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Blow-ups of the Toda Lattices and their Intersections with the Bruhat Cells

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Abstract. We study the topology of the set of singular points (blow-ups) in the solution of the nonperiodic Toda lattice defined on real split semisimple Lie algebra $\mathfrak{g}$. The set of blow-ups is called the Painlevé divisor. The isospectral manifold of the Toda lattice is compactified through the companion embedding which maps the manifold to the flag manifold associated with the underlying Lie algebra $\mathfrak{g}$. The Painlevé divisor is then given by the intersections of the compactified manifold with the Bruhat cells in the flag manifold. In this paper, we give explicit description of the topology of the Painlevé divisor for the cases of all the rank two Lie algebra, A_2, B_2, C_2, G_2, and A_3 type. The results are obtained by using the Mumford system and the limit matrices introduced originally for the periodic Toda lattice. We also give a Lie theoretic description of the Painlevé divisor of codimension one case, and propose several conjectures for the general case.

1. Introduction

It is well-known that the generalized (nonperiodic) Toda lattices associated with semisimple Lie algebra $\mathfrak{g}$ of rank l possess l polynomial invariants, *the Chevalley invariants*, which provide their integrability [**4, 12**]. The isospectral manifold determined by those polynomials defines a l-dimensional affine variety, and it can be compactified by adding the points associated with the *blow-ups* in the solution of the Toda lattice. Those points are defined as the zeros of τ-functions giving an explicit solution of the Toda lattice [**13**], and the set of zeros is sometimes called the *Painlevé divisor*. The number of τ-functions is given by the rank of the algebra, and each τ-function can be labeled by a dot in the corresponding Dynkin diagram. Then the Painlevé divisor consists of l components $\{\Theta_{\{k\}} : k = 1, \cdots, l\}$, and each $\Theta_{\{k\}}$ is associated with a root α_k in the Dynkin diagram. As in the case of periodic Toda [**1**], the singularities of the divisor are canonically associated with the Dynkin diagrams, i.e. $\Theta_J = \cap_{k\in J}\Theta_{\{k\}}$ for a subdiagram $J \subset \{1, \cdots, l\}$.

In [**8**], Flaschka and Haine considered a *companion* embedding map of the isospectral manifold into a flag manifold, and identified the Painlevé divisor as the

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intersection with certain Bruhat cells in the Bruhat decomposition,

$$G/B^+ = \bigcup_{w \in W} N^- w B^+ / B^+,$$

where G is the Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$, B^+ the Borel subgroup, N^- the unipotent subgroup and W the Weyl group of G. Then the compactification of the isospectral manifold can be obtained by gluing the Painlevé divisor in the flag manifold.

On the other hand, the real part of the compactified isospectral manifold was studied in [**5**], where the manifold was constructed by extending the work of Kostant in [**12**]. Theorem 3.2 in [**12**] describes part of the isospectral manifold of the Toda lattice in terms of one connected component of a split Cartan subgroup of G. There is a total of 2^l connected components which are labeled by a set of signs $\epsilon = (\epsilon_1, \cdots, \epsilon_l)$. In [**5**] instead all the connected components of a Cartan subgroup are involved. The upshot is that now a split Cartan subgroup $H_{\mathbb{R}}$, with all its connected components, becomes an open dense subset in the compactified isospectral manifold. This manifold is then described as a union of convex polytopes Γ_ϵ glued as in [**6**], and each connected component with the sign ϵ of the Cartan subgroup is the interior of the corresponding polytope Γ_ϵ. The convexity of the polytope Γ_ϵ can be shown by Atiyah's convexity theorem [**3**] with the torus embedding (conjugate to the companion embedding) in the flag manifold.

In this paper, we study the topological structure of the Painlevé divisor as the blow-ups of the Toda lattice on the polytopes. In Section 2, we provide a background information necessary for the present study which includes the isospectral manifold, the companion embedding to the flag manifold, the τ-functions and the Painlevé divisor.

In Section 3, we define the *limit matrices* to parametrize the Painlevé divisor. The limit matrix was first introduced in [**2**] for the periodic Toda lattice for a parametrization of the Birkhoff strata of the hyperelliptic Jacobi variety, and the existence of the limit matrix was shown based on Sato's theory of universal Grassmannians. We here give a direct proof of the existence of the limit matrix by using a factorization of the unipotent subgroup N^- (Proposition 3.2), and show that the companion embedding maps the limit matrix to the corresponding Bruhat cell.

In Section 4, we define the Mumford system for the A_l Toda lattice, which may be considered as an extension of the system used to parametrize the moduli space associated with the hyperelliptic Riemann surface and its Jacobian. The Mumford system gives an explicit coordinate for the Painlevé divisor through the limit matrix. Then we prove a topological equivalence between the top cell of A_k and certain Painlevé divisor of A_j with $j > k$ (Proposition 4.2).

In Section 5, we provide several explicit results for the Toda lattices on the Lie algebra $\mathfrak{g}$ of all rank 2 cases, A_2, B_2, C_2, G_2, and of type A_3.

Then in Section 6, we give a Lie theoretic description of the Painlevé divisor based on the results in [**5, 6**]. We first review the details of the construction of the compactified manifold by gluing the polytopes Γ_ϵ of the Cartan subgroup $H_{\mathbb{R}}$, and it is worth keeping in mind that $H_{\mathbb{R}}$ is not necessarily a Cartan subgroup in G but rather in another Lie group $\tilde{G}$ defined in Notation 6.2. We then define an "algebraic" version of the Painlevé divisor, denoted by $\tilde{\Theta}^a_{\{i\}}$, in terms of the simple root character χ_{α_i} defined on the Cartan subgroup. The characters χ_{α_k} can be expressed in terms of the characters χ_{ω_i} associated to the fundamental weights

ω_i which have similar properties to the τ-functions. Then we give Conjectures 6.5 and 6.18 that $\Theta_{\{i\}}$ and $\tilde{\Theta}^a_{\{i\}}$ become homeomorphic if small modifications on $\tilde{\Theta}^a_{\{i\}}$ are introduced. These conjectures about the structure of the Painlevé divisors are verified in all the rank 2 cases as well as in A_3 discussed in Section 5. The homology of the spaces constructed in terms of the root characters is computable with the same methods used in [**5**]. Conjectures 6.5 and 6.18 then would allow the computation of the homology of the Painlevé divisors.

2. Toda lattices and Painlevé divisor

The generalized (nonperiodic) Toda lattice equation related to real split semisimple Lie algebra $\mathfrak{g}$ of rank l is defined by the Lax equation, [**4, 12**],

$$\frac{dL}{dt} = [A, L] \tag{2.1}$$

where the Lax pair (L, A) are given by

$$\left\{ \begin{aligned} L(t) &= \sum_{i=1}^{l} b_i(t) h_{\alpha_i} + \sum_{i=1}^{l} (a_i(t) e_{-\alpha_i} + e_{\alpha_i}) \\ A(t) &= -\sum_{i=1}^{l} a_i(t) e_{-\alpha_i} \end{aligned} \right. \tag{2.2}$$

Here $\{h_{\alpha_i}, e_{\pm\alpha_i}\}$ is the Cartan-Chevalley basis of the algebra $\mathfrak{g}$ with the positive simple roots $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ which satisfy the relations,

$$[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm C_{j,i} e_{\pm\alpha_j}\ , \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j},$$

where $(C_{i,j})$ is the $l \times l$ Cartan matrix of the Lie algebra $\mathfrak{g}$. The Lax equation (2.1) then gives

$$\left\{ \begin{aligned} \frac{db_i}{dt} &= a_i \\ \frac{da_i}{dt} &= -\left(\sum_{j=1}^{l} C_{i,j} b_j \right) a_i \end{aligned} \right. \tag{2.3}$$

The integrability of the system can be shown by the existence of the Chevalley invariants, $\{I_k(L) : k = 1, \cdots, l\}$, which are given by the homogeneous polynomial of $\{(a_i, b_i) : i = 1, \cdots, l\}$. Then in this paper we are concerned with the topology of the *real* isospectral manifold defined by

$$Z(\gamma)_{\mathbb{R}} = \left\{ (a_1, \cdots, a_l, b_1 \cdots, b_l) \in \mathbb{R}^{2l} \ : \ I_k(L) = \gamma_k \in \mathbb{R},\ k = 1, \cdots, l \right\}.$$

The manifold $Z(\gamma)_{\mathbb{R}}$ can be compactified by adding the set of points corresponding to the *blow-ups* of the solution. The set of blow-ups has been shown to be characterized by the intersections with the Bruhat cells of the flag manifold G/B^+, which are referred to as the *Painlevé divisors*, and the compactification is described in the flag manifold. In order to explain some details of this fact, we first define the set $\mathcal{F}_\gamma$,

$$\mathcal{F}_\gamma := \{ L \in e_+ + \mathcal{B}^- \ : \ I_k(L) = \gamma_k, k = 1, \cdots, l \},$$

where $e_+ = \sum_{i=1}^{l} e_{\alpha_i}$, and $\mathcal{B}^-$ is the Lie algebra of B^-. Then there exists a unique element $n_0 \in N^-$ such that $L \in \mathcal{F}_\gamma$ can be conjugated to the normal form C_γ,

$L = n_0 C_\gamma n_0^{-1}$ [**11**]. In the case of $\mathfrak{g} = \mathfrak{sl}(l+1,\mathbb{R})$, C_γ is the companion matrix given by

$$C_\gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 0 & 1 \\ (-1)^l \gamma_l & \cdots & \cdots & -\gamma_1 & 0 \end{pmatrix},$$

where the Chevalley invariants are given by the elementary symmetric polynomials of the eigenvalues of L. Then we define:

DEFINITION 2.1. [**8**]: The companion embedding of $\mathcal{F}_\gamma$ is defined as the map,

$$\begin{array}{rccc} c_\gamma : & \mathcal{F}_\gamma & \longrightarrow & G/B^+ \\ & L & \longmapsto & n_0^{-1} \bmod B^+ \end{array}$$

where $L = n_0 C_\gamma n_0^{-1}$ with $n_0 \in N^-$.

The isospectral manifold $Z(\gamma)_\mathbb{R}$ can be considered as a subset of $\mathcal{F}_\gamma$ with the element L in the form of (2.2). The Toda lattice (2.1) then defines a flow on $\mathcal{F}_\gamma$ which is embedded as follows:

PROPOSITION 2.1. [**8**] *The Toda flow maps to the flag manifold as*

$$\begin{array}{rcccl} c_\gamma : & L(t) & \longmapsto & n_0^{-1} n(t) & \bmod B^+ \\ & & & = n_0^{-1} e^{tL^0} & \bmod B^+ \end{array}$$

where $L^0 = n_0 C_\gamma n_0^{-1}$, *and* $n(t) \in N^-, b(t) \in B^+$ *are given by the factorization of* $e^{tL^0} = n(t)b(t)$.

This Proposition is based on the solution formula using the factorization, i.e.

$$L(t) = n(t)^{-1} L^0 n(t) = b(t) L^0 b(t)^{-1}. \tag{2.4}$$

However one should note that the factorization is not always possible, and the general form is given by the Bruhat decomposition,

$$G = \bigcup_{w \in W} N^- w B^+ \ .$$

It has been also shown in [**8, 1**] that for a subset J of $\{1, \cdots, l\}$ the blow-up of the solution $L(t)$ at $t = t_J$ corresponds to the case

$$e^{t_J L^0} \in N^- w_J B^+, \quad \text{where} \quad w_J \neq id \ ,$$

where w_J is the longest element of the Weyl subgroup W_J associated with the Dynkin diagram labeled by J. Thus the Toda flow meets only those Bruhat cells, and we see that the Painlevé divisor, denoted by $\mathcal{D}_J$, characterizes the intersection of the Bruhat cell corresponding to the longest element $w_J \in W$ with the compactified isospectral manifold $\tilde{Z}(\gamma)_\mathbb{R}$, i.e.

$$\mathcal{D}_J = \tilde{Z}(\gamma)_\mathbb{R} \bigcap N^- w_J B^+ / B^+, \quad \text{with} \quad w_\emptyset = id. \tag{2.5}$$

Here $\tilde{Z}(\gamma)_\mathbb{R}$ is the closure of the image of the isospectral manifold under the companion embedding c_γ in (2.1), and it has a decomposition (intersection with the

Bruhat decomposition),

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \overline{c_\gamma(Z(\gamma)_{\mathbb{R}})} = \bigsqcup_{J\subset\{1,\cdots,l\}} \mathcal{D}_J .$$

The analytical structure of the blow-ups can be obtained by the τ-functions, which are defined by

$$b_k = \frac{d}{dt}\ln\tau_k, \qquad a_k = a_k^0 \prod_{j=1}^{l} (\tau_j)^{-C_{k,j}}, \tag{2.6}$$

From (2.3), the tau-functions then satisfy the bilinear equations,

$$\tau_k\tau_k'' - (\tau_k')^2 = \prod_{j\neq k} (\tau_j)^{-C_{k,j}}, \tag{2.7}$$

where $\tau_k'' = d^2\tau_k/dt^2$ and $\tau_k' = d\tau_k/dt$, and $\tau_0 = 1, \tau_{l+2} = 0$. Then the Painlevé divisor $\mathcal{D}_J$ can be defined as

$$c_\gamma(L(t)) \in \mathcal{D}_J \overset{\text{def}}{\Longleftrightarrow} \tau_k(t) = 0, \text{ iff } k \in J. \tag{2.8}$$

We also define the set Θ_J as a disjoint union of $\mathcal{D}_{J'}$,

$$\Theta_J := \bigsqcup_{J'\supseteq J} \mathcal{D}_{J'} .$$

Then we have a stratification of $\tilde{Z}(\gamma)_{\mathbb{R}}$,

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \Theta^{(0)} \supset \Theta^{(1)} \supset \cdots \supset \Theta^{(l)} = c_\gamma(C_\gamma), \quad \text{with} \quad \Theta^{(k)} = \bigcup_{|J|=k} \Theta_J .$$

The irreducibility of the Painlevé divisors $\Theta_{\{k\}}$ was shown in [**7**], where the analog of Riemann's singularity theorem for the compactified complex manifold $\tilde{Z}(\gamma)_{\mathbb{C}}$ was also discussed.

