# A QUOTIENT OF THE SET $[BG, BU(n)]$ FOR A FINITE GROUP $G$ OF SMALL RANK

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ABSTRACT. Let $G_p$ be a Sylow $p$-subgroup of the finite group $G$ and let $\mathrm{Char}_n^G(G_p)$ represent the set of degree $n$ complex characters of $G_p$ that are the restrictions of class functions on $G$. We prove that the natural map $\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$ is an epimorphism for all finite groups $G$ that do not contain a subgroup isomorphic to $(\mathbb{Z}/p)^3$ for any prime $p$. We show, furthermore, that $\psi_G$ is in fact an isomorphism for two types of finite groups $G$: those with periodic cohomology and those of odd order that do not contain a subgroup isomorphic to $(\mathbb{Z}/p)^3$ for any prime $p$.

## 1. INTRODUCTION

Our purpose is to investigate the relationship between the homotopy classes of maps from $BG$ for an finite group $G$ to $BU(n)$ and the degree $n$ characters of the Sylow $p$-subgroups of $G$. Throughout this paper we will let $G$ be a finite group, $p$ a prime dividing the order of $G$, and $G_p$ a Sylow $p$-subgroup of $G$. In addition we will use $\mathrm{Char}_n(G_p)$ to represent the set of degree $n$ complex characters of $G_p$ and will let $\mathrm{Char}_n^G(G_p)$ represent the subset consisting of those degree $n$ complex characters of $G_p$ that are the restrictions of class functions on $G$.

We will first describe a natural map

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p),$$

which will be discussed throughout this paper. Let us examine the following diagram where $\mathrm{Rep}(G, U(n)) = \mathrm{Hom}(G, U(n))/\mathrm{Inn}(U(n))$:

$$
\begin{array}{ccccc}
[BG, BU(n)] & \xrightarrow{\cong} & \prod_{p||G|}[BG, BU(n)_p^\wedge] & \xrightarrow{\mathrm{res}} & \prod_{p||G|}[BG_p, BU(n)_p^\wedge] \\
\bar{\psi}_G \downarrow & & \bar{\phi}_G \downarrow & & \cong \uparrow \\
\prod_{p||G|} \mathrm{Char}_n(G_p) & = & \prod_{p||G|} \mathrm{Char}_n(G_p) & \xleftarrow{\cong} & \prod_{p||G|} \mathrm{Rep}(G_p, U(n)).
\end{array}
$$

The isomorphism in the upper left is a result of work by Jackowski, McClure, and Oliver [13], while the isomorphism on the far right is a result of work by Dwyer and Zabrodsky [9] and the isomorphism in the lower right is a basic result in representation theory. $\mathrm{res}$ is the map induced by the inclusion of the Sylow p-subgroups $G_p$ into $G$. We now let maps $\bar{\psi}_G$ and $\bar{\phi}_G$ be the maps

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that make the diagram commute. We will show in the next section that the image of $\bar{\psi}_G$ and $\bar{\phi}_G$ both lie in the subset $\prod_{p||G|} \text{Char}_n^G(G_p) \subseteq \prod_{p||G|} \text{Char}_n(G_p)$. So we will let $\psi_G$ and $\phi_G$ be the maps $\bar{\psi}_G$ and $\bar{\phi}_G$ respectively, with the range restricted to $\prod_{p||G|} \text{Char}_n^G(G_p)$. Now that we have defined the map $\psi_G$, we state the following three theorems, which express the main results of this paper:

**Theorem 1.1.** *If $G$ is a finite group that does not contain a rank two elementary abelian subgroup, then*

$$[BG, BU(n)] \cong \prod_{p||G|} \text{Char}_n^G(G_p).$$

**Theorem 1.2.** *Let $G$ be a finite group that does not contain a rank three elementary abelian subgroup. If $|G|$ is odd, then*

$$[BG, BU(n)] \cong \prod_{p||G|} \text{Char}_n^G(G_p).$$

**Theorem 1.3.** *If $G$ is a finite group that does not contain a rank three elementary abelian subgroup, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \text{Char}_n^G(G_p)$$

*is an epimorphism.*

Before going on, we should notice that for the diagram above, the maps, excluding maps from $[BG, BU(n)]$, can each be separated into a product over all $p||G|$, yielding a separate diagram for each $p||G|$:

$$
\begin{array}{ccc}
[BG, BU(n)_p^\wedge] & \xrightarrow{\ \text{res}_p\ } & [BG_p, BU(n)_p^\wedge] \\
\bar{\phi}_{G,p} \downarrow & & \cong \uparrow \\
\text{Char}_n(G_p) & \xleftarrow{\ \cong\ } & \text{Rep}(G_p, U(n)).
\end{array}
$$

We will work with each prime dividing the order of $G$ separately. First we will show that the image of the map $\bar{\phi}_{G,p}$ is always contained in the subset $\text{Char}_n^G(G_p) \subseteq \text{Char}_n(G_p)$. Once we have done this, we will let $\phi_{G,p}$ be the map $\bar{\phi}_{G,p}$ with the range restricted to $\text{Char}_n^G(G_p)$. We will show the image is contained in $\text{Char}_n^G(G_p)$ by looking at the map $\text{res}_p$ and by introducing a homology decomposition of $BG$.

