A RIEMANNIAN INVARIANT, EULER STRUCTURES AND SOME TOPOLOGICAL APPLICATIONS

DAN BURGHELEA AND STEFAN HALLER

Abstract. In this paper:

(i) We define and study a new numerical invariant $R(X, g, \omega)$ associated with a closed Riemannian manifold (M, g), a closed one form ω and a vector field X with isolated zeros. When $X = -\operatorname{grad}_g f$ with $f : M \to \mathbb{R}$ a Morse function this invariant is implicit in the work of Bismut–Zhang. The definition of this invariant requires "geometric regularization".

(ii) We define and study the sets of Euler structures and co-Euler structures of a based pointed manifold (M, x_0). When $\chi(M) = 0$ the concept of Euler structure was introduced by V. Turaev. The Euler resp. co-Euler structures permit to remove the geometric anomalies from Reidemeister torsion resp. Ray-Singer torsion.

(iii) We apply these concepts to torsion related issues, cf. Theorems 3 and 4. In particular we show the existence of a meromorphic function associated to a pair $(M, \mathfrak{e}^*)$, consisting of a smooth closed manifold and a co-Euler structure, defined on the variety of complex representations of the fundamental group of M whose real part is the Ray–Singer torsion (corrected). This function generalizes the Alexander polynomial for the complement of a knot.

Contents

Date: October 10, 2003.

2000 *Mathematics Subject Classification.* 58xx, 57Rxx.

The second author is supported by the 'Fonds zur Förderung der wissenschaftlichen Forschung' (Austrian Science Fund), project number P14195-MAT.

1. Introduction

The invariant $R(X, g, \omega)$. Let M be a closed manifold and $\omega \in \Omega^1(M)$ a closed one form with real or complex coefficients.

(i) A pair of two Riemannian metrics g_1, g_2 determines the Chern–Simons class $\mathrm{cs}(g_1, g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d\Omega^{n-2}(M;\mathcal{O}_M)$ and then the numerical invariant

$$R(g_1, g_2, \omega) := \int_M \omega \wedge \mathrm{cs}(g_1, g_2).$$

(ii) A pair of two vector fields without zeros X_1, X_2 determines a homology class $c(X_1, X_2) \in H_1(M;\mathbb{Z})$, see section (3) below, and then a numerical invariant

$$R(X_1, X_2, \omega) = \langle[\omega], c(X_1, X_2)\rangle. \tag{1}$$

(iii) A pair consisting of a vector field without zeros and a Riemannian metric g determines a degree $n-1$ form $X^*\Psi(g) \in \Omega^{n-1}(M;\mathcal{O}_M)$ and therefore a numerical invariant

$$R(X, g, \omega) = \int_M \omega \wedge X^*\Psi(g). \tag{2}$$

Here $\Psi(g) \in \Omega^{n-1}(TM \setminus 0_M;\mathcal{O}_M)$ is the Mathai–Quillen form introduced in [8, section 7], cf. section 2 below.

The first purpose of this paper is to extend the invariant (iii) to the case of vector fields with isolated zeros, not necessary nondegenerate. Both smooth triangulations and Euler structures provide examples of such vector fields, cf sections 4, 5. If X has zeros then the integrand in (2) is defined only on $M \setminus \mathcal{X}$, $\mathcal{X}$ the set of zeros of X, and the integral (2) might be divergent. Fortunately it can be regularized by a procedure we will refer to as "geometric regularization" as described in section 3 and this leads to the numerical invariant $R(X, g, \omega)$ from the title (cf Theorem 1 below). One can also extend the invariant (ii) to vector fields with isolated zeros, cf section 3.

A first pleasant application of the invariant R and of the extension of (ii) is the extension of the Chern-Simons class from a pair of two Riemannian metrics g_1 and g_2 to a pair of two smooth triangulations τ_1 and τ_2 or to a pair of a Riemannian metric g and a smooth triangulation τ cf section (5). These classes permit to treat on "equal foot" a

Riemannian metric and a smooth triangulation when comparing subtle invariants like "torsion" defined using a Riemannian metric, and using a triangulation, in analogy with the comparison of such invariants for two metrics or two triangulations.

An other pleasant application of the invariant R is the derivation of a result of J. Marcsik [7], see Theorem 3, from a theorem by Bismut–Zhang, cf. section 7.

Euler structures. The second focus of this paper are Euler structures. These were introduced by Turaev cf [11] for manifolds M with trivial Euler–Poincaré characteristic $\chi(M) = 0$. It was noticed in [2] that the Euler structures can be defined for an arbitrary base pointed manifold (M, x_0) and that the definition is independent of the base point provided $\chi(M) = 0$. The set of Euler structures $\mathfrak{Eul}(M, x_0)$ is an affine version of $H_1(M;\mathbb{Z})$ in the sense that $H_1(M;\mathbb{Z})$ acts freely and transitively on $\mathfrak{Eul}(M, x_0)$.

We introduce the set $\mathfrak{Eul}^*(M, x_0)$ of co-Euler structures on which $H^{n-1}(M; \mathcal{O}_M)$ acts freely and transitively.