In the case of a given matrix (adjoint) representation, one can construct an explicit solution for $\{a_j(t)\}$ in the matrix $L(t)$. First we have the following Lemma:

LEMMA 2.1. *The diagonal element $b_{j,j}$ of the upper triangular matrix $b \in B^+$ in the factorization (2.1) is expressed by*

$$b_{j,j}(t) = \frac{D_j[\exp(tL^0)]}{D_{j-1}[\exp(tL^0)]}$$

where $D_j[\exp(tL^0)]$ is the determinant of the j-th principal minor of $\exp(tL^0)$, i.e.

$$D_j[\exp(tL^0)] = \left(e^{tL^0} v_1 \wedge \cdots \wedge v_j,\; v_1 \wedge \cdots \wedge v_j \right).$$

with the standard basis $\{v_i\}_{i=1}^{l}$ of $\mathbb{R}^n$ with some n.

Then using the formula in (2.4), we can obtain the solution $a_j(t)$ and the explicit representation of the τ-functions in terms of the determinants $D_j[\exp(tL^0)]$. Thus the τ-functions are the entire functions of t given by polynomials of exponential functions $\exp(\lambda_k t)$ with the eigenvalues λ_k of L^0. In fact, one can show that the $D_j[\exp(tL^0)]$ can be expressed as the Hankel determinant,

$$D_j[\exp(tL^0)] = \begin{vmatrix} D_1 & D_1' & \cdots & D_1^{(j-1)} \\ D_1' & D_1'' & \cdots & D_1^{(j)} \\ \vdots & \ddots & \ddots & \vdots \\ D_1^{(j-1)} & \cdots & \cdots & D_1^{(2j-2)} \end{vmatrix}, \quad j = 1, 2, \cdots, n,$$

where $D_1 = D_1[\exp(tL^0)] = \sum_{i=1}^n \rho_i \exp(\lambda_i t)$ for some $\rho_i \in \mathbb{R} \setminus \{0\}$. With this formula, one can study a detailed behavior of the τ functions [**9**].

Remark 2.2. On any Cartan subgroup of G there is another set of functions having similar properties to the τ functions. These are the root characters χ_{ω_i} associated to fundamental weights ω_i. For example, the simple root characters $\chi_i := \chi_{\alpha_i}$ can be expressed in terms of the χ_{ω_i} with the inverse of the Cartan matrix of the Lie algebra $\mathfrak{g}$. This is the same relation that exists between the a_i in (2.2) and the τ functions. The signs of the characters χ_i change when chamber walls $\chi_i = -1$ are crossed in a Cartan subgroup in analogy to what happens to the signs of the a_i when a Painlevé divisor is crossed. If χ_i^* denotes the root character of the simple root α_i corresponding to each separate chamber in the Cartan subgroup, then χ_i^* is continuous through α_i walls and through some α_j walls. The points on a Cartan subgroup where $\chi_i^* + 1 = 0$ are called the α_i-*negative wall* [**5**], which defines an "algebraic"version of the Painlevé divisors $\Theta_{\{i\}}$ in terms of the functions χ_i. This set is compactified and gives rise to a topological space $\tilde{\Theta}^a_{\{i\}}$ (see Section 6).

3. Limit matrices, Painlevé divisors and the companion embedding

Here we show that Painlevé divisors can be parametrized using limit matrices. These were first introduced in [**2**] for the case of the periodic Toda lattice. The main result in [**2**] is to show the existence of the limit matrix, say L_J, which is constructed by conjugating the Lax matrix $L(t)$ with a matrix in N^- and taking the limit $t \to t_J$ corresponding to the factorization $e^{t_J L^0} = \hat{n}(t_J) w_J \hat{b}(t_J)$ for $\hat{n} \in N^-$ and $\hat{b} \in B^+$. In our case of the nonperiodic Toda lattice limit matrices arise as a consequence of Theorem 3.3 of [**8**].

Definition 3.1. For fixed $J \subset \{1, \cdots, l\}$ we let P_J denote the parabolic subgroup of G containing B^+ and associated to J. One can define a projection

$$\pi_J : G/B^+ \to G/P_J.$$

The group N^- factors as $N^- = N_J^- N_J^+$ with $N_J^\pm := N^- \cap w_J N^\pm w_J^{-1}$. Hence any $n \in N^-$ can be written as $n = uy$ with $u \in N_J^-$ and $y \in N_J^+$ unique elements. We thus obtain factorizations (notation of Proposition 2.1): $n_0^{-1} n(t) = u(t) y(t)$, and $\pi_J(u(t) y(t) B^+) = u(t) P_J$.

Since the limit $n_0^{-1} n(t) B^+$ as $t \to t_J$ exists (see Proposition 2.1), it is of the form $\hat{u}(t_J) w_J B^+$ for some $\hat{u}(t_J) \in N_J^-$. Then we have

Proposition 3.1. *With notation as in Definition 3.1, the limit of $u(t)$ as $t \to t_J$ exists,*

$$\lim_{t \to t_J} u(t) = \hat{u}(t_J) \in N_J^-.$$

Proof. Since $\lim_{t\to t_J} n_0^{-1}n(t)B^+ = \hat{u}(t_J)w_JB^+$ with $\hat{u}(t_J) \in N_J^-$, by applying π_J we obtain

$$\lim_{t\to t_J} n_0^{-1}n(t)P_J = \hat{u}(t_J)P_J.$$

On the other hand $\pi_J(n_0^{-1}n(t)B^+) = \pi_J(u(t)y(t)B^+) = u(t)P_J$. Therefore, since the top N^- orbit in G/P_J can be identified with N_J^- we then obtain a limit inside this group: $\lim_{t\to t_J} u(t) = \hat{u}(t_J)$. □

DEFINITION 3.2. A limit matrix of L is an element L_J in the set $\mathcal{F}_\gamma$ of the form,

$$L_J = Ad(\hat{u}^{-1}(t_J))C_\gamma, \quad \text{with} \quad \hat{u}(t_J) \in N_J^-.$$

Let $u(t) = \hat{u}(t_J)\overline{u}(t)$. Then $\lim_{t\to t_J} \overline{u}(t) = e$, with e the identity, and we have

PROPOSITION 3.2. *The limit matrix is also expressed as*

$$\lim_{t\to t_J} Ad(y(t))L(t) = L_J(t_J).$$

Proof. We have

$$L(t) = Ad(n^{-1}(t)n_0)C_\gamma = Ad(y^{-1}(t)u^{-1}(t))C_\gamma = Ad(y^{-1}(t)\overline{u}^{-1}(t)\hat{u}^{-1}(t_J))C_\gamma$$

Hence $Ad(y(t))L(t) = Ad(\overline{u}(t))L_J$. We now take limit and use that $\overline{u}(t) \to e$ to conclude. □

The result can be summarized in the diagram,

$$\begin{array}{ccc}
L(t) & \xrightarrow{c_\gamma} & n_0^{-1}n(t) \ \mathrm{mod} B^+ \\
\downarrow & & \downarrow \\
Ad(y(t))L(t) & \longrightarrow & u(t)y(t) \ \mathrm{mod} B^+ \\
{\scriptstyle t\to t_J}\downarrow & & \downarrow{\scriptstyle t\to t_J} \\
L_J(t_J) & \xrightarrow{c_\gamma} & \hat{u}(t_J)w_J \ \mathrm{mod} B^+
\end{array}$$

REMARK 3.3. For each set J we can define a function $\phi_J : Z(\gamma)_\mathbb{R} \to Ad(N_J^-)C_\gamma$ given by $\phi_J(L) = Ad(y)L$. A limit matrix L_J is then an element in the boundary $\overline{\phi_J(Z(\gamma)_\mathbb{R})} \setminus \phi_J(Z(\gamma)_\mathbb{R})$. The closure takes place inside $Ad(N_J^-)C_\gamma$. This gives another description of $\mathcal{D}_J$ which allows one to define the compactification of the isospectral manifold $Z(\gamma)_\mathbb{R}$ using only the limit matrices. The companion embedding then takes a simple form. First note that any limit matrix L_J is contained in the N_J^- orbit of C_γ. Hence $L_J = Ad(\hat{u}^{-1}(t_J))C_\gamma$ where $\hat{u}(t_J) \in N_J^-$ is unique. For $J = \emptyset$, we just set $\hat{u}(t) = n_0^{-1}n(t)$. The companion embedding then maps L_J to $\hat{u}(t_J)w_JB^+$.

REMARK 3.4. In all our examples $y(t) = y_J(t)$ can be replaced with $x_J^{-1}(t)$ an element in N_J^- defined below in terms of a companion matrix associated to a Levi factor.

In the following, we determine the limit matrices for the case of $\mathfrak{g} = \mathfrak{sl}(l+1,\mathbb{R})$ (the general case will be discussed elsewhere). Let consider the set J be given by s consecutive numbers, say $\{i+1,\cdots,i+s\}$, $(i+s\le l)$. Then from (2.8) this implies that the divisor $\mathcal{D}_J$ consists of the points corresponding to the zeros of τ-functions, $\tau_k = 0$ for all $k\in J$. On the other hand, from (2.7), we can show

LEMMA 3.1. *For each $j \in J = \{i+1,\cdots,i+s\}$, $\tau_j(t)$ has the following form near its zero $t = t_J$,*

$$\tau_{i+k}(t) \simeq (t-t_J)^{m_k} + \cdots, \quad \text{with} \quad m_k = k(s+1-k),\ 1\le k\le s. \tag{3.1}$$

Proof. Substituting (3.1) into (2.7), and using $\tau_i(t_J)\neq 0$, we have $m_k = k(m_1+1-k)$. Then $\tau_{i+s+1}(t_J)\neq 0$ implies $m_1 = s$. □

Then using (2.6) one can find the blow-up structure of the functions (a_j,b_j). We note here that this structure is the same as the case of the smaller system $\mathfrak{sl}(s+1,\mathbb{R})$ with the total blow-up. The Lax matrix of this smaller system is just the submatrix (here the b-variables are modified from the original form in (2.2), e.g. $b_k - b_{k-1}\to b_k$),

$$L' = \begin{pmatrix} b_{i+1} & 1 & 0 & \cdots & 0 \\ a_{i+1} & b_{i+2} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & b_{i+s} & 1 \\ 0 & \cdots & \cdots & a_{i+s} & b_{i+s+1} \end{pmatrix}$$

Then one can put this matrix into a companion matrix by a unique element $x'_J \in N^-$, the set of $(s+1)\times(s+1)$ lower triangular matrices with 1's on the diagonals. The companion matrix $C'_J = {x'}_J^{-1}L'x'_J$ and x'_J are given by

$$C'_J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & 0 & 1 \\ (-1)^s\xi_{s+1} & \cdots & \cdots & -\xi_2 & \xi_1 \end{pmatrix}, \quad x'_J = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ * & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ * & \cdots & \cdots & * & 1 \end{pmatrix},$$

where ξ_k's are the polynomials of (a_j,b_j) in the Lax matrix. Since the Toda lattice is isospectral, those polynomials stays constants even when all of the elements (a_j,b_j) blows up. Then the limit matrix L_J is obtained by the limit of the conjugation of L with $x_J\in N_J^+$,

$$L_J = \lim_{t\to t_J} Ad(x_J^{-1}(t))L(t), \quad \text{with} \quad x_J = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & x'_J & \vdots & \vdots \\ \vdots & \ddots & \ddots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}.$$