## 2. A Homology Decomposition and Conjugation Families

We will use the subgroup decomposition as our homology decomposition, which can be described in the following manner. First let $\mathcal{C}$ be a collection of $p$-subgroups of $G$ closed under conjugation. Let $\mathcal{O}_\mathcal{C}$ be the $\mathcal{C}$-orbit category,

which is the category with objects $G/P$ for $P \in \mathcal{C}$ and with $G$-maps as the morphisms. Let $\mathcal{I}$ be the inclusion functor from $\mathcal{O}_\mathcal{C}$ to the category of $G$-spaces. Composing $\mathcal{I}$ with the Borel construction $(-)_{hG}$ gives a functor $\alpha_\mathcal{C} : \mathcal{O}_\mathcal{C} \to \mathbf{Spaces}$. Notice that $\alpha_\mathcal{C}(G/P)$ has the homotopy type of $BP$ for any $p$-subgroup $P \subseteq G$. This functor naturally induces another functor $b_\mathcal{C} : \mathrm{hocolim}\, \alpha_\mathcal{C} \to BG$; therefore, $b_\mathcal{C}$ gives a homology decomposition of $BG$ if and only if $\mathcal{C}$ is an ample collection of subgroups of $G$ (see [8]).

At this point an ample collection of subgroups of $G$ will be discussed, starting with three definitions.

**Definition 2.1.** Let $P \subseteq G$ be a $p$-subgroup. $P$ is said to be *p-radical* (or *p-stubborn*) if $N_G(P)/P$ has no non-trivial normal $p$-subgroups.

**Definition 2.2.** Let $P \subseteq G$ be a $p$-subgroup. $P$ is said to be *p-centric* if $Z(P)$ is a Sylow $p$-subgroup of $C_G(P)$.

**Definition 2.3** (see [11]). Let $P \subseteq G$ be a $p$-centric subgroup. $P$ is said to be *principal p-radical* if $N_G(P)/PC_G(P)$ has no non-trivial normal $p$-subgroups.

Now let the collection $\mathcal{C}$ be the set of all principal $p$-radical subgroups of $G$. Grodal has shown that this is an ample collection [11], allowing for a homology decomposition of $G$. Also it should be noted that any Sylow $p$-subgroup of $G$ is contained in the collection $\mathcal{C}$ and that any principal $p$-radical subgroup of $G$ is necessarily $p$-radical.

Next notice that the map $res_p$ factors through an inverse limit constructed via the homology decomposition described above:

$$res_p : [BG, BU(n)_p^\wedge] \stackrel{\alpha_p}{\to} \varprojlim_{G/P \in \mathcal{O}_\mathcal{C}} [BP, BU(n)_p^\wedge] \stackrel{\beta_p}{\to} [BG_p, BU(n)_p^\wedge].$$

The map $\alpha_p$ is induced by restriction and the map $\beta_p$ is a projection onto a particular element since $G_p \in \mathcal{C}$. The following diagram then commutes where the maps $\pi_p^1$ and $\pi_p^2$ are projection onto the the set of degree $n$ characters of $G_p$ and onto the set of representations of $G_p$ respectively:

$$\begin{array}{ccc}
[BG, BU(n)_p^{\wedge}] & == & [BG, BU(n)_p^{\wedge}] \\
\alpha_p \downarrow & & \mathrm{res}_p \downarrow \\
\varprojlim_{G/P \in \mathcal{O}_{\mathcal{C}}} [BP, BU(n)_p^{\wedge}] & \xrightarrow{\beta_p} & [BG_p, BU(n)_p^{\wedge}] \\
\cong \uparrow & & \cong \uparrow \\
\varprojlim_{G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Rep}(P, U(n)) & \xrightarrow{\pi_p^2} & \mathrm{Rep}(G_p, U(n)) \\
\cong \downarrow & & \cong \downarrow \\
\varprojlim_{G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Char}_n(P) & \xrightarrow{\pi_p^1} & \mathrm{Char}_n(G_p).
\end{array}$$

The isomorphisms on the left side follow easily from the earlier discussion of $p$-groups. We notice that the composite of the entire right hand column is the map $\bar{\phi}_{G,p}$ whose image we want to find. From the diagram it is obvious that we can instead find the image of the map $\pi_p^1$.

In order to look closer at the image of the map $\pi_p^1$, Alperin's fusion theory will be discussed. In particular the definition of weak conjugation family and an example will be given.

**Definition 2.4** (Alperin [1]). Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. A set $\mathcal{F}$ of pairs $\{(H,T)\}$, where $H \subseteq G_p$ and $T \subseteq N_G(H)$, is called a *weak conjugation family* provided that whenever $A$ and $B$ are subsets of $G_p$ and $B = g^{-1}Ag$ for $g \in G$, there are elements $(H_1, T_1), (H_2, T_2), \dots, (H_n, T_n)$ of $\mathcal{F}$ and elements $x_1, x_2, \dots, x_n, y$ of $G$ such that

(1) $B = (x_1 x_2 \cdots x_n y)^{-1} A (x_1 x_2 \cdots x_n y)$,
(2) $x_i \in T_i$ for $1 \leq i \leq n$ and $y \in N_G(G_p)$, and
(3) $A \subseteq H_1$, $(x_1 x_2 \cdots x_i)^{-1} A (x_1 x_2 \cdots x_i) \subseteq H_{i+1}$ for $1 \leq i \leq n-1$.