The set of co-Euler structures is defined as the set of equivalence classes of pairs (g, α) where $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ satisfies $d\alpha = E(g)$. Two pairs (g_1, α_1) and (g_2, α_2) are equivalent iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$.

$\mathfrak{Eul}^*(M, x_0)$ represent a smooth version (deRham version) of a sort of dual aspect of $\mathfrak{Eul}(M, x_0)$. In the case of a closed manifold M we show the existence of an affine version of Poincaré duality map $P : \mathfrak{Eul}^*(M, x_0) \to \mathfrak{Eul}(M, x_0) \otimes \mathbb{R}$, where $\mathfrak{Eul}(M, x_0) \otimes \mathbb{R}$ is a real version of $\mathfrak{Eul}(M, x_0)$, see below. We also define the coupling

$$\mathbb{T} : \mathfrak{Eul}(M, x_0) \times \mathfrak{Eul}^*(M, x_0) \to H_1(M; \mathbb{R})$$

based on a regularization very similar to the one for R (cf. section 3).

The interest of Euler and co- Euler structures come from the following. An element $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, x_0)$ removes the metric ambiguity of the Ray–Singer torsion and provides a scalar product (the analytic scalar product), i.e. a metric, in the complex line

$$(\det V)^{-\chi(M)} \otimes \det H^*(M; \rho), \tag{3}$$

for every complex representation $\rho \in \mathrm{Rep}(\pi_1(M, x_0); V)$. An element $\mathfrak{e} \in \mathfrak{Eul}(M, x_0)$ removes the triangulation ambiguity [1] and provides a scalar product (the combinatorial scalar product), i.e. a metric, in the line (3) (cf also [5]).

[1] and the additional ambiguity produced by the choice of a lift of each cell of the triangulation to the universal cover of the manifold

As a first pleasant application of the Euler and co-Euler structures is a reformulation of a result of Bismut–Zhang, proven [1], referred to as the Bismut–Zhang theorem, Theorem 7, section 6. Precisely, the analytic scalar product associated to $\mathfrak{e}^*$ is the same as the combinatorial scalar product associated to $\mathfrak{e}$ [2] multiplied by the absolute value of $\langle\Theta_\rho, \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*)\rangle \in \mathbb{C}^*$. Here $\Theta_\rho \in H^1(M;\mathbb{C}^*)$ is the cohomology class corresponding to $\det\circ\rho : H_1(M;\mathbb{Z}) \to \mathbb{C}^*$, cf. Theorem 7.

Let $\mathrm{Rep}(\Gamma; V)$ denote the affine algebraic variety of complex representations of the finitely presented group Γ on the complex vector space V. Denote by $\mathrm{Rep}^M(\Gamma; V)$, $\Gamma := \pi_1(M, x_0)$, the complex analytic set which is the union of components of $\mathrm{Rep}(\Gamma; V)$ for which the generic representation ρ has vanishing cohomology $H^*(M;\rho) = 0$. Consider a co-Euler structure $\mathfrak{e}^* = [g, \alpha]$, i.e. $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ and $d\alpha = E(g)$. Assign to each ρ the the corrected Ray–Singer torsion

$$T_{\mathrm{an}}(\nabla, g, \mu) \cdot e^{\frac{1}{2}\int_M \omega(\nabla,\mu)\wedge\alpha} \in \mathbb{R}_+, \tag{4}$$

where ∇ is a flat connection on the associated bundle $F_\rho \to M$ whose holonomy representation at the base point x_0 is ρ, μ is a Hermitian structure on F_ρ and $T_{\mathrm{an}}(\nabla, g, \mu)$ denotes the Ray–Singer torsion of the deRham complex $(\Omega^*(M; F_\rho), d^*_\nabla)$ equipped with the scalar product induced by g and a Hermitian metric μ. The closed one form $\omega(\nabla, \mu)$ is the Kamber–Tondeur closed one form associated to (∇, μ), cf. [1], [3] and section 2. The above quantity (4) is independent of μ and the representatives ∇ and (g, α) of ρ and $\mathfrak{e}^*$. It provides a real valued function on $\mathrm{Rep}^M(\Gamma; V) \setminus \Sigma(M)$, where $\Sigma(M)$ is the subset of representations ρ for which $\dim H^*(M; \rho)$ is not locally minimal.

The Bismut–Zhang theorem in the reformulation mentioned above permits to construct a meromorphic complex valued function on the complex analytic space $\mathrm{Rep}^M(\Gamma; V)$. The zeros and poles of this function are contained in $\Sigma(M)$ and the absolute value, when restricted to $\mathrm{Rep}^M(\Gamma; V) \setminus \Sigma(M)$, is the real valued function described above. This complex valued meromorphic extension of the "corrected" Ray–Singer torsion has a number of interesting applications and implications which will be discussed in subsequent work. In a forthcoming paper a complex holomorphic line bundle over $\mathrm{Rep}(\Gamma; V)$ will be associated to M which is a homotopy invariant, and a meromorphic section constructed. When restricted to $\mathrm{Rep}^M(\Gamma; V)$ such section is the meromorphic function stated above.