Let us now give an example to illustrate the construction:

EXAMPLE 3.5. The A_3 Toda lattices: The Lax matrix is given by

$$L = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ a_1 & b_2 & 1 & 0 \\ 0 & a_2 & b_3 & 1 \\ 0 & 0 & a_3 & b_4 \end{pmatrix}, \quad \sum_{k=1}^{4} b_k = 0.$$

The limit matrices L_J are determined as follows:

a) $J = \{1\}$: Then $\tau_1(t) \sim t_* = (t - t_{\{1\}})$ implies that $a_1 \sim t_*^{-2}, a_2 \sim t_*, b_1 \sim t_*^{-1}, b_2 \sim t_*^{-1}$ and others are regular. The limit matrix is then obtained by the limit $x_{\{1\}}^{-1} L x_{\{1\}} \to L_{\{1\}}$ as $t_* \to 0$,

$$L_{\{1\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\xi_2 & \xi_1 & 1 & 0 \\ \eta_1 & 0 & b_3 & 1 \\ 0 & 0 & a_3 & b_4 \end{pmatrix}, \quad \text{with} \quad x_{\{1\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\xi_1 = b_1 + b_2,\ \xi_2 = b_1 b_2 - a_1,\ \eta_1 = -a_2 b_1$ are the parameters for the divisor $\mathcal{D}_{\{1\}}$.

b) $J = \{2\}$: With $\tau_2 \sim t_* = (t - t_{\{2\}})$, we have $a_1 \sim t_*, a_2 \sim t_*^{-2}, a_3 \sim t_*, b_2 \sim t_*^{-1}, b_3 \sim t_*^{-1}$, and the limit matrix is given by

$$L_{\{2\}} = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \eta_1 & -\xi_2 & \xi_1 & 1 \\ 0 & \eta_2 & 0 & b_4 \end{pmatrix}, \quad \text{with} \quad x_{\{2\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -b_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\xi_1 = b_2 + b_3,\ \xi_2 = b_2 b_3 - a_2,\ \eta_1 = -a_1 b_3,\ \eta_2 = -a_3 b_2$ are the parameters for the divisor $\mathcal{D}_{\{2\}}$.

c) $J = \{3\}$: This case is similar to the case $J = \{1\}$, and we have

$$L_{\{3\}} = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ a_1 & b_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \eta_1 & -\xi_2 & \xi_1 \end{pmatrix}, \quad \text{with} \quad x_{\{3\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -b_3 & 1 \end{pmatrix}$$

where $\xi_1 = b_3 + b_4,\ \xi_2 = b_3 b_4 - a_3,\ \eta_1 = -a_2 b_4$ are the parameters for the divisor $\mathcal{D}_{\{3\}}$.

d) $J = \{1,2\}$: We construct $L_{\{1,2\}}$ from $L_{\{1\}}$ with $x_{\{1,2\}}$,

$$L_{\{1,2\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \xi_3' & -\xi_2' & \xi_1' & 1 \\ \eta_1' & 0 & 0 & b_4 \end{pmatrix}, \quad \text{with} \quad x_{\{1,2\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \xi_2 & -\xi_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\xi_1' = \xi_1 + b_3,\ \xi_2' = \xi_2 + \xi_1 b_3,\ \eta_1' = \eta_1 + \xi_2 b_3$ with ξ_1, ξ_2, η_1 in $L_{\{1\}}$ are then the parameters for the divisor $\mathcal{D}_{\{1,2\}}$. This can be of course done with a matrix $x_{\{2,1\}}$ from $L_{\{2\}}$.

e) $J = \{2,3\}$: This is similar to the previous case d), and we have

$$L_{\{2,3\}} = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta_1' & \xi_3' & -\xi_2' & \xi_1' \end{pmatrix}, \quad \text{with} \quad x_{\{2,3\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \xi_2 & -\xi_1 & 1 \end{pmatrix}$$

where $\xi_1' = \xi_1 + b_2,\ \xi_2' = \xi_2 + \xi_1 b_2,\ \eta_1' = \eta_1 + \xi_2 b_2$ with ξ_1, ξ_2, η_1 in $L_{\{2\}}$ are then the parameters for the divisor $\mathcal{D}_{\{2,3\}}$.

f) $J = \{1,3\}$: We construct the limit matrix $L_{\{1,3\}}$ from $L_{\{1\}}$ by using $x_{\{1,3\}} = x_{\{3\}}$.

$$L_{\{1,3\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\xi_2 & \xi_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta_1' & 0 & -\xi_2' & \xi_1' \end{pmatrix},$$

where $\xi_1' = b_3 + b_4,\ \xi_2' = b_3 b_4 - a_3,\ \eta_1' = -\eta_1 b_3$.

4. The A_l Toda lattice and the Mumford system

In [**14**], Mumford gave a parametrization of the theta divisor for a hyperelliptic Jacobian with triples of polynomials determined by the factorization of the corresponding hyperelliptic curve. This is related to the periodic Toda lattice, but the idea can be also applied to the present case of nonperiodic Toda lattice on $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$:

Definition 4.1. The Mumford system for the spectral curve $F_l(\lambda) = \det(\lambda I - L)$ of degree $l+1$ is the triples of polynomials $(u_d(\lambda), v_d(\lambda), w_d(\lambda))$ determined by

$$F_l(\lambda) = u_d(\lambda) w_d(\lambda) + v_d(\lambda),$$

where u_d is a monic polynomial of degree d, v_d is a polynomial of degree less than d with the condition $v_d(\mu_k) = F_l(\mu_k)$ for the roots of $u_d(\lambda) = 0$, and w_d is a monic polynomial of degree $l + 1 - d$.

One can write u_d and v_d in the form,

$$\begin{cases} u_d(\lambda) & = \prod_{k=1}^{d} (\lambda - \mu_k), \\ v_d(\lambda) & = \sum_{k=1}^{d} F_l(\mu_k) \prod_{j \neq k} \dfrac{\lambda - \mu_j}{\mu_k - \mu_j}. \end{cases}$$

When $d = l$ (the rank of the matrix), the μ-variables can globally parametrize the isospectral manifold $Z(\gamma)_{\mathbb{R}}$ by taking an explicit relation with the original variables (a_k, b_k) in L, for example, choose the l-th principal minor of L to be $u_l(\lambda)$. One can also define an integrable system for the Mumford system as

$$\begin{cases} \dfrac{du}{dt} & = v, \\ \dfrac{dv}{dt} & = u \left[\dfrac{vw}{u}\right]_+ - vw, \\ \dfrac{dw}{dt} & = -\left[\dfrac{vw}{u}\right]_+, \end{cases} \tag{4.1}$$

where $[f(\lambda)]_+$ indicates the polynomial part of $f(\lambda)$ (see [**14, 15**] for the periodic case). The integrability is a direct consequence of the isospectrality, i.e. fixing the curve $F_l(\lambda) = uw + v$. It is also interesting to note that the system has a Lax form,

$$\frac{dM}{dt} = [M, B], \quad \text{with} \quad M = \begin{pmatrix} h & u \\ w & -h \end{pmatrix}, \quad B = \frac{1}{2h} \begin{pmatrix} 0 & v \\ b & 0 \end{pmatrix},$$

where $h^2 = v$ and $b = [vw/u]_+$. Then the first equation in (4.1) gives the system,

$$\frac{d\mu_k}{dt} = -\frac{F_l(\mu_k)}{\prod_{j\neq k}(\mu_k - \mu_j)}, \quad k = 1, \cdots, d.$$

Using the Lagrange interpolation formula,

$$\sum_{k=1}^{d} \frac{\mu_k^n}{\prod_{j\neq k}(\mu_k - \mu_j)} = \begin{cases} 0 & \text{if } n < d-1, \\ 1 & \text{if } n = d-1. \end{cases}$$

we obtain, after integration,

$$\sum_{k=1}^{d} \int_{\mu_0}^{\mu_k} \frac{\lambda^n d\lambda}{F_l(\lambda)} = \begin{cases} c_n & n < d-1, \\ -t + c_{d-1} & n = d-1. \end{cases}$$

with some constants μ_0 and $c_k, k = 1, \cdots, d-1$. In particular, the system with $d = 1$ gives

$$\frac{d\mu_1}{dt} = -F_l(\mu_1),$$

whose solution has $l+1$ fixed points at $\mu_1 = \lambda_k$ for $k = 1, \cdots, l+1$, and blows up when $\mu_1 > \max_k(\lambda_k)$ or $\mu_1 < \min_k(\lambda_k)$. One can also show the following Proposition on the topology of certain 1-dimensional Painlevé divisors $\Theta_J(A_k)$ of the A_k Toda lattice:

Proposition 4.1. *Let $J_{k-1} \subset \{1, \cdots, k\}$ be either $\{1, \cdots, k-1\}$ or $\{2, \cdots, k\}$. Then the Painlevé divisors $\Theta_{J_{k-1}}(A_k)$ are all homeomorphic to circle, i.e.*

$$\Theta_{J_{k-1}}(A_k) \cong S^1, \quad \text{for} \quad k = 1, 2, \cdots,$$

where $J_0 = \emptyset$.

Proof. Since the homeomorphism between the divisors with $J = \{1, \cdots, k-1\}$ and $J = \{2, \cdots, k\}$ is obvious, we consider the case with $J = \{1, \cdots, k-1\}$. In this case, the limit matrix has the form,

$$L_J = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 1 & 0 \\ (-1)^{k-1}\xi_k & \cdots & \cdots & \cdots & \xi_1 & 1 \\ \eta & 0 & \cdots & \cdots & 0 & b_{k+1} \end{pmatrix},$$

where ξ_i are the coefficients of the polynomial $|\lambda I - L'| = \lambda^k + \sum_{i=1}^{k}(-1)^i \xi_i \lambda^{k-i}$ with L' given by the first $k \times k$ part of the Lax matrix L, and $\eta = -a_k b_1 \cdots b_{k-1}$ (in the limit $t \to t_J$). Then from the Mumford system $F_k(\lambda) := |\lambda I - L_J| = u_1 w_1 + v_1$, we have

$$\eta = -F_k(\mu_1), \quad \text{with} \quad \mu_1 = b_{k+1},$$

where $v_1 = -\eta$. This indicates that the Painlevé divisor $\mathcal{D}_J(A_k)$ has just one connected component of $\mathbb{R}$, and adding the highest divisor $\Theta_{\{1,\cdots,k\}}(A_k)$ we see that the closure $\Theta_J(A_k)$ is homeomorphic to S^1. This completes the proof. □

We can also show the following on higher dimensional divisors, PROPOSITION 4.2. *Let $J_n^k \subset \{1, \cdots, k+n\}$ be either $\{1, \cdots, n\}$ or $\{k+1, \cdots, k+n\}$. The Painlevé divisors $\mathcal{D}_{J_n^k}(A_{k+n})$ are all homeomorphic to the top cell of the A_k Toda lattice, i.e.*

$$\mathcal{D}_{\emptyset}(A_k) \cong \mathcal{D}_{J_n^k}(A_{k+n}), \quad \text{for} \quad n \geq 1.$$

Proof. Let J be $\{1, \cdots, n\}$. Then the limit matrix L_J is given by

$$L_J = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is the $(n+1) \times (n+1)$ companion matrix of the corresponding block in the matrix L, A_2 is the $(n+1) \times k$ matrix having zero entries except 1 at the bottom left corner (i.e. $(k,1)$-entry), A_3 is the $k \times (n+1)$ matrix having zero entries except η at the top left corner ($(1,1)$-entry), and A_4 is the $k \times k$ submatrix of Lax matrix,

$$A_4 = \begin{pmatrix} b_{n+2} & 1 & \cdots & \cdots & 0 \\ a_{n+2} & b_{n+3} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{k+n} & 1 \\ 0 & \cdots & \cdots & a_{k+n} & b_{k+n+1} \end{pmatrix}.$$

Then from the factorization of $F_{k+n}(\lambda) = |\lambda I - L_J| = u_k(\lambda) w_k(\lambda) + v_k(\lambda)$, we have the Mumford system,

$$u_k = |\lambda I - A_4|, \quad v_k = -\eta |\lambda I - B_4|, \quad w_k = |\lambda I - A_1|,$$

where B_4 is the $(k-1) \times (k-1)$ submatrix of A_4 by deleting the first row and column vectors. Thus we have

$$\eta |\mu_j I - B_4| = -F_{k+n}(\mu_j), \quad \text{for} \quad j = 1, \cdots, k,$$