The example we will be using of a weak conjugation family is given in the following theorem of Goldschmidt. In order to state the theorem we need to give two more definitions.

**Definition 2.5** (Alperin [1]). Let $Q$ and $R$ be Sylow $p$-subgroups of a finite group $G$ and let $H = Q \cap R$. $H$ is said to be a *tame intersection* if $N_Q(H)$ and $N_P(H)$ are Sylow $p$-subgroups of $N_G(H)$.

**Definition 2.6.** A finite group $G$ is called *$p$-isolated* if it contains a proper subgroup $H \subset G$ such that if $p \mid |H|$ and for any $g \in G \setminus H$, $p \nmid |H \cap gHg^{-1}|$. In this case $H$ is called a strongly $p$-embedded subgroup of $G$.

**Theorem 2.7** (Goldschmidt [10, Theorem 3.4]). *Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. Let $\mathcal{F}_0$ be the set of all pairs $(H, N_G(H))$ where $H \subseteq G_p$ such that there exists a Sylow $p$-subgroup $P$ of $G$ with the following properties:*

*(1) $H = G_p \cap P$ a tame intersection,*
*(2) $C_{G_p}(H) \subseteq H$,*
*(3) $H$ a Sylow $p$-subgroup of $O_{p',p}(N_G(H))$, and*
*(4) $H = P$ or $N_G(H)/H$ is $p$-isolated.*

*$\mathcal{F}_0$ is a weak conjugation family.*

The next proposition will begin to relate Alperin's fusion theory with the inverse limit that has been used in this paper. This will allow us to find the image of the map $\pi_p^1$.

**Proposition 2.8.** *Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. The set $\mathcal{F}$ consisting of all pairs $(H, N_G(H))$, where $H \subseteq G_p$ is a principal $p$-radical subgroup of $G$, is a weak conjugation family.*

Proof: We will show that $\mathcal{F}$ is a weak conjugation family by showing that as a set it contains $\mathcal{F}_0$, the weak conjugation family of Goldschmidt. Let $(H, N_G(H)) \in \mathcal{F}_0$. We first notice that $H$ is a $p$-centric subgroup of $G$ since $C_{G_p}(H)$ is a Sylow $p$-subgroup of $C_G(H)$ by the fact that $H$ is a tame intersection. Since $H$ is a Sylow $p$-subgroup of $O_{p',p}(N_G(H))$ and $H$ is $p$-centric, $H$ must also be principal $p$-radical (see [11, Remark 10.12]). □

**Theorem 2.9.** *Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. If $\mathcal{C}$ is the collection of all $p$-subgroups $H \subseteq G$ that are prinicipal $p$-radical, then the projection map*

$$\pi_p^1 : \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \operatorname{Char}_n(P) \to \operatorname{Char}_n(G_p)$$

*is one to one and is onto the subset $\operatorname{Char}_n^G(G_p) \subseteq \operatorname{Char}_n(G_p)$.*

Proof: First we will show that the image of $\pi_p^1$ is contained in $\operatorname{Char}_n^G(G_p)$. Suppose that $\gamma \in \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \operatorname{Char}_n(P)$. Define a map $\pi_P$ for each $P \in \mathcal{C}$ as the projection from the inverse limit of the complex character of $P$. Notice that $\pi_p^1 = \pi_{G_p}$. Next define $\chi_P = \pi_P(\gamma)$ and let $\chi$ mean $\chi_{G_p}$. Let $\mathcal{F}_p$ be the set consisting of all pairs $(H, N_G(H))$ where $H \in \mathcal{C}$ and $H \subseteq G_p$. By the last proposition, $\mathcal{F}_p$ is a weak conjugation family.

Suppose that $a, b \in G_p$ such that $\exists\, g \in G$ with $b = gag^{-1}$. By the definition of weak conjugation family, there exists $H_1, H_2, \ldots, H_m \in \mathcal{C}$ with $H_i \subseteq G_p$ for $1 \le i \le n$, and $x_1, x_2, \ldots x_m, y \in G$ such that

- $b = (x_1 x_2 \cdots x_n y)^{-1} a (x_1 x_2 \cdots x_n y)$,
- $x_i \in N_G(H_i)$ for $1 \le i \le n$,
- $y \in N_G(G_p)$, and
- $a \subseteq H_1$, $(x_1 x_2 \cdots x_i)^{-1} a (x_1 x_2 \cdots x_i) \subseteq H_{i+1}$ for $1 \le i \le n-1$.

For each $1 \leq i \leq n$ we notice that $\chi|_{H_i}$ respects fusion in $N_G(H_i)$ and $\chi$ respects fusion in $N_G(G_p)$. From this we see that the following three statements hold:

- $\chi(a) = \chi(x_1^{-1} a x_1)$,
- $\chi((x_1 x_2 \cdots x_i)^{-1} a (x_1 x_2 \cdots x_i)) = \chi((x_1 x_2 \cdots x_{i+1})^{-1} a (x_1 x_2 \cdots x_{i+1}))$ for each $1 \leq i \leq n-1$, and
- $\chi((x_1 x_2 \cdots x_n)^{-1} a (x_1 x_2 \cdots x_n)) = \chi(b)$.