Main results. Suppose M is a closed manifold of dimension n. Given a Riemannian metric g denote by $E(g) \in \Omega^n(M; \mathcal{O}_M)$ the Euler form

[2] see section 6for the definitions of these terms

and by $\Psi(g) \in \Omega^{n-1}(TM\setminus 0_M; \mathcal{O}_M)$ the Mathai–Quillen form associated to g. If X_1 and X_2 are two vector fields with isolated zeros we get an element

$$c(X_1, X_2) \in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z}))$$

whose boundary equals the zeros of X_1 and X_2, weighted with their indices, see section 2.

Theorem 1. *Let M be a closed connected manifold.*

(i) *Suppose $\omega \in \Omega^1(M)$ is a real or complex valued closed one form, g a Riemannian metric and X a vector field with isolated zeros. Let f be a smooth real or complex valued function with $\omega = df$ in the neighborhood of the zero set $\mathcal{X}$ of X. Then the number*

$$R(X,g,\omega;f) := \int_{M\setminus\mathcal{X}} (\omega - df)\wedge X^*\Psi(g) - \int_M fE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x) f(x)$$

is independent of f and will therefore be denoted by $R(X,g,\omega)$.

(ii) *If g_1 and g_2 are two Riemannian metrics, then*

$$R(X,g_2,\omega) - R(X,g_1,\omega) = \int_M \omega\wedge \mathrm{cs}(g_1,g_2).$$

(iii) *If X_1 and X_2 are two vector fields with isolated zeros then*

$$R(X_2,g,\omega) - R(X_1,g,\omega) = \int_{c(X_1,X_2)} \omega.$$

(iv) *If ω_1 and ω_2 are two closed one forms so that $\omega_2 - \omega_1 = dh$ then*

$$R(X,g,\omega_2) - R(X,g,\omega_1) = -\int hE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x).$$

(v) *If K is a complex analytic space and $\omega(z)$ is a holomorphic family of closed complex valued one forms, $z \in K$, then the assignment $z \rightsquigarrow R(X,g,\omega(z))$ is a holomorphic function in z.*

In section 3 we will prove statements (i) through (iv). More precisely they are the contents of Lemma 1, Proposition 1 and Proposition 2. The proof of (v) follows from the linearity in ω of the invariant $R(X,g,\omega)$.

An Euler structure on a base pointed manifold (M,x_0) is an equivalence class of pairs (X,c), where X is a vector field with isolated singularities and c is a singular one chain with integral coefficients whose boundary equals $\sum_{x\in\mathcal{X}} \mathrm{IND}(x)x - \chi(M)x_0$, where $\mathcal{X}$ denotes the zero set of X. Two such pairs (X_1,c_1) and (X_2,c_2) are equivalent if c_2 differs from $c_1 + c(X_1, X_2)$ by a boundary. We write $\mathfrak{Eul}(M, x_0)$ for the set of Euler structures based at x_0. This is an affine version of $H_1(M;\mathbb{Z})$ in that $H_1(M;\mathbb{Z})$ acts freely and transitively on it. Considering chains c with real coefficients we get an affine version of $H_1(M;\mathbb{R})$ which we denote by $\mathfrak{Eul}(M, x_0) \otimes \mathbb{R}$.

The set $\mathfrak{Eul}^*(M, x_0)$ of co-Euler structures is defined as the set of equivalence classes of pairs (g, α) where $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ satisfies $d\alpha = E(g)$. Two pairs (g_1, α_1) and (g_2, α_2) are equivalent iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$. The cohomology $H^{n-1}(M; \mathcal{O}_M)$ acts on $\mathfrak{Eul}(M, x_0)$ freely and transitively by $[g, \alpha] + [\beta] := [g, \alpha - \beta]$.

Theorem 2. *Let (M, x_0) be a closed connected base pointed manifold.*

(i) *Let $\pi_0(\mathfrak{X}(M, x_0))$ denote the set of connected components of vector fields which just vanish in x_0 equipped with the C^∞ topology. If $\dim M > 2$ then we have a bijection:*

$$\pi_0(\mathfrak{X}(M, x_0)) \to \mathfrak{Eul}(M, x_0), \qquad [X] \mapsto [X, 0]$$

(ii) *Let $\pi_0(\mathfrak{X}_0(M))$ denote the set of connected components of nowhere vanishing vector fields equipped with the C^∞ topology. If $\chi(M) = 0$ and $\dim M > 2$ we have a surjection:*

$$\pi_0(\mathfrak{X}_0(M)) \to \mathfrak{Eul}(M, x_0), \qquad [X] \mapsto [X, 0].$$

(iii) *There exists a bijection*

$$P : \mathfrak{Eul}^*(M, x_0) \to \mathfrak{Eul}(M, x_0) \otimes \mathbb{R},$$

which is equivariant in the sense that $P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta)$, for all $\beta \in H^{n-1}(M; \mathcal{O}_M)$. Here $\mathrm{PD} : H^{n-1}(M; \mathcal{O}_M) \to H_1(M;\mathbb{R})$ denotes Poincaré duality.