The left-hand side of this equation has the same form for all the cases with fixed k, and the right-hand side gives a real one-dimensional affine curve for each $\mu_j \in \mathbb{R}$ of degree $k+n$. This implies that all the divisors $\mathcal{D}_{J_n^k}(A_{k+n})$ have the same parametrization, so that they are all homeomorphic. □

Since the boundaries of each $\mathcal{D}_{J_n^k}(A_{k+n})$ seems to have the same structure for $n \geq 1$, we expect

CONJECTURE 4.2. *The Painlevé divisors $\Theta_{J_n^k}(A_{k+n})$ for $n \geq 1$ are all homeomorphic, i.e.*

$$\Theta_{J_1^k}(A_{k+1}) \cong \cdots \cong \Theta_{J_n^k}(A_{k+n}).$$

5. Examples for rank 2 and 3

5.1. The A_2-Toda lattice. The Lax matrix is a 3×3 matrix given by

$$L = \begin{pmatrix} b_1 & 1 & 0 \\ a_1 & b_2 & 1 \\ 0 & a_2 & b_3 \end{pmatrix}, \quad \text{with} \quad \sum_{k=1}^{3} b_k = 0,$$

and the spectral curve $F_2(\lambda) = \det(\lambda I - L)$ is

$$F_2(\lambda) = \lambda^3 + I_1 \lambda - I_2,$$

where the Chevalley invariants $I_k(L)$ are given by

$$I_1(L) = b_1 b_2 + b_2 b_3 + b_1 b_3 - a_1 - a_2, \quad I_2(L) = b_1 b_2 b_3 - a_1 b_3 - a_2 b_1.$$

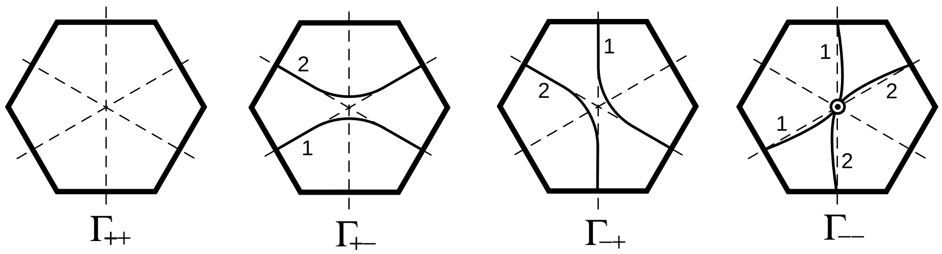

FIGURE 1. The A_2 hexagons $\Gamma_{\epsilon_1\epsilon_2}$. The numbers 1 and 2 mark the Painlevé divisors $\Theta_{\{1\}}$ and $\Theta_{\{2\}}$. The center point (double circle) in Γ_{--} indicates the divisor $\Theta_{\{1,2\}}$.

To parametrize the isospectral manifold $Z(\gamma)_{\mathbb{R}}$, we consider for example the following Mumford system with the choice of the triples,

$$\begin{cases} u_2(\lambda) & = \prod_{k=1}^2(\lambda-\mu_k) = \begin{vmatrix} \lambda - b_2 & -1 \\ -a_2 & \lambda - b_3 \end{vmatrix}, \\ v_2(\lambda) & = F_2(\mu_1)\dfrac{\lambda-\mu_2}{\mu_1-\mu_2} + F_2(\mu_2)\dfrac{\lambda-\mu_1}{\mu_2-\mu_1}, \\ w_2(\lambda) & = \dfrac{1}{u_2(\lambda)}(F_2(\lambda)-v_2(\lambda)) = \lambda + w_0. \end{cases}$$

Then in terms of (μ_1,μ_2) the Chevalley invariants are given by

$$I_1 = -(\mu_1+\mu_2)^2 + \mu_1\mu_2 - a_1, \quad I_2 = -\mu_1\mu_2(\mu_1+\mu_2) - a_1 b_3,$$

which leads to

$$a_1(\mu_k - b_3) = -F_2(\mu_k), \quad k = 1, 2.$$

Also from (4.1), we have the Toda flow in the variable (μ_1,μ_2),

$$\frac{d\mu_k}{dt} = (-1)^k \frac{F_2(\mu_k)}{\mu_1-\mu_2}, \quad k = 1, 2,$$

which is also obtained by setting $a_1(\mu_k - b_3) = (-1)^{k+1}(\mu_1-\mu_2)d\mu_k/dt$. The system has 6 fixed points with $(\mu_1,\mu_2) = (\lambda_i,\lambda_j)$, $1 \le i \ne j \le 3$, and for each set of the signs (ϵ_1,ϵ_2) with $\epsilon_i = \mathrm{sign}(a_i(0))$ the integral manifold gives a hexagon, denoted by $\Gamma_{\epsilon_1\epsilon_2}$ as in Fig.1. In particular, one can easily see that there is no blow-up in Γ_{++} (note that $I_1(L) = \gamma_1$ makes all the variables be bounded, if both a_1 and a_2 are positive). Those four hexagons are glued together along with their boundaries according to the standard action of the Weyl group S_3 on the signs (ϵ_1,ϵ_2), and the compactified manifold is topologically equivalent to a connected sum of two Klein bottles $\mathbb{K}$ [**9**]. This can be seen by counting the Euler characteristic, $6(vertices) - 12(edges) + 4(hexagons) = -2$ and the nonorientability (see [**5**] for the general argument on the compactification based on the Weyl group action).

The Painlevé divisor $\mathcal{D}_{\{1\}}$ corresponding to $\tau_1 = 0$ can be parametrized by the limit matrix,

$$L_{\{1\}} = \begin{pmatrix} 0 & 1 & 0 \\ -\xi_2 & \xi_1 & 1 \\ \eta_1 & 0 & b_3 \end{pmatrix}, \tag{5.1}$$

where $\xi_1 = b_1 + b_2, \xi_2 = b_1 b_2 - a_1$ and $\eta_1 = -a_2 b_1$. The matrix $L_{\{1\}}$ is obtained by the limit,

$$L_{\{1\}} = \lim_{t \to t_1} x_{\{1\}}^{-1}(t) L(t) x_{\{1\}}(t), \quad \text{with} \quad x_{\{1\}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the spectral curve $F_2(\lambda)$ gives the algebraic relations (the Chevalley invariants),

$$\xi_1 + b_3 = 0, \ I_1(L_{\{1\}}) = \xi_2 + \xi_1 b_3, \ I_2(L_{\{1\}}) = \eta_1 + \xi_2 b_3,$$

which leads to

$$\begin{aligned} \mathcal{D}_{\{1\}} &= \left\{ (\xi_1, \xi_2, \eta_1, b_3) \in \mathbb{R}^4 \ : \ \xi_1 = -b_3, I_1 = \gamma_1, I_2 = \gamma_2 \ \right\} \\ &= \left\{ (\eta_1, b_3) \in \mathbb{R}^2 \ : \ \eta_1 = -F_2(b_3) = -b_3^3 - \gamma_1 b_3 + \gamma_2 \right\}. \end{aligned}$$

We thus show that the closure of $\mathcal{D}_{\{1\}}$ is homeomorphic to a circle S^1, and it intersects with three subsystems corresponding to $(a_2 = 0, b_3 = \lambda_k)$ for $k = 1, 2, 3$. The Mumford equation (4.1) can be used to provide a dynamics on $\mathcal{D}_{\{1\}}$ with $\mu_1 = b_3$ and $\eta_1 = d\mu_1/dt$,

$$\frac{d\mu_1}{dt} = -F_2(\mu_1).$$

In Figure 1, $\Theta_{\{1\}}$ is shown as a curve with the label "1". The $\Theta_{\{2\}}$ has the similar structure. Thus we obtain:

Proposition 5.1. *The compactified manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ and the Painlevé divisor have the following topology,*

$$\tilde{Z}(\gamma)_{\mathbb{R}} = \Theta_{\emptyset} \cong \mathbb{K} \sharp \mathbb{K}, \quad \Theta_{\{1\}} \cong \Theta_{\{2\}} \cong S^1.$$

We also note by taking out the divisors $\Theta_{\{1\}}$ and $\Theta_{\{2\}}$ from $\tilde{Z}(\gamma)_{\mathbb{R}}$ that the top cell $\mathcal{D}_{\emptyset} = \tilde{Z}(\gamma)_{\mathbb{R}} \cap N^- B^+ / B^+$ is diffeomorphic to a torus $\mathbb{T}$ with a hole of a disk $\mathbb{D}$, i.e.

$$\mathcal{D}_{\emptyset} \cong \mathbb{T} \setminus \mathbb{D}.$$

5.2. The C_2 Toda lattice. Since the B_2 Toda lattice has the same structure as the C_2 case, we discuss only the latter one. The Lax matrix for C_2 Toda lattice is given by a 4×4 matrix,

$$L = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ a_1 & b_2 & 1 & 0 \\ 0 & 2a_2 & -b_2 & 1 \\ 0 & 0 & a_1 & -b_1 \end{pmatrix}$$

whose spectral curve $F_2(\lambda) = \det(\lambda I - L)$ is

$$F_2(\lambda) = \lambda^4 - I_1 \lambda^2 + I_2$$

with the Chevalley invariants $I_k(L)$,

$$I_1 = b_1^2 + b_2^2 + 2a_1 + 2a_2, \ I_2 = (b_1 b_2 - a_1)^2 + 2b_1^2 a_2.$$

The corresponding polytope $\Gamma_{\epsilon_1 \epsilon_2}$ with the signs $\epsilon_k = \text{sign}(a_k)$ is given by a octagon with eight vertices associated with the fixed point of the system, $a_1 = a_2 = 0$. Those vertices are expressed as $(b_1, b_2) = (\sigma_1 \lambda_i, \sigma_2 \lambda_j)$ for $\sigma_k \in \{\pm\}, i \neq j \in \{1, 2\}$. Gluing those octagons along their boundaries, we find that the compactified manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ is topologically equivalent to a connected sum of three Klein bottles $\mathbb{K}$. Again just

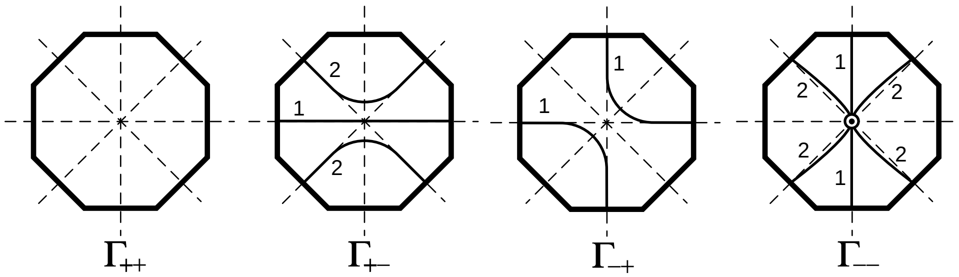

FIGURE 2. The C_2 octagons $\Gamma_{\epsilon_1\epsilon_2}$ and the Painlevé divisors $\Theta_{\{1\}}, \Theta_{\{2\}}$ and $\Theta_{\{1,2\}}$.

count the Euler characteristic, $8(vertices) - 16(edges) + 4(octagons) = -4$, and the nonorientability leads to the result.