Putting these statements together, we see that $\chi(a) = \chi(b)$; therefore, $\chi$ respects fusion in $G$ and so is contained in $\mathrm{Char}_n^G(G_p)$.

To show that $\pi_p^1$ is a one to one correspondence, it is enough to show the existence of an inverse mapping

$$\sigma_{G_p} : \mathrm{Char}_n^G(G_p) \to \varprojlim_{G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Char}_n(P).$$

Let $\chi \in \mathrm{Char}_n^G(G_p)$. We will define $\sigma_{G_p}$ by giving the characters $\chi_P$, which will be $\pi_P \circ \sigma_{G_p}(\chi)$ for each $P \in \mathcal{C}$. Fix $P \in \mathcal{C}$. There exists $g \in G$ such that $P \subseteq g^{-1} G_p g$. So let $\chi_P(h) = \chi(g h g^{-1})$. Since $\chi$ respects fusion in $G$, it is easy to see that this definition of $\chi_P = \pi_P \circ \sigma_{G_p}(\chi)$ is well defined. This definition then gives the following definition of $\sigma_{G_p}$:

$$\sigma_{G_p}(\chi) = \varprojlim_{G/P \in \mathcal{O}_{\mathcal{C}}} \chi_P.$$

It is obvious from the definition that $\pi_p^1 \circ \sigma_{G_p}$ is the identity mapping. We now have to show only that $\sigma_{G_p} \circ \pi_p^1$ is the identity mapping. Let $\gamma$ be an element of the inverse limit and let $\chi = \pi_{G_p}(\gamma)$. Given $P \in \mathcal{C}$, observe that $\pi_P(\gamma)$ must be $\chi_P$ defined above by the nature of the $\mathcal{C}$-orbit category $\mathcal{O}_{\mathcal{C}}$. This observation shows that $\pi_P \circ \sigma_{G_p} \circ \pi_p^1 = \pi_P$ for each $P \in \mathcal{C}$; therefore, $\sigma_{G_p} \circ \pi_p^1$ must be the identity mapping. $\square$

Applying Theorem 2.9 shows that the map $\pi_p^1$ is an isomorphism when the range is restricted to $\mathrm{Char}_n^G(G_p)$. Using this result we see that we can use the maps $\phi_{G,p}$, $\phi_G$, and $\psi_G$ instead of the corresponding maps $\bar{\phi}_{G,p}$, $\bar{\phi}_G$, and $\bar{\psi}_G$. We also see that if the map $\alpha_p$ is a monomorphism or an epimorphism, so is the map $\phi_{G,p}$. In order to study the map $\alpha_p$, we must apply obstruction theory.

## 3. Obstruction Theory

Recall the functor $\alpha_{\mathcal{C}} : \mathcal{O}_{\mathcal{C}} \to \mathbf{Spaces}$, which we encountered in Section 2. We will now compose $\alpha_{\mathcal{C}}$ with the functor $\mathrm{Hom}(-, BU(n)_p^\wedge)$ to get a functor $\mathcal{D}_{\mathcal{C}} : \mathcal{O}_{\mathcal{C}} \to \mathbf{Spaces}$, which takes $G/P$ to a space with the homotopy type of $\mathrm{Hom}(BP, BU(n)_p^\wedge)$. We will see that the obstructions to the map $\alpha_p$ being a monomorphism or an epimorphism lie in the upper limits $\lim^i \pi_j \mathcal{D}_{\mathcal{C}}$.

To see where the obstructions lie, we will need to apply the cosimplicial replacement functor $\prod^*$. There are natural isomorphisms $\pi_t \prod^* \mathcal{D}_\mathcal{C} \cong \prod^* \pi_t \mathcal{D}_\mathcal{C}$ for each $t \geq 0$ and $\lim^s \pi_t \mathcal{D}_\mathcal{C} \cong \pi^s \prod^* \pi_t \mathcal{D}_\mathcal{C}$ for each $s, t \geq 0$ [4, chapter XI]. These together give the natural isomorphism $\lim^s \pi_t \mathcal{D}_\mathcal{C} \cong \pi^s \pi_t \prod^* \mathcal{D}_\mathcal{C}$ for each $s, t \geq 0$.

An element $\gamma \in \lim_{\leftarrow G/P \in \mathcal{O}_\mathcal{C}} [BP, BU(n)_p^\wedge]$ can be thought of as an element of $\pi_0 \mathrm{Tot}_1 \prod^* \mathcal{D}_\mathcal{C}$. Thinking of it as an element of $\pi_0 \mathrm{Tot}_1 \prod^* \mathcal{D}_\mathcal{C}$, the obstructions to $\gamma$ being in the image of the map $\alpha_p$ lie in the cohomotopy groups $\pi^{i+1} \pi_i \Pi^* \mathcal{D}_\mathcal{C} \cong \pi^{i+1} \Pi^* \pi_i \mathcal{D}_\mathcal{C}$ for $i \geq 1$. Similarly, the obstructions to $\gamma$ having a unique preimage in $[BG, BU(n)_p^\wedge]$ lie in the cohomotopy groups $\pi^i \pi_i \Pi^* \mathcal{D}_\mathcal{C} \cong \pi^i \Pi^* \pi_i \mathcal{D}_\mathcal{C}$ for $i \geq 1$.