(iv) *The assignment $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) := P(\mathfrak{e}^*) - \mathfrak{e}$*

$$\mathbb{T} : \big(\mathfrak{Eul}(M, x_0) \otimes \mathbb{R}\big) \times \mathfrak{Eul}^*(M, x_0) \to H_1(M;\mathbb{R})$$

is a corrected version of the invariant R. More precisely suppose $\mathfrak{e} = [X, c]$, $\mathfrak{e}^* = [g, \alpha]$. Then for every $[\omega] \in H^1(M;\mathbb{R})$ we have

$$\langle [\omega], \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) \rangle = \int_M \omega \wedge (X^*\Psi(g) - \alpha) - \int_c \omega$$

where $\omega \in \Omega^1(M)$ is any representative of $[\omega]$ which vanishes locally around x_0 and locally around the zeros of X.

Statements (i) and (ii) are essentially due to Turaev and is the contents of Propositions 3 and 4 in section 4. The proof of (iii) and (iv) can be found at the end of section 4, cf. Proposition 5.

Suppose N be a closed manifold of dimension $n-1$ and suppose $\varphi : N \to N$ is a diffeomorphism. Let $\varphi_i : H^i(N;\mathbb{C}) \to H^i(N;\mathbb{C})$ denote the homomorphisms induced in cohomology, let $P^i_\varphi(z) = \det(z\varphi_i - \mathrm{Id})$ and let

$$\zeta_\varphi(z) := \frac{\prod_{i \text{ even}} P^i_\varphi(z)}{\prod_{i \text{ odd}} P^i_\varphi(z)} \tag{5}$$

denote the Lefschetz zeta function of φ.

The mapping torus $M = N_\varphi$, obtained from $I \times N$ by gluing $\{1\} \times N$ to $\{0\} \times N$ via φ, is equipped with a closed one form ω defined by

$$\omega := p^* ds. \tag{6}$$

Here ds is the volume form on S^1 and the map $p : M \to S^1$ is induced from the first factor projection $p : [0,1] \times N \to [0,1]$.

Choose a Riemannian metric g on M and consider the Laplace–Beltrami operators $\Delta^q : \Omega^q(M) \to \Omega^q(M)$ as well as the vector field $X = -\operatorname{grad}_g \omega$. Denote by $L_X : \Omega^q(M) \to \Omega^q(M)$ the Lie derivative and let $L^\sharp_X$ denote its formal adjoint with respect to g. Denote by $\Delta^q_\omega(t) : \Omega^q(M) \to \Omega^q(M)$, the Witten Laplacian defined by

$$\Delta^q_\omega(t) := \Delta^q + t(L_X + L^\sharp_X) + t^2 ||\omega||^2 \,\mathrm{Id}\,. \tag{7}$$

The Witten Laplacians, are second order elliptic selfadjoint operators. They are obviously non-negative definite and zero order perturbations of the Laplace–Beltrami operators. For sufficiently large t, actually all t with $P^i_\varphi(e^t) \neq 0$ for all i, they are strictly positive definite hence invertible.

For t with $P^i_\varphi(e^t) \neq 0$ for all i we introduce the Ray–Singer torsion:

$$\log T_{\mathrm{an}}(\omega, g)(t) := \frac{1}{2} \sum_i (-1)^{i+1} i \log\det \Delta^q_\omega(t) \tag{8}$$

The proof of the next theorem is contained in section 7.

Theorem 3 (J. Marsick)**.** *Let N be a closed manifold, $\varphi : N \to N$ a diffeomorphism and let g be a Riemannian metric on the mapping torus $M := N_\varphi$. With the definitions above*

$$\log |\zeta_\varphi(e^t)| = \log T_{\mathrm{an}}(\omega, g)(t) + tR(X, g, \omega) \tag{9}$$

for every $t \in \mathbb{R}$ which satisfies $P^i_\varphi(e^t) \neq 0$ for all i.

Let (M, x_0) be a closed manifold with base point and set $\Gamma := \pi_1(M, x_0)$. Every $\rho \in \mathrm{Rep}(\Gamma; V)$ gives rise to a homomorphism $\det \circ \rho : \Gamma \to \mathbb{C}^*$ which descends to a homomorphism $H_1(M;\mathbb{Z}) \to \mathbb{C}^*$ and thus defines a cohomology class $\Theta_\rho \in H^1(M;\mathbb{C}^*)$. This defines a homolomorphic function

$$\Theta : \mathrm{Rep}(\Gamma;V) \to H^1(M;\mathbb{C}^*), \quad \Theta(\rho) := \Theta_\rho.$$

Theorem 4. *Let (M,x_0) be a closed smooth manifold and let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M,x_0)$ be a co-Euler structure and o_H an orientation of the cohomology vector space $H^*(M;\mathbb{R})$. Then there exists a complex meromorphic function $\mathcal{T}_{\mathfrak{e}^*,o_H}$ on $\mathrm{Rep}^M(\Gamma;V)$ with the following properties:*

(i) *The poles and zeros of $\mathcal{T}_{\mathfrak{e}^*,o_H}$ are contained in $\Sigma(M)$.*