The Painlevé divisor $\Theta_{\{1\}}$ is now parametrized by the limit matrix

$$L_{\{1\}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\xi_2 & \xi_1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \eta_1 & 0 & -\xi_2 & -\xi_1 \end{pmatrix}$$

where $\xi_1 = b_1 + b_2, \xi_2 = b_1 b_2 - a_1$ and $\eta_1 = -2a_2 b_1^2$. Then the Chevalley invariants $I_k(L)$ are expressed by

$$I_1 = \xi_1^2 - 2\xi_2, \quad I_2 = \xi_2^2 - \eta_1$$

from which we obtain

$$\eta_1 = \frac{1}{4}((\xi_1^2 - I_1)^2 - 4I_2).$$

This implies that the $\Theta_{\{1\}}$ is homeomorphic to S^1 and intersects with four subsystems corresponding to $\xi_1 = \sigma(\lambda_1 \pm \lambda_2)$ with $\sigma \in \{\pm 1\}$ and with the divisor $\Theta_{\{2\}}$ in Γ_{--} (see Figure 2),

Unlike the case of A_2 Toda lattice, the divisor $\Theta_{\{2\}}$ has a different structure. The corresponding limit matrix $L_{\{2\}}$ is given by

$$L_{\{2\}} = \begin{pmatrix} b_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \eta_2 & \xi_3 & 0 & 1 \\ 0 & -\eta_2 & 0 & -b_1 \end{pmatrix}$$

where $\xi_3 = b_2^2 + 2a_2, \eta_2 = a_1 b_2$. The invariants I_k are then given by

$$I_1 = b_1^2 + \xi_3, \quad I_2 = \xi_3 b_1^2 - 2\eta_2 b_1,$$

and we obtain

$$\eta_2 = -\frac{1}{b_1} F_2(b_1).$$

Because of the singularity in this equation at $b_1 = 0$, the $\Theta_{\{2\}}$ is shown to be homeomorphic to a figure eight, where each circle intersects two subsystems corresponding to either $b_1 = |\lambda_k|$ or $b_1 = -|\lambda_k|$ with $k = 1, 2$. The node of the figure eight corresponds to the divisor $\Theta_{\{1,2\}}$ (see Figure 2). We thus obtain, PROPOSITION 5.2. *The topology of the isospectral manifold of C_2 and the Painlevé divisor is given by*

$$\tilde{Z}(\gamma)_{\mathbb{R}} \cong \mathbb{K} \sharp \mathbb{K} \sharp \mathbb{K}\,, \quad \Theta_{\{1\}} \cong S^1, \quad \Theta_{\{2\}} \cong S^1 \vee S^1.$$

The B_2 Toda lattice has the same structure, but $\Theta_{\{1\}}$ and $\Theta_{\{2\}}$ have the opposite structure.

5.3. The G_2 Toda lattice. We use the following one for the Lax matrix,

$$L = \begin{pmatrix} b_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ a_1 & b_2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & a_2 & b_1 - b_2 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 2a_1 & 0 & 1 & 0 & 0 \\ 0 & \cdot & 0 & 2a_1 & -b_1 + b_2 & 1 & 0 \\ 0 & \cdot & \cdot & 0 & a_2 & -b_2 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & a_1 & -b_1 \end{pmatrix}.$$

The spectral curve is then given by

$$F_2(\lambda) = \lambda(\lambda^2(\lambda^2 + I_1)^2 + I_2),$$

where I_1 and I_2 are the Chevalley invariants given by homogeneous polynomials of $(a_1, \cdots, b_2)$. Each polygon $\Gamma_{\epsilon_1,\epsilon_2}$ in the isospectral manifold has 12 vertices corresponding to $a_1 = a_2 = 0$ which is also the order of the Weyl group. Those polygons are glued to obtain the compactified manifold which is topologically equivalent to a sum of five Klein bottles. The Euler characteristic is $12(vertices) - 24(edges) + 4(polygons) = -8$.

The Painlevé divisor $\mathcal{D}_{\{1\}}$ is parametrized by the limit matrix $L_{\{1\}}$,

$$L_{\{1\}} = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -\xi_2 & \xi_1 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \eta & 0 & 0 & \xi_1^2 - 4\xi_2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\eta & 0 & 0 & -\xi_2 & -\xi_1 \end{pmatrix},$$

where $\xi_1 = b_1 + b_2$, $\xi_2 = b_1 b_2 - a_1$ and $\eta = 2b_1 a_1 a_2$ in the limit $t \to t_{\{1\}}$ with $\tau_1(t) \sim t - t_{\{1\}}$. Here we have used the conjugating matrix $x_{\{1\}}$ as,

$$x_{\{1\}} = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -b_1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & b_2 - b_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2a_1 & b_2 - b_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b_1 & 1 \end{pmatrix},$$

which can be obtained from the structure of τ-functions in (2.6). Then the invariants I_1, I_2 are given by

$$I_1 = 3\xi_2 - \xi_1^2, \quad I_2 = (4\xi_2 - \xi_1^2)\xi_2^2 + 2\eta\xi_1.$$

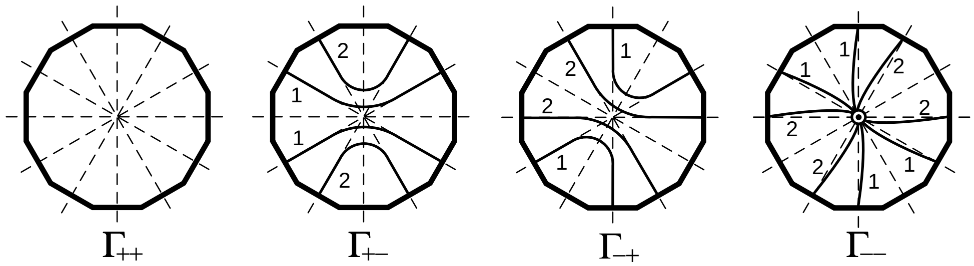

FIGURE 3. The G_2 polygons $\Gamma_{\epsilon_1\epsilon_2}$ with the Painlevé divisors $\Theta_{\{1\}}, \Theta_{\{2\}}$ and $\Theta_{\{1,2\}}$.

Eliminating ξ_2, we obtain

$$\eta = \frac{1}{2\xi_1}\left(-\frac{1}{27}(\xi_1^2+I_1)^2(\xi_1^2+4I_1)+I_2\right).$$

which has two connected components, and each component intersects three times with the boundaries of the polytopes Γ_{+-}, Γ_{-+} and Γ_{--} (see Figure 3).

The limit matrix corresponding to the Painlevé divisor $\mathcal{D}_{\{2\}}$ is given by

$$L_{\{2\}} = \begin{pmatrix} \xi_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \eta & -\xi_2 & \xi_1 & 1 & 0 & \cdot & 0 \\ 0 & -2\eta & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2\eta & -\xi_2 & -\xi_1 & 1 \\ 0 & 0 & 0 & 0 & -\eta & 0 & -\xi_1 \end{pmatrix}$$

with the conjugating matrix $x_{\{2\}}$,

$$x_{\{2\}} = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & -b_2 & 1 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1-b_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here the new variables are $\xi_1 = b_1,\ \xi_2 = b_2(b_1-b_2)-a_2$ and $\eta = a_1 b_2$ in the corresponding limit with $\tau_2(t) \to 0$. With those variables, the invariants are

$$I_1 = \xi_2 - \xi_1^2, \quad I_2 = 3\eta^2 - 2\xi_1\eta(\xi_2+2\xi_1^2) - \xi_1^2\xi_2^2,$$

from which we have two curves $\eta = \eta_\pm(\xi_1)$,

$$\eta_\pm = \frac{1}{3}\left(\xi_1(3\xi_1^2+I_1) \pm \sqrt{2\xi_1^2(5\xi_1^4+4I_1\xi_1^2+I_1^2)+I_2}\right).$$

Those curves indicate that there are two connected components of the divisor $\mathcal{D}_{\{2\}}$ and each component has three intersections with the subsystems. Topologically then the divisors $\mathcal{D}_{\{1\}}$ and $\mathcal{D}_{\{2\}}$ are the same, and adding the divisor $\mathcal{D}_{\{1,2\}}$ one can conclude that the closure of both divisors are topologically equivalent to a figure eight. Thus we have

PROPOSITION 5.3. *The topology of the G_2 Toda isospectral manifold and the divisor is given by*

$$\tilde{Z}(\gamma)_{\mathbb{R}} \cong \overbrace{\mathbb{K} \sharp \cdots \sharp \mathbb{K}}^{5}, \quad \Theta_{\{1\}} \cong \Theta_{\{2\}} \cong S^1 \vee S^1.$$

5.4. The A_3 Toda lattice. In the example 3.5, we gave the limit matrices for the Painlevé divisors. Here we discuss the topology of the divisors by computing explicitly the isospectral sets of those matrices, i.e.

$$F_3(\lambda) = \lambda^4 + I_1\lambda^2 - I_2\lambda + I_3,$$

where the Chevalley invariants $I_k(L_J), k = 1,2,3$ are now expressed in terms of the parameters in the limit matrices. Here we use the same parametrizations in Example 3.5:

a) $J = \{1\}$: We take the polynomial $u_2(\lambda)$ in the Mumford system as $u_2(\lambda) = \begin{vmatrix} \lambda - b_3 & -1 \\ -a_3 & \lambda - b_4 \end{vmatrix}$, i.e. $\mu_1 + \mu_2 = b_3 + b_4 = -\xi_1,\ \mu_1\mu_2 = b_3b_4 - a_3$. Then the Chevalley invariants are given by

$$\begin{cases} I_1 &= \xi_2 - (\mu_1 + \mu_2)^2 + \mu_1\mu_2, \\ I_2 &= \eta_1 - \mu_1\mu_2(\mu_1 + \mu_2) + \xi_2(\mu_1 + \mu_2), \\ I_3 &= \xi_2\mu_1\mu_2 + \eta_1 b_4. \end{cases}$$

Eliminating ξ_2, we find

$$\eta_1(\mu_k - b_4) = -F_3(\mu_k), \quad k = 1, 2.$$

As was shown in Proposition 4.2, comparing this with the top cell of the A_2 Toda lattice in Subsection 5.1, one can see

$$\mathcal{D}_{\{1\}} \cong \mathbb{T} \setminus \mathbb{D}.$$

b) $J = \{2\}$: We take $u_2(\lambda) = (\lambda - b_1)(\lambda - b_4)$, i.e. $\mu_1 = b_1, \mu_2 = b_4$. Then we have, using $\xi_1 = -(\mu_1 + \mu_2)$,

$$\begin{cases} I_1 &= \mu_1\mu_2 - (\mu_1 + \mu_2)^2 + \xi_2, \\ I_2 &= \xi_2(\mu_1 + \mu_2) - \mu_1\mu_2(\mu_1 + \mu_2) + \eta_1 + \eta_2, \\ I_3 &= \mu_1\mu_2\xi_2 + \mu_1\eta_2 + \mu_2\eta_1, \end{cases}$$

which lead to

$$\eta_k = (-1)^k \frac{F_3(\mu_k)}{\mu_1 - \mu_2}.$$

c) $J = \{3\}$: This case is similar to the one with $J = \{1\}$, and we have the same formulae of the Chevalley invariants in the variables μ_1, μ_2 which are defined as $u_2(\lambda) = \begin{vmatrix} \lambda - b_1 & -1 \\ -a_3 & \lambda - b_2 \end{vmatrix}$, i.e. $\mu_1 + \mu_2 = b_1 + b_2 = -\xi_1,\ \mu_1\mu_2 = b_1b_2 - a_1$.

d) $J = \{1,2\}$: Here we take $u_1(\lambda) = \lambda - b_4$ for the Mumford system, i.e. $\mu_1 = b_4$ and $v_1 = -\eta_1'$. Then the Chevalley invariants are

$$I_1 = \xi_2' - \mu_1^2, \quad I_2 = \xi_3' + \xi_2'\mu_1, \quad I_3 = \xi_3'\mu_1 - \eta_1',$$

and from $v_1 = -\eta_1'$, we obtain

$$\eta_1' = -F_3(\mu_1),$$

which implies that $\mathcal{D}_{\{1,2\}}$ intersects with four boundaries of the polytopes, and the closure, $\Theta_{\{1,2\}}$ is homeomorphic to a circle.