Recall that for any representation $\rho : P \to U(m)$ with $P$ a $p$-group,

$$\pi_n(\mathcal{D}_\mathcal{C}(G/P), B\rho) \cong \pi_n(BC_{U(m)}(Im(\rho))) \otimes \mathbb{Z}_p^\wedge \text{ (see [9])}.$$

Since the centralizer of a finite $p$-subgroup of $U(m)$ is the product of various $U(i)$, it is clear that $BC_{U(m)}(Im(\rho))$ is simply connected; therefore,

$$\pi^2 \pi_1 \Pi^* \mathcal{D}_\mathcal{C} = 0 = \pi^1 \pi_1 \Pi^* \mathcal{D}_\mathcal{C}.$$

The obstructions we need to investigate then lie in the cohomotopy groups

$$\pi^i \pi_j X = \pi^i \pi_j \Pi^* \mathcal{D}_\mathcal{C} = \lim^i \pi_j \mathcal{D}_\mathcal{C}$$

for $j + 1 \geq i \geq j \geq 2$. In particular, if $\lim^{i+1} \pi_i \mathcal{D}_\mathcal{C} = 0$ for $i \geq 2$, then the map $\alpha_p$ is an epimorphism, and if $\lim^i \pi_i \mathcal{D}_\mathcal{C} = 0$ for $i \geq 2$, then the map $\alpha_p$ is an monomorphism. We will next discuss situations where these higher limits are indeed trivial.

## 4. Vanishing Obstruction Groups

Before we proceed in showing when these higher limits vanish, we first need a couple of definitions. These will allow us to use standard theory first introduced by Jackowski, McClure, and Oliver [12, 13].

**Definition 4.1.** A functor $F : \mathcal{O}_\mathcal{C} \to Ab$ is called an *atomic* functor if it vanishes on all but possibly a single isomorphism class of objects. An atomic functor will be said to be concentrated on an isomorphism class on which it does not vanish.

**Definition 4.2** (Definition 4.7 of [13]). For any prime $p$, finite group $G$, and $\mathbb{Z}_{(p)}(G)$-module $M$, let $F_M : \mathcal{O}_{\mathcal{S}_p(G)} \to \mathbb{Z}_{(p)} - mod$ be the atomic functor concentrated on the free orbit $G/1$ with $F_M(G/1) = M$. Set $\Lambda^*(G; M) = \lim^* F_M$.

The following lemma follows directly from work by Jackowski, McClure, and Oliver, in particular from the first proof of Lemma 5.4 in [12].

**Lemma 4.3.** *Suppose $\mathcal{C}$ is a subcollection of $\mathcal{S}_p(G)$ for a finite group $G$ such that for any functor $F : \mathcal{O}_{\mathcal{S}_p(G)} \to \mathbb{Z}_{(p)} - mod$, $\lim^* F \cong \lim^* F|_{\mathcal{O}_{\mathcal{C}}}$. Also assume that $F' : \mathcal{O}_{\mathcal{C}} \to \mathbb{Z}_{(p)} - mod$ is an atomic functor concentrated on the isomorphism class of $G/P$ for some $P \in \mathcal{C}$. Then*

$$\lim{}^* F' \cong \Lambda^*(N_G(P)/P; F'(G/P)).$$

We will be filtering our functor by atomic functors in order to see where $\Lambda^*(\Gamma, M)$ vanishes. The relevant lemma is a result of work by Grodal [11]. (See also [13, Prop. 4.11] and [5, Proposition 5.8].)

**Lemma 4.4** (See [11]). *Let $\Gamma$ be a finite group and $M$ a finitely generated $\mathbb{Z}_{(p)}(\Gamma)$-module. If $m$ is a non-negative integer such that $\mathrm{rk}_p(\Gamma) \leq m$, then $\Lambda^j(\Gamma, M) = 0$ for any $j > m$. In particular, if $p^{m+1} \nmid |\Gamma|$, then $\Lambda^j(\Gamma, M) = 0$ for any $j > m$.*

Lemma 4.4 gives us the following proposition when a functor is filtered by atomic functors.

**Proposition 4.5.** *Let $G$ be a finite group and $p$ a prime number dividing $|G|$. Let $\mathcal{C}$ be the set of all subgroups of $G$ that are principal $p$-radical. If $m$ is a positive integer such that for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq m$, then for any functor $F : \mathcal{O}_{\mathcal{C}} \to \mathbb{Z}_{(p)} - mod$, $\lim^j F = 0$ for any $j > m$.*

Using this proposition in conjunction with the work of the previous section, we get the following result.

**Theorem 4.6.** *Let $G$ be a finite group and $p$ a prime number dividing $|G|$. Let $\mathcal{C}$ be the set of all subgroups of $G$ that are principal $p$-radical. If for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq 2$, then the map $\alpha_p$ described earlier is an epimorphism. If for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq 1$, then the map $\alpha_p$ is an isomorphism.*

In the rest of this paper, we will use Theorem 4.6 and various results about finite groups to prove the three theorems presented in the introduction.