(ii) *For a generic representation ρ we have*

$$|\mathcal{T}_{\mathfrak{e}^*,o_H}(\rho)| = T_{\mathrm{an}}(\nabla,g,\mu)e^{\frac{1}{2}\int_M \omega(\nabla,\mu)\wedge\alpha},$$

where ∇ is a any flat connection on the associated bundle F_ρ, μ is any Hermitian structure on F_ρ and where $[g,\alpha]$ is any representative of $\mathfrak{e}^*$.[3]

(iii) *For $\beta \in H^{n-1}(M;\mathcal{O}_M)$ we have*

$$\mathcal{T}_{\mathfrak{e}^*+\beta,o_H} = \mathcal{T}_{\mathfrak{e}^*,o_H} \cdot \langle \Theta, \mathrm{PD}(\beta)\rangle.$$

By changing the orientation one might change $\mathcal{T}_{\mathfrak{e}^*,o_H}$ up to multiplication by ± 1.

When we do not want to specify the orientation o_H we can write $\mathbb{T}_{\mathfrak{e}^*}$ for the resulting meromorphic function up to a sign ambiguity.

This meromorphic function carries relevant topological and geometric information even in the simplest possible situations. For example let N be a simply connected closed manifold, $\varphi : N \to N$ a diffeomorphism and let $M = N_\varphi$ denote the mapping torus. Then $\Gamma := \pi_1(M) = \mathbb{Z}$. Let λ_i denote the eigenvalues of

$$\varphi^{even} : \bigoplus_{k \text{ even}} H^k(M;\mathbb{R}) \to \bigoplus_{k \text{ even}} H^k(M;\mathbb{R})$$

and similarly let ν_j denote the eigenvalues of

$$\varphi^{odd} : \bigoplus_{k \text{ odd}} H^k(M;\mathbb{R}) \to \bigoplus_{k \text{ odd}} H^k(M;\mathbb{R})$$

Let ω denote the closed one form defined in (6) and let g be a Riemannian metric on M. Finally let $\mathfrak{e}^*$ denote the co-Euler structure Poincaré dual to the Euler structure given by $-\mathrm{grad}_g(\omega)$. Then

$$\mathrm{Rep}^M(\Gamma;\mathbb{C}) = \mathrm{Rep}(\Gamma;\mathbb{C}) = \mathbb{C}^* = \mathbb{C}\setminus 0,$$

[3] The integral $\int_M \omega(\nabla,\mu)\wedge\alpha$ may be divergent and has to be regularized, cf. section 4.

$\Sigma(M) = \{\lambda_i, \nu_j\}$ and using Theorem 4 we get from Theorem 3

$$\mathcal{T}_{\mathfrak{e}^*,o_H}(z) = \pm\frac{\prod_i(z\lambda_i - 1)}{\prod_j(z\nu_j - 1)}.$$

2. Chern and Mathai–Quillen form

Let $\pi : E \to M$ be a rank k real vector bundle, and $\tilde{\nabla} := (\nabla, \mu)$ be a pair consisting of a connection ∇ and μ, a parallel Hermitian structure, i.e. a fiber wise scalar product. Let $\mathcal{O}_E$ denote the orientation bundle of E, a flat real line bundle over M. There is a canonic $\mathrm{Vol} \in \Omega^k(E; \pi^*\mathcal{O}_E)$, which vanishes when contracted with horizontal vectors and which assigns to a k-tuple of vertical vectors "their volume times their orientation". Moreover let ξ denote the Euler vector field on E which assigns to a point $x \in E$ the vertical vector $-x \in T_xE$. Mathai and Quillen have introduced, cf. [8], the differential form

$$\Psi(\tilde{\nabla}) := \frac{\Gamma(k/2)}{(2\pi)^{k/2}|\xi|^k} i_\xi \mathrm{Vol} \in \Omega^{k-1}(E \setminus 0_E; \pi^*\mathcal{O}_E).$$

Clearly $\Psi(\tilde{\nabla})$ has the following properties which follow immediately from the definition.

(i) $\Psi(\tilde{\nabla})$ is the pullback of a form on $(E \setminus 0_E)/\mathbb{R}_+$.

(ii) If $E(\tilde{\nabla}) \in \Omega^k(M; \mathcal{O}_E)$ denotes the Euler form of $\tilde{\nabla}$ then:
$$d\Psi(\tilde{\nabla}) = \pi^*E(\tilde{\nabla}). \tag{10}$$

(iii) If $\mathrm{cs}(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{k-1}(M; \mathcal{O}_E)/d(\Omega^{k-2}(M; \mathcal{O}_E))$ denotes the Chern–Simons class then:
$$\Psi(\tilde{\nabla}_2) - \Psi(\tilde{\nabla}_1) = \pi^* \mathrm{cs}(\tilde{\nabla}_1, \tilde{\nabla}_2) \tag{11}$$

(iv) Suppose $E = TM$ is equipped with a Riemannian metric g, $\nabla_g = (\tilde{\nabla}_g, g)$ is the Levi–Civita pair and X is a vector field with isolated zero x. Let B_ϵ denote the ball of radius ϵ around x, with respect to some chart. Then
$$\lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon)} X^*\Psi(\tilde{\nabla}_g) = \mathrm{IND}(x), \tag{12}$$
where $\mathrm{IND}(x)$ denotes the Hopf index of X at x.