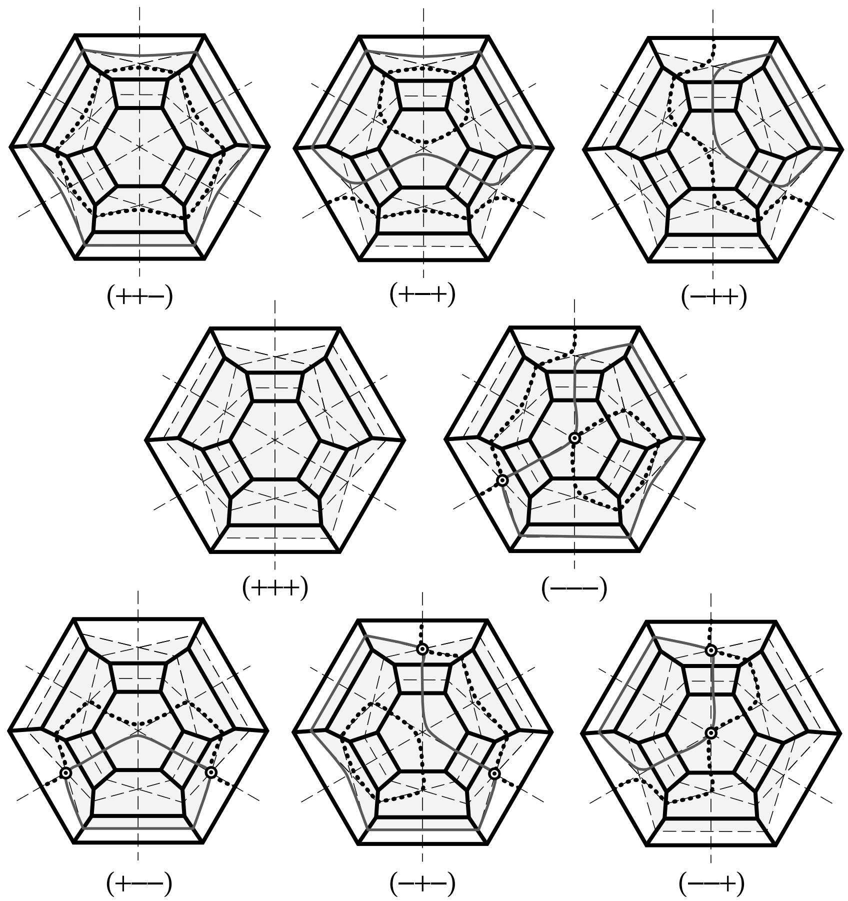

FIGURE 4. The A_3 polytopes Γ_ϵ marked by $\epsilon = (\epsilon_1\epsilon_2\epsilon_3)$ and the Painlevé divisors $\Theta_{\{1\}}$ (the solid grey curves), $\Theta_{\{2\}}$ (the dotted curves) and $\Theta_{\{1,2\}}$ (the double circles).

e) $J = \{2,3\}$: We get exactly the same result as the previous case with $\mu_1 = b_1$.
f) $J = \{1,3\}$: The Chevalley invariants are given by

$$I_1 = \xi_1\xi_1' + \xi_2 + \xi_2', \quad I_2 = \xi_1\xi_2' + \xi_2\xi_1', \quad I_3 = \xi_2\xi_2' - \eta_1'.$$

Using $\xi_1 + \xi_1' = 0$ and eliminating ξ_1', ξ_2', we obtain

$$\eta_1' = \frac{1}{4\xi_1^2}\left(\xi_1^2(\xi_1^2 + I_1)^2 - I_2^2\right) - I_3.$$

This equation indicates that $\mathcal{D}_{\{1,3\}}$ has two connected components, each of which intersects with three boundaries of the polytopes. Each boudary corresponds to a point $(\eta_1' = 0, \lambda_i + \lambda_j)$ for $i \neq j$. Then we can see

$$\Theta_{\{1,3\}} \cong S^1 \vee S^1.$$

The results are summarized in Figure 4 where the Painlevé divisors $\Theta_{\{1\}}, \Theta_{\{2\}}$ and $\Theta_{\{1,2\}}$ are shown as the solid grey curves, the dotted curves and the double circles.

The $\Theta_{\{3\}}$ has a similar structure to the $\Theta_{\{1\}}$. One can see from this Figure that each portion of the $\Theta_{\{1\}}$ on a Γ_ϵ is homeomorphic to either hexagon or octagon, and we have 4 hexagons in Γ_ϵ with $\epsilon = (++-), (-++), (+--), (--+)$, and 3 octagons with $\epsilon = (+-+), (-+-), (---)$. Then the Euler characteristic can be computed as follows: The total number of vertices are given by $12 = (4\times 6+3\times 8)/4$ by identifying 4 vertices of the polygons, the edges are $24 = (4\times 6+3\times 8)/2$ in total, and we have 7 faces, i.e. the Euler characteristic is $12-24+7=-5$. One can also see the non-orientability of the divisor, so that the $\Theta_{\{1\}}$ is topologically equivalent to a connected sum of 7 real projective planes $\mathbb{P}$ (or 3 Klein bottles plus a projective plane). For $\Theta_{\{2\}}$, we have 4 squares and 4 hexagons. However two squares in Γ_{---} are attached at a point of the divisor $\Theta_{\{1,2,3\}}$, and thus the $\Theta_{\{2\}}$ gives a singular variety. By detaching those two squares, one can compute the Euler characteristic in the same way as above, and we obtain $12-24-10=-2$. This shows that the desingularized variety of $\Theta_{\{2\}}$ is homeomorphic to the compactified manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ for the A_2 Toda lattice (in the next section we give a further discussion on the desingularization in Lie theoretic point of view). Thus we have

PROPOSITION 5.4. *The Painlevé divisors for the A_3 Toda lattice have the following topology,*

$$\begin{array}{ll}\Theta_{\{1\}} \cong \Theta_{\{3\}} \cong \mathbb{K} \sharp \mathbb{K} \sharp \mathbb{K} \sharp \mathbb{P}, & \Theta^\vee_{\{2\}} \cong \mathbb{K} \sharp \mathbb{K}, \\ \Theta_{\{1,2\}} \cong \Theta_{\{2,3\}} \cong S^1, & \Theta_{\{1,3\}} \cong S^1 \vee S^1,\end{array}$$

where $\Theta^\vee_{\{2\}}$ is the desingularization of $\Theta_{\{2\}}$ by a resolution at the divisor $\Theta_{\{1,2,3\}}$.

The singular structure on the divisor $\Theta_{\{1,3\}}$ has been also found in the case of periodic Toda lattice [**10**].

6. An algebraic version of the Painlevé divisor

Here we discuss the Painlevé divisor in the framework of the Lie theory. We first review and summarize some Lie theoretic notation.

6.1. Notations and Definitions.

NOTATION 6.1. Lie algebras: Recall that $\mathfrak{g}$ denote a real split semisimple Lie algebra of rank l and we are fixing a split Cartan subalgebra $\mathfrak{h}$ with root system Δ, a positive system Δ_+ determining the Borel subgroup B^+ of G. The corresponding set of simple roots is $\Pi := \{\alpha_i : i = 1, \cdots, l\}$ as in Section 2 where we just denoted $\Pi = \{k = 1, \cdots, l\}$.

The Weyl group W is thus generated by the simple reflections s_{α_i}, $i = 1, \cdots, l$. For any $S \subset \Pi$, we define the subgroup generated by S,

$$W_S = \langle\ s_{\alpha_i}\ :\ \alpha_i \in S\ \rangle$$

This is the Weyl group of a parabolic Lie subgroup and it is standard to refer to W_S as a *parabolic subgroup* of W.

NOTATION 6.2. Lie groups: We let $G_{\mathbb{C}}$ denote the connected adjoint Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and G the connected Lie subgroup correspondintg to $\mathfrak{g}$. Denote by $\tilde{G}$ the Lie group $\{g \in G_{\mathbb{C}}\ :\ Ad(g)\mathfrak{g} \subset \mathfrak{g}\}$. A split Cartan of $\tilde{G}$ with Lie algebra $\mathfrak{h}$ will be denoted by $H_{\mathbb{R}}$; this Cartan subgroup has exactly 2^l connected components and the component of the identity is denoted by $H = \exp(\mathfrak{h})$. We let $\chi_i := \chi_{\alpha_i}$ denote the roots characters defined on $H_{\mathbb{R}}$.

EXAMPLE 6.3. If $G = Ad(SL(n,\mathbb{R}))$, then $\tilde{G}$ is isomorphic to $SL(n,\mathbb{R})$ for n odd and to $Ad(SL(n,\mathbb{R})^{\pm})$ for n even. This example is the underlying Lie group for the Toda lattices as shown in [**5**].

DEFINITION 6.4. The negative walls: Recall that the compactified isospectral manifold $\tilde{Z}(\gamma)_{\mathbb{R}}$ of the Toda lattice is described in [**5**] as a closure in G/B^+ of a generic $H_{\mathbb{R}}$ orbit. Hence there is an embedding $f : H_{\mathbb{R}} \to \tilde{Z}(\gamma)_{\mathbb{R}} \subset G/B^+$.

The exponential map $\exp : \mathfrak{h} \to H$ separates H, and consequently every connected component of $H_{\mathbb{R}}$, into chambers. If χ_i is a simple root characters relative to a fixed dominant chamber then χ_{α_i} can be extends to an adjacent chamber by $s_{\alpha_j}\chi_{\alpha_i} = \chi_{\alpha_i}\chi_{\alpha_j}^{-C_{i,j}}$. This defines a single function χ_i^* on an open dense subset of $H_{\mathbb{R}}$ which equals $\chi_{w(\alpha_i)}$ on each $w-$chamber for $w \in W$ (denoted by $\phi_{w,i}$ in Definition 5.4 of [**5**]). The functions $|\chi_i^*|$ are well defined and continuous throughout $H_{\mathbb{R}}$ and the χ_i^* are well defined and continuous at all the α_i walls and some of the α_j walls. For example, if σ is a permutation, then the corrersponding chamber in $SL(3,\mathbb{R})$ looks like $\{(r_{\sigma(1)}, r_{\sigma(2)}, r_{\sigma(3)}) : |r_1| > |r_2| > |r_3|\}$ and $\chi_1^* = r_{\sigma(1)}r_{\sigma(2)}^{-1}$, $\chi_2^* = r_{\sigma(2)}r_{\sigma(3)}^{-1}$.

The functions $\chi_i^*+1 = 0$ on $H_{\mathbb{R}}$ then determine a topological subspace of $\tilde{Z}(\gamma)_{\mathbb{R}}$ whose closure we denote $\tilde{\Theta}_{\{i\}}$. Similarly a subset $J \subset \Pi$ determines a topological space $\tilde{\Theta}_J$ by equations $\chi_i^* + 1 = 0$ for $\alpha_i \in J$. We call $\tilde{\Theta}_J$ the *negative wall* associated with the set J (see Subsection 6.3 for another definition in the language of [**6**] which does not explicitly involve the Cartan subgroup).

CONJECTURE 6.5. *There is a surjective continuous map* $f : \Theta_J \to \tilde{\Theta}_J^a$. *This map is a homeomorphism in an open dense subset of* Θ_J. *Whenever* $\tilde{\Theta}_J^a$ *happens to be homeomorphic to a non-singular manifold then* f *is a homeomorphism.*

EXAMPLE 6.6. In the case of $\mathfrak{sl}(3)$ all $\Theta_{\{i\}}$ and $\tilde{\Theta}_{\{i\}}^a$ are homeomorphic. They are both homeomorphic to a circle (see Example 6.15 below). For $\mathfrak{sl}(4)$ again $\Theta_{\{i\}}$ is homeomorphic to $\tilde{\Theta}_{\{i\}}^a$ for $i = 1, 3$ (details in Example 6.12 and Proposition 5.4). However $\Theta_{\{2\}}$ and $\tilde{\Theta}_{\{2\}}^a$ are not homeomorphic. The situation is described in Example 6.19 together with Proposition 5.4 and is as follows. It is possible to desingularize $\Theta_{\{2\}}$ so that the compact connected surface $\hat{\Theta}_{\{2\}}$ which is obtained is non-orientable with Euler characteristic -2. Then there are maps $\hat{\Theta}_{\{2\}} \to \Theta_{\{2\}} \to \tilde{\Theta}_{\{2\}}$ and $\hat{\Theta}_{\{2\}}$ now resolves the singularities of both $\Theta_{\{2\}}$ and $\tilde{\Theta}_{\{2\}}$. Conjecture 6.5 needs to be sharpened by modifying $\tilde{\Theta}_J^a$ slightly so that one always has homeomorphisms. Below we propose such a modification for the case when J consists of one simple root.

It is now easy to see that $\Theta_{\{1\}}$ and $\tilde{\Theta}_{\{1\}}^a$ agree in the case of A_3. Figure 4 shows the eight polytopes Γ_ϵ corresponding to 2^l connected components of $H_{\mathbb{R}}$. In fact what is shown is the boundary of each polytope and the intersection of $\Theta_{\{i\}}$ for $i = 1, 2$. However, the negative walls are also depicted by the same picture. The only modification consists in drawing the dotted lines or the solid grey lines through the center of the hexagons. The actual negative walls are obtained by joining the dotted line or solid grey line to the center of the polytope through straight lines generating cones. Hence $\tilde{\Theta}_{\{1\}}^a$ intersected with each polytope consists of a disk in the form of a cone joining the center of the polytope with the path described on the boundary of the polytope by the solid grey line. Gluings are described in detail in Definition 6.9. What results is a smooth compact surface.

In order to introduce modifications to $\tilde{\Theta}^a_J$ we need to describe its structure in more detail. We do this by using the description of a manifold M given in [**6**] which is homeomorphic to $\tilde{Z}(\gamma)_{\mathbb{R}}$. We review the construction of M and then define new topological spaces Θ^a_J in the case when J conists of one simple root.

