$$\beta_m(n, z_n) = \frac{a^{2m+1}}{(4\pi)^{2m}(m!)^2} \left(\prod_{k=1}^{m} (1 + \tilde{z}_n/k) \right)^{-2}.$$

By (46) we obtain that the product in the above formula equals

$$1 + O[(\log m)/m].$$

Hence

$$\beta_m(n, z_n) = \frac{a^{2m+1}}{(4\pi)^{2m}(m!)^2}\left[1 + O(\frac{\log m}{m})\right].$$

Finally we are going to explain that $\beta_m(n, z_n)$ gives the main term in the asymptotics of $\beta(n, z)$. For that we prove that

$$\beta_{\nu+1}(n, z_n) \leq \frac{a^2}{8}\frac{\log m}{m}\beta_\nu(n, z_n) \quad \forall \nu \geq m. \tag{47}$$

Fix $\nu \geq m$. Let $\tau : A_{\nu+1} \to A_\nu$ be the map that is defined as follows: if $x = (x_1, \dots, x_{2\nu+3}) \in A_{\nu+1}$ and $x_{r+2} = -2$ is the first negative step in x ($1 \leq r \leq 2m - 1$ because $n = 2m + 1$) then $\tau(x) = (x_1, \dots, x_r, x_{r+3}, \dots, x_{2\nu+3})$. Fix $y = (y_1, \dots, y_{2\nu+1}) \in A_\nu$ then the set $\tau^{-1}(\{y\})$ has r elements, namely

$$\tau^{-1}(\{y\}) = \{(y_1, \dots, y_j, -2, 2, y_{j+1}, \dots, y_{2\nu+1}), \quad j = 2, \dots, r + 1\}.$$

Therefore, in view of (43), we have that

$$\sum_{x\in\tau^{-1}(y)} B(x; n, z) = B(y; n, z)\sum_{j=2}^{r+1} D(j, z), \tag{48}$$

where, for $j = 2i$ and $j = 2i + 1$ respectively, we have

$$D(2i, z) = \frac{a^2}{[\pi(2n - 4i + 2) + z][\pi 4i + z]} \leq \frac{a^2}{16(m - i + 1)i}$$

$$D(2i + 1, z) = \frac{a^2}{[\pi 4i + z][\pi(2n - 4i - 2) + z]} \leq \frac{a^2}{16i(m - i)}$$

Now it is easy to see by that $\sum_{j+2}^{r+1} \leq \frac{a^2}{8}(\log m)/m$. Hence by (48) we have

$$\beta_{\nu+1}(n, z) = \sum_{y\in A_\nu}\sum_{x\in\tau^{-1}(y)} B(x; n, z) \leq \beta_\nu(n, z)\cdot\frac{a^2}{8}(\log m)/m.$$

Thus (47) is proven and this completes the proof of Lemma 7.

Lemma 7 together with Lemma 6 imply (28). Theorem 1 is proven.

5. Comments

The proof of Theorem 1 can be easily modified to see that the following statement holds.

Theorem 8. *Let $L = L^0 + V$ where the potential V is defined by*

$$P(x) = ae^{-2\pi ix} + be^{2\pi ix}, \quad Q(x) = \overline{P(x)} \quad a, b \in \mathbb{R}. \tag{49}$$

Then there exists $N_* = N_*(a, b)$ such that for $|n| > 2N_*$ we have $\gamma_n = 0$ if n is even, and if $|n| = 2m + 1$ then

$$\gamma_{2m+1} = \frac{2a^{m+1}b^m}{(4\pi)^{2m}(m!)^2}\left[1 + O(\frac{\log m}{m})\right], \quad m > N_* \tag{50}$$

$$\gamma_{-(2m+1)} = \frac{2a^m b^{m+1}}{(4\pi)^{2m}(m!)^2}\left[1 + O(\frac{\log m}{m})\right], \quad m > N_*. \tag{51}$$

Observe that for large enough $|n|$ all even (periodic) gaps are empty, while all odd (antiperiodic) gaps are nonempty. In fact, the same is true for all n.

Theorem 9. *Consider 1D Dirac operator $L = L^0 + V$, where the potential V is defined by the functions*

$$P(x) = a + be^{2\pi ix}, \quad Q(x) = \overline{P(x)}, \quad a, b \in \mathbb{R}. \tag{52}$$

Then all periodic or antiperiodic eigenvalues of L are of multiplicity one, thus all spectral gaps of L are nonempty.

If we consider the case where the potential is given by (49), then all periodic eigenvalues are of multiplicity 2, and all antiperiodic eigenvalues are of multiplicity 1.

We'll present complete proofs of the statements of this section elsewhere.

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