(v) For $M = \mathbb{R}^n$, $E := TM$ equipped with $g_{ij} = \delta_{ij}$, $\tilde{\nabla}_g$ the Levi–Civita pair and in the coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ one has:

$$\Psi(\tilde{\nabla}_g) = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum_{i=1}^n (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \wedge \dots \wedge \widehat{d\xi_i} \wedge \dots \wedge d\xi_n.$$

Let $\xi : E \to M$ be a complex vector bundle equipped with a flat connection. Given two Hermitian structures μ_1 and μ_2 denote by $V(\mu_1, \mu_2)$ the positive real valued function given at $y \in M$ by the volume with respect with the scalar product defined by $(\mu_2)_y$ of a parallelepiped provided by an orthonormal frame with respect to $(\mu_1)_y$.

A trivialization $\epsilon : M \times \mathbb{C}^n \to E$ of the bundle ξ will be called a frame of ξ. Frames in the vector bundle ξ exists only if the bundle is trivial.

Similarly for any two framings ϵ_1 and ϵ_2 denote by $D(\epsilon_1, \epsilon_2)$ the complex valued function which at $y \in M$ is given by the determinant of the matrix representing the frame $\epsilon_2(y)$ in terms of the frame represented bu $\epsilon_1(y)$.

Note that each frame ϵ induces unique Hermitian structure (which make the frame ϵ_y an orthonormal base) called $\mu(\epsilon)$ and we have

$$\Re(D(\epsilon_1, \epsilon_2)) = V(\mu(\epsilon_1), \mu(\epsilon_2)). \tag{13}$$

Suppose that the bundle ξ is equipped with a flat connection ∇. To any Hermitian structure in $\xi : E \to M$ following Kamber–Tondeur one associates the real valued closed (i.e. locally exact) differential form $\omega(\nabla, \mu) \in \Omega^1(M)$ defined as follows: For any $x \in M$ choose U a contractible open neighborhood and denote by $\tilde{\mu}_x$ the Hermitian structure in $E|_U \to U$ obtained by parallel transport of μ_x. This Hermitian structure is well defined since U is one connected and the connection is flat.

Define $\omega(\nabla, \mu) := dV_x/V_x$ as being the logarithmic differential of the non-zero function $V_x : U \to \mathbb{R}$ defined by $V_x = V(\mu, \tilde{\mu}_x)$.

If the bundle $\xi : E \to M$ is trivial and ϵ is a frame then by the same procedure (and by replacing V by D) one associates the complex valued closed one form $\omega(\nabla, \epsilon) \in \Omega^1(M; \mathbb{C})$.

The following properties hold:

$$\begin{aligned}
\Re(\omega(\nabla, \epsilon)) &= \omega(\nabla, \mu(\epsilon)) \\
\omega(\nabla, \epsilon_1) - \omega(\nabla, \epsilon_2) &= d\log(D(\epsilon_1, \epsilon_2)) \\
\omega(\nabla, \mu(\epsilon_1)) - \omega(\nabla, \mu(\epsilon_2)) &= d\log(V(\mu(\epsilon_1), \mu(\epsilon_2)))
\end{aligned}$$

3. The invariant $R(X, g, \omega)$. The geometric regularization

Suppose M is a closed manifold of dimension n, g a Riemannian metric and $X : M \to TM \setminus 0_M$ a vector field without zeros. Suppose $\omega \in \Omega^1(M; \mathbb{R})$ or $\Omega^1(M; \mathbb{C})$ is a closed form. Define

$$R(X, g, \omega) := \int_M \omega \wedge X^*\Psi(g), \tag{14}$$

which will be a real or complex number. It is not hard to check that for any function h

$$R(X, g, \omega + dh) - R(X, g, \omega) = -\int_M hE(g),$$

and for any two Riemannian metrics g_1 and g_2

$$R(X, g_2, \omega) - R(X, g_1, \omega) = \int_M \omega \wedge \mathrm{cs}(g_1, g_2).$$

These properties are straightforward consequences of (10) and (11).

Suppose X_1 and X_2 are two vector fields without zeros. Let $p : I \times M \to M$ denote the projection, where $I = [1, 2]$. Consider a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\} \times M$, $i = 1, 2$. The zero set $\mathbb{X}^{-1}(0)$ is a closed one dimensional canonically oriented submanifold of $I \times M$. Hence it defines a homology class in $I \times M$, which turns out to be independent of the chosen homotopy $\mathbb{X}$. We thus define $c(X_1, X_2) := p_*(\mathbb{X}^{-1}(0)) \in H_1(M; \mathbb{Z})$. One can show that

$$R(X_2, g, \omega) - R(X_1, g, \omega) = \int_{c(X_1, X_2)} \omega$$

This property will be verified below in a slightly more general case.