DEFINITION 6.7. Let $\mathcal{E}$ be the set of signs $\mathcal{E} = \{(\epsilon_1, \cdots, \epsilon_l) : \epsilon_k \in \{\pm\}\}$. Then we define an action of W on $\mathcal{E}$ by setting $s_i\epsilon = \epsilon'$ where

$$\epsilon'_j = \epsilon_j \epsilon_i^{C_{j,i}},$$

which can be deduced from the W-action on the root character χ_i with $\epsilon_i = \mathrm{sign}(\chi_i)$. The fact that this defines an action which corresponds to the action of W on the set of connected components of a split Cartan subgroup of the real semisimple Lie group $\tilde{G}$ can be found in [**5**].

For any $S \subset \Pi$ we let $\mathbb{D}(S)$ denote the set of all Dynkin diagrams that have the simple roots in S marked by $+$ or $-$. We also define an action of the group W_S on this set by making $w \in W_S$ act on the signs associated to the simple roots in S as prescribed above. For example $\circ_- - \circ_+ - \circ \in \mathbb{D}(S)$ with $S = \{\alpha_1, \alpha_2\}$ and $s_1(\circ_- - \circ_- - \circ) = \circ_- - \circ_+ - \circ$.

We now obtain actions of W_S on $\mathcal{E} \times W$ and on $\mathbb{D}(S) \times W$ given by $\sigma(\epsilon, w) = (\sigma\epsilon, w\sigma^{-1})$ and $\sigma(\delta, w) = (\sigma\delta, w\sigma^{-1})$. The orbits of the W_S action on $(\epsilon, w) \in \mathcal{E} \times W$ and $(\delta, w) \in \mathbb{D}(S) \times W$ are denoted by $[\epsilon, w]_S$ and $[\delta, w])_S$ respectively with the sub-index S dropped when the set S is clear from the context. These W_S orbits in the case of $\mathbb{D}(S) \times W$ are the *full set of colored Dynkin diagrams* introduced in section 4 of [**5**]. The orbits of W_S on $\mathcal{E} \times W$ with $S = \{\alpha_i\}$ are used below to parametrize the walls $\chi_i^* \pm 1 = 0$ intersected with a fixed polytope. The walls $\chi_i^* \pm 1 = 0$ in $\tilde{Z}(\gamma)_{\mathbb{R}}$ can be parametrized by $\mathbb{D}(S) \times W$ with $S = \{\alpha_i\}$.

6.2. Review of the description of $\tilde{Z}(\gamma)_{\mathbb{R}}$ in terms of the polytopes Γ_ϵ. Here we discuss the detailed description of negative walls in the connection to the Painlevé divisors Θ_J. Let us first summarize the construction of the isospectral manifold of the Toda lattice given in [**6**]. Starting with a polytope Γ, other polytopes Γ_ϵ are constructed where $\epsilon \in \mathcal{E}$. These polytopes then form a compact smooth manifold when they are glued together through their boundaries. We now review the details.

In terms of the description given in [**5**], each Γ_ϵ has interior that can be made to correspond to a connected component of a split Cartan subgroup of the real semisimple split Lie group $\tilde{G}$. *Chambers* and *walls* then refer to the action of W on a Cartan subgroup, and the internal chamber walls in the polytopes Γ_ϵ correspond to walls of this action ($\chi_i^* \pm 1 = 0$). When $\chi_i^* = -1$ then the chamber at the other side of the wall need not be the one obtained by application of $s_{\alpha_i} = s_i$.

DEFINITION 6.8. Consider Γ a convex polytope consisting of the convex hull of a W orbit of a regular element x_o in $\mathfrak{h}$. We first denote C'_e the dominant chamber in $\mathfrak{h}$ intersected with Γ and $\overline{C'}_e$ the corresponding closure, and also denote $C'_w = w(C'_e)$. We define $C_w = \{w\} \times C'_w$ and its closure $\overline{C}_w = \{w\} \times \overline{C'}_w$. The $'^{,\cdots}$ will refer to subsets of Γ, and we have the convention:

$$\{\cdots\}^{\cdots} = \{w\} \times \{\cdots\}'^{\cdots}$$

in all our notation concerning walls. For each simple root α_i we may consider the corresponding α_i (internal) chamber wall intersected with $\overline{C'}_w$. Denote this set by $[w]^{\prime,\alpha_i,IN}$. Each external wall of the convex hull of Wx_o is parametrized by a simple roots α_i. We denote an *external* wall of Γ by $[w]^{\prime,\alpha_i,OUT}$ if it intersects all the *internal* chamber walls except for $[w]^{\prime,\alpha_i,IN}$.

For any $J \subset \Pi$ we define the subsets of $\overline{C'}_w$ of dimension $|\Pi \setminus J|$,

$$\begin{cases} [w]^{\prime,J,\Theta} &= \bigcap_{\alpha_i \in J} [w]^{\prime,\alpha_i,\Theta}, \quad \text{if } J \neq \emptyset \ , \\ [w]^{\prime,J,\Theta} &= C'_w \ , \qquad\qquad\quad \text{if } J = \emptyset \ , \end{cases}$$

where Θ is either OUT or IN. Thus we have the decomposition,

$$\overline{C'}_w = \bigcup_{\substack{J \subset \Pi \\ \Theta \in \{OUT, IN\}}} [w]^{\prime,J,\Theta}.$$

DEFINITION 6.9. We will need to use the action of W on the set of signs $\mathcal{E}$ of Definition 6.7. We now define gluing maps between the chamber walls denoted by $\{\epsilon\} \times [w]^{\cdots} = \{\epsilon\} \times \{w\} \times [w]^{\prime\cdots}$ as follows: For the internal walls, we define

$$\begin{array}{rccc} g_{w,i,IN}: & \{\epsilon\} \times [w]^{\alpha_i,IN} & \longrightarrow & \{s_{\alpha_i}\epsilon\} \times [ws_{\alpha_i}]^{\alpha_i,IN} \\ & (\epsilon, w, x) & \longmapsto & (s_{\alpha_i}\epsilon, ws_{\alpha_i}, x) \end{array}$$

where note $ws_{\alpha_i}w^{-1}x = x$. For the external walls, we define

$$\begin{array}{rccc} g_{w,i,OUT}: & \{\epsilon\} \times [w]^{\alpha_i,OUT} & \longrightarrow & \{\epsilon^{(i)}\} \times [w]^{\alpha_i,OUT} \\ & (\epsilon, w, x) & \longmapsto & (\epsilon^{(i)}, w, x) \end{array}$$

where $\epsilon^{(i)} = (\epsilon_1, \cdots, -\epsilon_i, \cdots, \epsilon_l)$.

We denote $\tilde{M}$ the disjoint union of all the chambers endowed with different signs,

$$\tilde{M} = \bigcup_{\substack{w \in W \\ \epsilon \in \mathcal{E}}} \{w\epsilon\} \times \overline{C}_{w^{-1}}.$$

We also denote M the topological space obtained from the disjoint union in $\tilde{M}$ by gluing along the internal and external walls using the maps $g_{w,i,IN}$ and $g_{w,i,OUT}$. There is then a map

$$z : \tilde{M} \to M.$$

6.3. The negative walls. We now give a precise definition of the negative wall. Let us first define:

DEFINITION 6.10. First denote

$$\tilde{\Gamma}_\epsilon = \bigcup_{w \in W} \{w\epsilon\} \times \overline{C}_{w^{-1}} \ .$$

We now let Γ_ϵ denote the image of $\tilde{\Gamma}_\epsilon$ in M. Then after the identifications in M, this space becomes a copy of Γ.

NOTATION 6.11. Set $\epsilon' = w^{-1}\epsilon$ and recall the action of W and its subgroups on pairs (ϵ', w) where $\epsilon \in \mathcal{E}$ and $w \in W$ (Definition 6.7). Note that an α_i wall $[w]^{\alpha_i,IN}$ which is the intersection of two closed chambers $\{\epsilon'\} \times \overline{C}_w$ and $\{s_i\epsilon'\} \times \overline{C}_{ws_i}$ can be simply parametrized by the coset of w in $[w] \in W/ < s_i >$. To keep track of signs we need to consider the two pairs (ϵ', w) and $(s_i\epsilon', ws_i)$. This constitutes the orbit of (ϵ', w) under the action of $W_{\{s_i\}}$. We have already denoted this orbit by $[\epsilon', w]$ in Definition 6.7 and now this orbit $[w^{-1}\epsilon, w]$ will also be used as a parameter to denote the corresponding internal wall in Γ_ϵ. This wall is called *negative* for α_i if $\epsilon' = (\epsilon'_1, \dots, \epsilon'_l)$ has $\epsilon'_i = -1$.

For a set $J \subset \Pi$ one can also consider the orbit of W_J denoted $[\epsilon', w]_J$ which will now denote the intersection:

$$[\epsilon', w]_J = z\left(\bigcap_{\sigma\in W_J} \{\sigma\epsilon'\} \times \overline{C}_{w\sigma^{-1}}\right)$$

This parametrizes an intersection of several walls. We call this J multi-wall intersection $J-$*negative* or just negative if it is such that for all $\alpha_i \in J$ there is $\sigma \in W_J$ such that $(\sigma\epsilon')_i = -1$ where (ϵ', w) is a representative where w has minimal length in its coset in W/W_J.

For any $\alpha_i \in \Pi$ we consider the set $R_{\epsilon,J}$ given by

$$R_{\epsilon,J} = \{[\epsilon', w] : [\epsilon', w] \text{ is negative }\}$$

When $J = \{\alpha_i\}$ we will just write $R_{\epsilon,i}$

Consider the subspace of M given by

$$\tilde{\Theta}^a_J = z\left(\bigcup_{\epsilon\in\mathcal{E}, w\in R_{\epsilon,J}} [w^{-1}\epsilon, w]_J\right).$$

EXAMPLE 6.12. We can now describe the topology of $\tilde{\Theta}^a_{\{1\}}$. Since $\tilde{\Theta}^a_{\{1\}}$ is smooth it will suffice to compute its Euler characteristic. That $\tilde{\Theta}^a_{\{1\}}$ is not orientable will follow.

Walls in a fixed Γ_ϵ are parametrized as in Notation 6.11. If we want to parametrize walls independently of each separate polytope, we consider colored Dynkin diagrams as in [**5**]. Intersections of walls are obtained by coloring more simple roots with $-$s or $+$s. Thus we consider walls as parametrized by the full set of colored Dynkin diagrams (Definition 6.7).

The negative walls in $\Theta_{\{1\}}$ can then be listed: $[\circ_- - \circ - \circ, e]$, $[\circ_- - \circ - \circ, 2]$, $[\circ_- - \circ - \circ, 3]$ $[\circ_- - \circ - \circ, 23]$,$[\circ_- - \circ - \circ, 12]$, $[\circ_- - \circ - \circ, 32]$, $[\circ_- - \circ - \circ, 312]$ $[\circ_- - \circ - \circ, 123]$, $[\circ_- - \circ - \circ, 232]$, $[\circ_- - \circ - \circ, 1232]$, $[\circ_- - \circ - \circ, 2312]$, $[\circ_- - \circ - \circ, 12132]$.

Now boundaries must be considered. For example the boundaries of $[\circ_- - \circ - \circ, e]$ are $[\circ_- - \circ_- - \circ, e]$, $[\circ_- - \circ - \circ_-, e]$, $[\circ_- - \circ_+ - \circ, e]$, $[\circ_- - \circ - \circ_+, e]$. Therefore all these walls are part of $\Theta_{\{1\}}$. However $[\circ_- - \circ - \circ, 2]$ produces $[\circ_- - \circ_- - \circ, 2]$. Since the Weyl group W_S now includes s_2 then we can write this wall as $[\circ_+ - \circ_- - \circ, e]$ because $s_2(\circ_- - \circ_- - \circ) = \circ_+ - \circ_- - \circ$. Therefore the wall $[\circ_+ - \circ_- - \circ, e]$ must also be included in $\tilde{\Theta}^a_{\{1\}}$. Taking this into account we can easily count all the cells of $\tilde{\Theta}^a_{\{1\}}$. All the 1-cells of the form $[\circ_{\epsilon_1} - \circ_{\epsilon_2} - \circ, w]$ except $\epsilon_1 = \epsilon_2 = +$ appear. This gives $3 \times |W/W_{\{s_1,s_2\}}| = 3 \times 4$ such cells. We also get all the cells of the form $[\circ_- - \circ - \circ_\pm, w]$, a total of $2 \times |W/W_{\{s_1,s_3\}}| = 2 \times 6$. Hence a total of 24 cells of dimension one. Finally there are 7 cells of dimension 0. The Euler characteristic obtained is $12 - 24 + 7 = -5$. Since $\tilde{\Theta}^a_{\{1\}}$ is homeomorphic to a smooth compact surface, this completely describes its topology.