## 5. Finite Groups, part I

We will start by proving Theorem 1.1. We begin this process with the following lemma, which shows that the obstructions vanish when a Sylow $p$-subgroup of $G$ has a center of small index in itself.

**Lemma 5.1.** *Let $G$ be a finite group, $p$ a prime number dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. Let $m$ be an integer such that $p^m = [G_p : Z(G_p)]$. If $P$ is a $p$-centric subgroup of $G$, then $p^m \nmid [N_G(P) : P]$. In particular, if $G_p$ is abelian, then the only $p$-centric subgroups of $G$ are the Sylow $p$-subgroups.*

Proof: Let $P \subseteq G$ be a $p$-centric subgroup, which is not a Sylow $p$-subgroup of $G$. We may assume $P \subset G_p$, which implies that $Z(G_p) \subseteq C_G(P) \cap G_p$. Let $H = C_G(P) \cap G_p$, which is a Sylow $p$-subgroup of $C_G(P)$. It is clear that $Z(P) \subseteq H$, which implies that $Z(P) = H$ since $P$ is $p$-centric;

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therefore, $Z(G_p) \subseteq Z(P) \subseteq P$. Suppose that $Z(G_p) = P$, then $G_p \subseteq C_G(P)$, contradicting $P$ being $p$-centric. So we see that $Z(G_p)$ is strictly contained in $P$. Now by hypothesis, $p^m = [G_p : Z(G_p)]$; thus, by the strict containment $Z(G_p) \subset P$, $[G_p : P] < p^m$. This clearly implies $p^m \nmid [G : P]$. Then since $[N_G(P) : P]$ divides $[G : P]$, $p^m \nmid [N_G(P) : P]$. □

Proof of Theorem 1.1: As was discussed earlier, it is enough to show that for each prime $p$ that divides $|G|$, $\phi_{G,p}$ is an isomorphism. We assume that $G$ does not contain a rank two elementary abelian subgroup. Let $G_p$ be a Sylow $p$-subgroup of $G$. If $G_p$ is cyclic, it is abelian; therefore, the only $p$-centric subgroups of $G$ are the Sylow $p$-subgroups. By Theorem 4.6, the map $\alpha_p$ is an isomorphism as is the map $\phi_{G,p}$. On the other hand, if $G_p$ is not a cyclic group, then $p = 2$ and $G_p$ is a generalized quaternion group. Suppose in this case that $P \subseteq G_p$ is a $p$-centric subgroup of $G$. We see that either $P$ is the cyclic group of index two in $G_p$, $P$ is cyclic of order four and intersects the cyclic group of index two in $G_p$ in a cyclic group of order two, or $P$ itself is a generalized quaternion group. If $P$ is of index two in $G_p$, $4 \nmid [N_G(P) : P]$. In the second case, $N_{G_p}(P)$ is a quaternion group of order eight and $4 \nmid [N_G(P) : P]$. If $P$ is generalized quaternion and not a Sylow $p$-subgroup, it follows that $N_{G_p}(P)$ is a generalized quaternion group with $[N_{G_p}(P) : P] = 2$ while if $P$ is a Sylow $p$-subgroup, $[N_{G_p}(P) : P] = 1$. We have now proven in each case that if $P$ is a $p$-centric subgroup of $G$, then $p^2 \nmid [N_G(P) : P]$; furthermore, the map $\alpha_p$ is an isomorphism. From our previous discussion, we see that the map $\phi_{G,p}$ is also an isomorphism. □

The proof of Theorem 1.2 follows from the next lemma.

**Lemma 5.2** (See [15, 6.3.10]). *Let $G$ be a finite group of odd order. If $p \| G \|$ and $P$ is a principal $p$-radical subgroup of $G$ such that $\mathrm{rk}(P) \leq 2$, then $P$ is a Sylow $p$-subgroup of $G$.*

Proof: Let $p \| G \|$ and $P$ be a principal $p$-radical subgroup of $G$ such that $\mathrm{rk}(P) \leq 2$. Suppose $P$ is not a Sylow $p$-subgroup of $G$. Let $W = N_G(P)/C_G(P)$. The derived subgroup $W'$ of $W$ is a $p$-subgroup (see [15, 6.3.10]). For a subgroup $H$ of $N_G(P)$, we will denote by $\bar{H}$ the image of $HC_G(P)$ in $W$. Now since $P$ is principal $p$-radical, $O_p(W) = \bar{P}$. This implies $W' \subseteq \bar{P}$. So $W/\bar{P} = N_G(P)/PC_G(P)$ is an abelian group. Since $P$ is not a Sylow $p$-subgroup of $G$, $p | [N_G(P) : P]$. $P$ is principal $p$-radical; therefore, $Z(P)$ is a Sylow $p$-subgroup of $C_G(P)$. This shows that $p | [N_G(P) : PC_G(P)]$; thus, $N_G(P)/PC_G(P)$ has a non-trivial normal $p$-subgroup. This contradicts $P$ being a principal $p$-radical subgroup of $G$, so $P$ is a Sylow $p$-subgroup of $G$. □

According to Lemma 5, if $G$ is a rank two group of odd order with $p \| G \|$, then the map $\alpha_p$ is an isomorphism, and the map $\phi_{G,p}$ is also an isomorphism. It follows from the introduction that if $G$ is a rank two group of odd order, then the map $\phi_G$ is also an isomorphism, thus proving Theorem 1.2.