The above properties suggest the definition of the invariant R in the case X has isolated zeros even when the integral in (14) is divergent. This definition will be referred to as the *geometric regularization* of (14). We do not assume that the zeros of X are non-degenerate. Let $\mathcal{X}$ denote the zero set of X. Choose a function f so that $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$. Then

$$R(X, g, \omega; f) := \int_{M \setminus \mathcal{X}} \omega' \wedge X^*\Psi(g) - \int_M fE(g) + \sum_{x \in \mathcal{X}} \mathrm{IND}(x) f(x)$$

makes perfect sense. The next lemma establishes the proof of Theorem 1(i).

Lemma 1. *The quantity $R(X, g, \omega; f)$ does not depend on the choice of f.*

Proof. Suppose f_1 and f_2 are two functions such that $\omega_i' := \omega - df_i$, $i = 1, 2$ both vanish in a neighborhood U of $\mathcal{X}$, $i = 1, 2$. For every $x \in \mathcal{X}$ we choose a chart and let $B_\epsilon(x)$ denote the disk of radius ϵ around x. Put $B_\epsilon := \bigcup_{x \in \mathcal{X}} B_\epsilon(x)$.

For ϵ small enough $B_\epsilon \subset U$ and $f_2 - f_1$ is constant on each $B_\epsilon(x)$. Using (10), Stokes' theorem and (12) we get

$$\begin{aligned}
&R(X, g, \omega; f_2) - R(X, g, \omega; f_1) = \\
&= -\int_{M\setminus\mathcal{X}} d\big((f_2 - f_1) \wedge X^*(\Psi(g)\big) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= -\lim_{\epsilon\to 0}\int_{\partial(M\setminus B_\epsilon)} (f_2 - f_1) \wedge X^*\Psi(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= -\sum_{x\in\mathcal{X}}(f_2 - f_1)(x) \lim_{\epsilon\to 0}\int_{\partial(M\setminus B_\epsilon(x))} X^*\Psi(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= 0
\end{aligned}$$

and thus $R(X, g, \omega; f_1) = R(X, g, \omega; f_2)$. □

Definition 1. In view of the previous lemma we define $R(X, g, \omega) := R(X, g, \omega; f)$, where f is any function so that $\omega - df$ vanishes locally around $\mathcal{X}$.

From the very definition we immediately verify Theorem 1(iv) which we restate as

Proposition 1. *For every function h we have:*

$$R(X, g, \omega + dh) - R(X, g, \omega) = -\int_M hE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x) \tag{15}$$

For any vector field with isolated zeros $\mathcal{X}$ we set

$$e_X := \sum_{x\in\mathcal{X}} \mathrm{IND}(x)x,$$

a singular zero chain in M.

Suppose we have two vector fields X_1 and X_2 with non-degenerate zeros. Consider the vector bundle $p^*TM \to I \times M$, where $I := [1, 2]$ and $p : I\times M \to M$ denotes the natural projection. Choose a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\} \times M$, $i = 1, 2$. The zero set of $\mathbb{X}$ is a canonically oriented one dimensional submanifold with boundary. Hence it defines a singular one chain which, when pushed forward via p, is a one chain $c(\mathbb{X})$ in M, satisfying

$$\partial c(\mathbb{X}) = e_{X_2} - e_{X_1}.$$

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two non-degenerate homotopies from X_1 to X_2. Then certainly $\partial(c(\mathbb{X}_2) - c(\mathbb{X}_1)) = 0$, but we actually have

$$c(\mathbb{X}_2) - c(\mathbb{X}_1) = \partial\sigma, \tag{16}$$

for a two chain σ. Indeed, consider the vector bundle $q^*TM \to I \times I \times M$, where $q : I \times I \times M \to M$ denotes the natural projection. Choose a section of q^*TM which is transversal to the zero section, restricts to $\mathbb{X}_i$ on $\{i\} \times I \times M$, $i = 1, 2$ and which restricts to X_i on $\{s\} \times \{i\} \times M$ for all $s \in I$ and $i = 1, 2$. The zero set of such a section then gives rise to σ satisfying (16).

So for two vector fields with non-degenerate zeros this construction yields a one chain $c(X_1, X_2)$, well defined up to a boundary, satisfying $\partial c(X_1, X_2) = e_{X_2} - e_{X_1}$.

Let us extend this to vector fields with isolated singularities. Suppose X is a vector field with isolated singularities. For every zero $x \in \mathcal{X}$ we choose an embedded ball B_x centered at x, assuming all B_x are disjoint. Set $B := \bigcup_{x \in \mathcal{X}} B_x$. Choose a vector field with non-degenerate zeros X' that coincides with X on $M \setminus B$. Let $\mathcal{X}'$ denote its zero set. For every $x \in \mathcal{X}$ we have

$$\mathrm{IND}_X(x) = \sum_{y \in \mathcal{X}' \cap B_x} \mathrm{IND}_{X'}(y).$$

So we can choose a one chain $\tilde{c}(X, X')$ supported in B which satisfies $\partial \tilde{c}(X, X') = e_{X'} - e_X$. Since $H_1(B; \mathbb{Z})$ vanishes the one chain $\tilde{c}(X, X')$ is well defined up to a boundary.