Note that the actual boundary maps involved in a homology computation are as described in section 4 of [**5**].

6.4. A graph associated to the negative walls in Γ_ϵ. Let us define a graph to describe the negative walls for a fixed $\alpha_i \in \Pi$.

DEFINITION 6.13. The graph $\mathbb{G}(\epsilon)$:
We consider a graph $\mathbb{G}(\epsilon)$ having vertices (ϵ', w) with $\epsilon' = w^{-1}\epsilon$.

(a) If all $\epsilon_j = -$ then all the pairs (ϵ', w), $(s_j\epsilon', ws_j)$ are edges.

We now describe the edges when not all ϵ_j are negative.
First for all semisimple Lie algebras of rank $l \leq 3$:
(ϵ', w), $(s_j\epsilon', ws_j)$ is an edge if and only if one of the following is satisfied:

(b) $i \neq j$, $C_{j,i} \neq -2$, $\epsilon'_j = +$
(c) $i \neq j$, $C_{j,i} = -2$, $\epsilon'_j = -$
(d) $i \neq j$, $C_{i,j} = 0$, $\epsilon'_i = -$

Note that if we fix i and j then the corresponding subdiagram of the Dynkin diagram has rank two. We will show below that these conditions lead to the correct description of the divisors Θ_i in the rank two cases. The condition d) of Definition 6.13 will correspond to the case of $A_1 \times A_1$. If we consider only Lie algebra of types A, D, E and G_2, the conditions simplify to:

(a') $\alpha_j \in \Pi(\epsilon)$
(b') $i \neq j$, $\epsilon'_j = +$
(c') $i \neq j$, s_j commutes with s_i and $\epsilon'_i = -$

The case $\alpha_i = \alpha_2$ in B_2 is some kind of exception which requires a separate rule given in condition c).

In general,if the rank is n given a subset $S \subset \Pi$ and $\epsilon \in \mathcal{E}$ we denote ϵ_S the restriction formed by the ordered $|S|$-tuple consisting only of the ϵ_k with $\alpha_k \in S$.

We assume that all the edges of the graph have been defined for rank $|S| < n$. The pair (ϵ', w), $(s_j\epsilon', ws_j)$ is an edge if there is $S \subset \Pi$ with $|S| < n$, $s_i, s_j \in S$ and there is $\sigma \in W_S$ with $w = w_1\sigma$, $\ell(w_1) + \ell(\sigma) = \ell(w)$ and (ϵ'_S, σ), $(s_j\epsilon'_S, \sigma s_j)$ form an edge in the case of the split Lie subalgebra determined by S.

We now break up $R_{\epsilon,i}$ as a disjoint union of subsets consisting of negative walls belonging to the same connected component of the graph $\mathbb{G}(\epsilon)$. We thus obtain a set $I(\epsilon)$ consisting of subsets of $R_{\epsilon,i}$. The disjoint union $\bigcup_{\alpha \in I(\epsilon)} \alpha$ equals $R_{\epsilon,i}$.

DEFINITION 6.14. The graph $\mathbb{G}$: We now define a graph whose vertices are the elements $\alpha \in I(\epsilon)$ for $\epsilon \in \mathcal{E}$. If $\alpha \in I(\epsilon_1)$ and $\beta \in I(\epsilon_2)$, then there is an edge joining α to β if and only if there is w such that

(i) $[w^{-1}\epsilon_1, w] \in \alpha$, $[w^{-1}\epsilon_2, w] \in \beta$
(ii) Denoting $w^{-1}\epsilon_1 = \epsilon'$ then we have: $w^{-1}\epsilon_2 = (\epsilon')^{(i)}$

EXAMPLE 6.15. Consider the case of A_2 and $J = \{\alpha_1\}$. If $\epsilon = (--)$ condition a) in Definition 6.13 applies; however, as it turns out, condition b) alone will suffice in this case. We have the following edges indicated by $\to$ connecting the only two negative walls. $((--), e) \to ((-+), s_1) \to ((-+), s_1s_2)$. We have the following set of negative walls $R_{(--),1} = \{[(--), e], [(-+), s_1s_2]\}$. We obtain that $I(--)$ consists of one single element $\alpha^{--} = \{[(--), e], [(-+), s_1s_2]\}$.

For $\epsilon = (-+)$ we obtain $R_{(-+),1} = \{[(-+), e], [(-+), s_2]\}$ and $I(-+)$ consists of one single element $\alpha^{-+} = \{[(-+), e], [(-+), s_2]\}$

For $\epsilon = (+-)$ we have $R_{(+-),1} = \{[(--), s_2] \to [(--), s_1 s_2]\}$ and again there is one single element α^{+-}.

The graph $\mathbb{G}$ consists of a "cycle" $\alpha^{--} \to \alpha^{-+} \to \alpha^{+-} \to \alpha^{--}$. For example, there is an edge $\alpha^{--} \to \alpha^{-+}$ because $[(-+), e] \in \alpha^{--} \cap \alpha^{-+}$. If one consider the topological space consisting of the corresponding walls then what results is a circle in agreement with what was found in Propopsition 5.1. This corresponds to Figure 1 where the divisor indicated by the number 1 is replaced with two walls -straight lines- joining at the center of the hexagons. The edges of $\mathbb{G}$ correspond to intersections with the boundaries of the hexagons Γ_ϵ, that is with "subsystems".

EXAMPLE 6.16. Consider the case of G_2 and $J = \{s_1\}$ with $\epsilon = (+-)$. The negative walls are parametrized by

$$[(--), s_2], [(-+), s_2 s_1 s_2], [(--), s_1, s_2], [(-+), s_1 s_2 s_1 s_2].$$

Note that $[(--), s_2], [(-+), s_2 s_1 s_2]$ are in the same connected component of $\mathbb{G}(\epsilon)$ since

$$(--, s_2) \to (-+, s_2 s_1) \to (-+, s_2 s_1 s_2);$$

where $\to$ indicates an edge. However this process reaches a dead-end when we apply s_1 once more since one obtains $(--, s_2 s_1 s_2 s_1)$ but s_2 cannot be applied at this point because $\epsilon_2 = -1$. The connected component of the graph $\mathbb{G}(+-)$ which contains $[(--), s_2]$ then consists of

$$(--, s_2) \to (-+, s_2 s_1) \to (-+, s_2 s_1 s_2) \to (--, s_2 s_1 s_2 s_1).$$

From here

$$\alpha^{+-} = \{[(--), s_2], [(-+), s_2 s_1 s_2]\}$$

and similarly there is another set of negative walls

$$\beta^{+-} = \{[(--), s_1 s_2], [(-+), s_1 s_2 s_1 s_2]\}\,.$$

We have $I(+-) = \{\alpha^{+-}, \beta^{+-}\}$.

For $\epsilon = (-+)$, $\Pi(\epsilon) = \{s_1\}$ one obtains $\alpha^{-+} = \{[(-+), e], [(-+), s_2]\}$, $\beta^{+-} = \{[(--), s_1 s_2 s_1 s_2], [(--), s_2 s_1 s_2 s_1 s_2]\}$. For $\epsilon = (--)$ all the negative walls are in a single connected component. However, here, unlike what happens in the A_2 example one requires condition a) of Definition 6.13 with $\Pi(--) = \{s_1, s_2\}$. This allows the application of s_2 independently of the sign ϵ_2'. We have $\alpha^{--} = \{[(--), e], [(-+), s_1, s_2], [(--), s_2 s_1 s_2], [(-+), s_2 s_1 s_2 s_1 s_2]\}$ and $I(--) = \{\alpha^{--}\}$.

The graph $\mathbb{G}$ has nodes given by $\{\alpha^{+-}.\beta^{+-}, \alpha^{-+}, \beta^{-+}, \alpha^{--}\}$. The edges are $\alpha^{+-} \to \alpha^{-+}$, $\beta^{+-} \to \beta^{-+}$ and $\alpha^{--} \to x$ for $x = \alpha^{+-}, \beta^{+-}, \alpha^{-+}, \beta^{-+}$. This gives a total of six edges.

When one considers the topology of the sets of walls involved and the edges are regarded as the only gluings: α^{--} consists of two intersecting line segments and all the others consist of segments. What then results is a figure 8. The 6 edges are the intersection of this figure 8 with the boundaries of the polytopes Γ_ϵ (subsystems). Note that segments associated to α^{+-} and β^{+-} are being regarded as *disjoint*. However the two segments forming α^{--} are not disjoint because they form part of one single connected component of $\mathbb{G}(--)$. The topological space associated to these graphs and the negative walls will be made precise below for a general semisimple Lie algebra.

EXAMPLE 6.17. We now consider the case of B_2, $J = \{s_2\}$ and $\epsilon = (+,-)$. We have edges $(+-,e) \to (+-,s_1) \to (+-,s_1s_2)$ but $(+-,s_1s_2)$ is a dead-end because s_1 cannot be applied since $\epsilon_1 = +$ and $C_{1,2} = -2$. We also have an edge $(+-,e) \to (+-,s_2)$ which leads to a dead-end for the same reason. This gives a set $\alpha^{+-} = \{[(+-),e],[(+-),s_1]\}$. Another connected component of the graph produces $\beta^{+-} = \{[(+-),s_2s_1],[(+-),s_1s_2s_1]\}$.

For $\epsilon = (--)$ one obtains $\alpha^{--} = \{[(--),e],[(--),s_1s_2s_1]\}$ and for $\epsilon = (-+)$ $\alpha^{-+} = \{[(-+),s_1],[(-+),s_2s_1]\}$. Hence a graph results with edges $\alpha^{--} \to x$ and $\alpha^{-+} \to x$ where $x = \alpha^{+-}, \beta^{+-}$ giving a total of four edges in $\mathbb{G}$. Again we consider the topology of the sets of walls involved and, as in the previous examples, the edges in $\mathbb{G}$ are regarded as the only gluings between these segments. We obtain four segments corresponding to the elements in $I(\epsilon)$ giving rise to a circle that intersects the boundaries of the Γ_ϵ at four points (the edges of the graph $\mathbb{G}$). This corresponds to $\Theta_{\{1\}}$ in Proposition 5.2.

6.5. The spaces of negative walls Θ_i^a. Fix an element $\alpha \in I(\epsilon)$. We consider the disjoint union

$$\bigcup_{\alpha\in I(\epsilon),[\epsilon',w]\in\alpha,\epsilon\in\mathcal{E}} \{\alpha\} \times [w^{-1}\epsilon, w].$$

We define gluings for any pair α, β which are joined by an edge of the graph $\mathbb{G}$.

$$\begin{array}{rccc} g: & (\alpha,[\epsilon',w]) & \longrightarrow & [(\beta,\epsilon'^{(i)}w\sigma^{-1}]) \\ & (\alpha,\epsilon',w,x) & \longmapsto & (\beta,\epsilon'^{(i)},w,x) \end{array}$$

CONJECTURE 6.18. *There is a homeomorphism $g : \Theta_i \to \Theta_i^a$.*

EXAMPLE 6.19. The topology of $\Theta^a_{\{2\}}$ in the case of A_3 can be computed explicitly and shown to correspond to $\Theta_{\{2\}}$. We first compute the Euler characteristic of $\tilde{\Theta}^a_{\{2\}}$ using the method in Example 6.12. One obtains twelve 2-cells, twenty four 2 cells and seven 1-cells giving the same Euler characteristic as in the case of $\tilde{\Theta}^a_{\{1\}}$. However the sets $I(\epsilon)$ for $\epsilon = (+-+)$ and $\epsilon = (-+-)$ contain two elements. This can be seen in Figure 4 where the corresponding paths of dotted lines are disconnected. The recipe for the construction of $\Theta^a_{\{2\}}$ corresponds to separating the two cones obtained by joining these paths to the center of each of these polytopes. This introduces two additional points! Hence the Euler characteristic for $\Theta^a_{\{2\}}$ becomes -3.

One now notes that $\Theta^a_{\{2\}}$ remains singular as can be seen in Figure 4 where in the boundary of the polytope Γ_{---} there are two disconnected paths of dotted lines. It is possible to resolve this singularity by simply separating the center of this polytope into two separate points. This gives rise to a compact surface of Euler characteristic -2 since one additional point is added. The resulting surface can be seen to be non-orientable. We thus obtain that $\Theta^a_{\{2\}}$ is homeomorphic to $\Theta_{\{2\}}$. The compactification of the isospectral manifold of A_2 reappears but only as a resolution of singularities of the Painlevé divisor.

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