This only leaves Theorem 1.3 to be proven. As we have been doing in order to show that the map $\phi_G$ is onto, we will be showing that the map $\phi_{G,p}$ is onto for each prime $p$ dividing $|G|$. We will need to approach the prime 2 differently then the odd primes. Thus, we will break Theorem 1.3 into the following two propositions.

**Proposition 5.3.** *Let $G$ be a finite group and $p > 2$ an odd prime dividing $|G|$. If $G$ does not contain a rank three elementary abelian $p$-subgroup, then $\phi_{G,p} : [BG, BU(n)_p^\wedge] \to \mathrm{Char}_n^G(G_p)$ is an epimorphism.*

**Proposition 5.4.** *If $G$ be a finite group of even order not containing a rank three elementary abelian 2-subgroup, then $\phi_{G,2} : [BG, BU(n)_2^\wedge] \to \mathrm{Char}_n^G(G_2)$ is an epimorphism.*

It is clear that Theorem 1.3 follows immediately from these two propositions. We will spend the next two sections proving these two propositions.

## 6. 3-Groups of Maximal Class

We will prove Proposition 5.3 first. Before we do, we need to focus on a type of group that will arise in this proof. For Section 6 we will focus on the case where a Sylow 3-subgroup of $G$ is a 3-group of maximal class with the purpose of proving the following lemma.

**Lemma 6.1.** *Let $G$ be a finite group and $G_3$ be a Sylow 3-subgroup of $G$ such that $G_3$ is a 3-group of maximal class and $|G_3| \geq 3^5$. If $P \subseteq G_3$ is a 3-centric subgroup of $G$, then $[N_{G_3}(P) : P] \leq 3^2$.*

Proof: We will let the series

$$Z(G_3) = C_{n-1} \subset C_n \subset \cdots \subset C_3 \subset C_2 \subset G_3$$

be the lower central series of $G_3$. This is defined inductively by $C_2 = [G_3, G_3]$ and $C_i = [C_{i-1}, G_3]$ for $i > 2$. Since $G_3$ is of maximal class, $n$ is the integer such that $|G_3| = 3^n$. We also introduce another subgroup of $G_p$, which will be denoted by $C_1$. The subgroup $C_1$ is defined by the property that $C_1/C_4$ is the centralizer in $G_3/C_4$ of $C_2/C_4$ (see [3, section 2]). By these definitions we note that each $C_i$ is characteristic in $G_p$, thus giving the increasing sequence of $n-1$ distinct proper subgroups of $G_p$:

$$Z(G_3) = C_{n-1} \subset C_n \subset \cdots \subset C_3 \subset C_2 \subset C_1 \subset G_3.$$

By the work of Blackburn [3], either $C_1$ is abelian or the commutator subgroup $C_2$ is abelian and $[C_1, C_2] = Z(G_3)$. Now assume that $P$ is a proper subgroup of $G_3$ that is a 3-centric subgroup of $G$. We will look at two cases: the first when $P \subseteq C_1$ and the second when $P \nsubseteq C_1$.

Case I: Assume that $P \subseteq C_1$. Since $P$ is a 3-centric subgroup of $G$, $Z(C_1)$ must be a proper subgroup of $P$. Notice that if either $C_1$ is abelian or the commutator subgroup $C_2$ is abelian and $[C_1, C_2] = Z(G_3)$, then $C_3 \subseteq Z(C_1)$. This inclusion implies that $[G_3 : P] \leq 3^2$.

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Case II: Assume that $P \nsubseteq C_1$. Let $N = N_{G_3}(P)$. Let $t$ be an element of $P \setminus (P \cap C_1)$. It can be shown that $|C_{G_3}(t)| = 3^2$ and that $C_{G_3}(t) \subseteq P$; therefore, any subgroup of $G_3$, which contains $P$, is of maximal class.

Suppose that $P$ is abelian. This implies that $P = C_{G_3}(t)$ and that $N$ is generated by $t$ and $C_{n-2}$; thus $[N : P] = 3$.

Now suppose that $P$ is not abelian. Since $N$ is of maximal class, $[N, N] \subseteq C_2$; therefore, $[N, N]$ is abelian. Seeing that $P \lhd N$ and $P$ is not abelian, $[N : P] = 3. \square$

## 7. FINITE GROUPS, PART II

We need to mention one more lemma before we proceed with the proof of Proposition 5.3. This Lemma deals with the case where a Sylow $p$-subgroup of $G$ is a metacyclic $p$-group and $p$ is an odd prime. It follows from the work of J. Dietz [6] and of Martino and Priddy [14].

**Lemma 7.1** (See [6], and [14, Theorem 2.7]). *Let $G$ be a finite group, $p$ an odd prime divisor of $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. If $G_p$ is a metacyclic group, $|G_p| > p^3$, and $P \subseteq G_p$ is a principal $p$-radical subgroup of $G$, then $P = G_p$.*

Proof: Suppose $G_p$ is a metacyclic group with $|G_p| > p^3$. Martino and Priddy [14] have shown that such a $G_p$ is a Swan Group. In the proof they proved that if $H \subset G_p$ is a proper subgroup with $C_{G_p}(H) = Z(H)$, then

$$(N_{G_p}(H)/HC_{G_p}(H)) \cap O_p(\text{Out}(H)) \neq \{1\}.$$

So no proper subgroup $H \subset G_p$ can be a principal $p$-radical subgroup of $G$. $\square$

Proof of Proposition 5.3: Recall that $G$ is a finite group that does not contain a rank three elementary abelian subgroup and $p$ is an odd prime dividing $|G|$. Let $G_p$ be a Sylow $p$-subgroup of $G$ and let $n$ be an integer such that $|G_p| = p^n$. Since Theorem 1.1 takes care of the case where $\text{rk}(G_p) = 1$, we may assume that $\text{rk}(G_p) = 2$. It has been shown by Blackburn [2] that one of the following holds for $G_p$ (see also [7]):

(1) $n < 5$,
(2) $n \geq 5$ and $G_p$ is metacylic,
(3) $n \geq 5$, $p = 3$, and $G_3$ is a 3-group of maximal class,
(4) $n \geq 5$ and $G_p = \langle a, b, c | a^p = b^p = c^{p^{n-2}} = 1, [a, b] = c^{p^{n-3}}, c \in Z(G_p) \rangle$, or
(5) $n \geq 5$, $e \neq 0$ is a quadratic nonresidue mod $p$, and
$G_p = \langle a, b, c | a^p = b^p = c^{p^{n-2}} = [b, c] = 1, [a, b^{-1}] = c^{ep^{n-3}}, [a, c] = b \rangle.$

Recall that it is enough to show that in each of these cases, if $P \subseteq G_p$ is a principal $p$-radical subgroup of $G$, then $[N_{G_p}(P) : P] \leq p^2$. Lemma 7.1 gives

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this result for case 2, while case 3 was shown in Section 6. To proceed with the other three cases, recall that by Lemma 5.1 it is enough to show that $[G_p, Z(G_p)] \leq p^3$. The first case follows immediately since $G_p$ must have a non-trivial center. Notice in both of the remaining cases that the center of $G_p$ contains the element $c^p$ implying that $|Z(G_p)| \geq p^{n-3}$. This concludes these two cases as well as the proof of Proposition 5.3. $\square$

Proof of Proposition 5.4: Recall that $G$ is a finite group of even order that does not contain a rank three elementary abelian subgroup. As before let $G_2$ be a Sylow 2-subgroup of $G$. Suppose that $P \subset G_2$ is a principal 2-radical subgroup of $G$, which is not itself a Sylow 2-subgroup of $G$. Notice that $N_G(P)/PC_G(P) \subseteq \operatorname{Out}(P)$ is both non-trivial and not a 2-group. In particular this means that $P$ is a 2-group of rank one or two with an automorphism of odd order. Richard Thomas has classified all such 2-groups [16, 17]. According to his classification $P$ must be one of the following:

(1) $P \cong Q_8 * D_8$,
(2) $P \cong Q_8 \times Q_8$,
(3) $P \cong Q_8 \wr \mathbb{Z}_2$,
(4) $P \cong Q_8 * C$ where $\operatorname{Out}(C)$ is a 2-group, $\operatorname{Out}(C) \lhd \operatorname{Out}(P)$, and $C \neq D_8$,
(5) $P \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$, or
(6) $P \cong U_{64} \in \operatorname{Syl}_2(\operatorname{PSU}_3(\mathbb{F}_4))$.

Our notation is as follows: $Q_8$ means the quaternion 8 group, $D_8$ means the dihedral group of size 8, $*$ refers to the central product, and $\wr$ refers to the wreath product.

Since $P$ is a principal 2-radical subgroup of $G$, we know from the definition that $N_G(P)/PC_G(P) \cap O_2(\operatorname{Out}(P)) = \{1\}$; therefore, let $S \in \operatorname{Syl}_2(N_G(P)/PC_G(P))$ and $T \in \operatorname{Syl}_2(\operatorname{Out}(P)/O_2(\operatorname{Out}(P)))$. It is clear then that $S$ must be isomorphic to a subgroup of $T$. In order to show that $\operatorname{rk}_2(N_G(P)/P) \leq 2$, it is enough to show that $\operatorname{rk}_2(T) \leq 2$. We will do this on a case by case basis using the list above. In cases 1 and 2, $T \cong D_8$, giving $\operatorname{rk}_2(T) = 2$. In cases 3, 4, and 5, $T \cong \mathbb{Z}_2$, implying that $\operatorname{rk}_2(T) = 1$. In the final case, $T \cong \mathbb{Z}_4$, implying that $\operatorname{rk}_2(T) = 1$. We see that for any possible $P \subseteq G_2$ that is a principal 2-radical subgroup of $G$, $\operatorname{rk}_2(N_G(P)/P) \leq 2$. This concludes the proof of Proposition 5.4. $\square$

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