Given two vector fields X_1 and X_2 with isolated zeros we choose perturbed vector fields X_1' and X_2' as above and set

$$c(X_1, X_2) := \tilde{c}(X_1, X_1') + c(X_1', X_2') - \tilde{c}(X_2, X_2').$$

Then obviously $\partial c(X_1, X_2) = e_{X_2} - e_{X_1}$. Using $H_1(B; \mathbb{Z}) = 0$ again, one checks that different choices for X_1' and X_2' yield the same $c(X_1, X_2)$ up to a boundary.

Summarizing, for every pair of vector fields X_1 and X_2 with isolated zeros we have constructed a one chain

$$c(X_1, X_2) \in C_1(M; \mathbb{Z}) / \partial(C_2(M; \mathbb{Z})),$$

which satisfies $\partial c(X_1, X_2) = e_{X_2} - e_{X_1}$.

Definition 2. For two Riemannian metrics g_1, g_2 and a closed one form ω set

$$R(g_1, g_2, \omega) := \int_M \omega \wedge \mathrm{cs}(g_1, g_2). \tag{17}$$

For two vector fields X_1, X_2 and a closed one form ω set

$$R(X_1, X_2, \omega) := \int_{c(X_1, X_2)} \omega. \tag{18}$$

Remark 1. Even though $\mathrm{cs}(g_1, g_2)$ is only defined up to an exact form this ambiguity does not affect the integral (17). Similarly, even though $c(X_1, X_2)$ is only defined up to a boundary this ambiguity does not affect the integral (18).

The next proposition is a reformulation of Theorem 1(ii) and Theorem 1(iii).

Proposition 2. *Let M be a closed manifold, ω a closed one form, g, g_1, g_2 Riemannian metrics and let X, X_1, X_2 be vector fields with isolated zeros. Then*

$$R(X, g_2, \omega) - R(X, g_1, \omega) = R(g_1, g_2, \omega) \tag{19}$$

and

$$R(X_2, g, \omega) - R(X_1, g, \omega) = R(X_1, X_2, \omega). \tag{20}$$

Proof. Let's prove (19). Choose f so that $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$, the zero set of X. Using $X^*(\Psi(g_2) - \Psi(g_1)) = \mathrm{cs}(g_1, g_2)$ modulo exact forms, Stokes' theorem and $d\,\mathrm{cs}(g_1, g_2) = E(g_2) - E(g_1)$ we conclude

$$\begin{aligned}
&R(X, g_2, \omega) - R(X, g_1, \omega) = \\
&= \int_{M\setminus\mathcal{X}} \omega' \wedge X^*\big(\Psi(g_2) - \Psi(g_1)\big) - \int_M f\big(E(g_2) - E(g_1)\big) \\
&= \int_M \omega \wedge \mathrm{cs}(g_1, g_2) - \int_M df \wedge \mathrm{cs}(g_1, g_2) - \int_M f\big(E(g_2) - E(g_1)\big) \\
&= \int_M \omega \wedge \mathrm{cs}(g_1, g_2) \\
&= R(g_1, g_2, \omega).
\end{aligned}$$

Now let's turn to (20). Let $\mathcal{X}_i$ denote the zero set of X_i, $i = 1, 2$. Assume first that the vector fields X_1 and X_2 are non-degenerate and that there exists a non-degenerate homotopy $\mathbb{X}$ from X_1 to X_2 whose zero set is contained in a simply connected $I \times V \subseteq I \times M$. Choose a function f such that $\omega' := \omega - df$ vanishes on V. Then

$$R(X_1, X_2, \omega) = \int_{\mathbb{X}^{-1}(0)} p^*df = \sum_{x\in\mathcal{X}_2} \mathrm{IND}_{X_2}(x) f(x) - \sum_{x\in\mathcal{X}_1} \mathrm{IND}_{X_1}(x) f(x),$$

where $p : I \times M \to M$ denotes the natural projection. Let $\tilde{p} : p^*TM \to TM$ be the natural vector bundle homomorphism over p. Using the last equation, Stokes' theorem and $d(\mathbb{X}^*\tilde{p}^*\Psi(g)) = p^*E(g)$ we get:

$$\begin{aligned} & R(X_2, g, \omega) - R(X_1, g, \omega) = \\ &= \int_{I\times(M\setminus V)} d\big(p^*\omega' \wedge \mathbb{X}^*\tilde{p}^*\Psi(g)\big) + R(X_1, X_2, \omega) \\ &= -\int_{I\times M} p^*(\omega' \wedge E(g)) + R(X_1, X_2, \omega) \\ &= R(X_1, X_2, \omega) \end{aligned}$$

For the last equality note that $\omega' \wedge E(g) = 0$ for dimensional reasons.

Still assuming that X_1 and X_2 have non-degenerate zeros we next treat the case of a general non-degenerate homotopy $\mathbb{X}$, whose zero set is not necessarily contained in a simply connected subset. Perturbing the homotopy slightly we may assume that no component of its zero set lies in a single $\{s\} \times M$. Then we certainly find $0 = t_0, \dots, t_k = 1$ so that Y_{t_i}, the restriction of $\mathbb{X}$ to $\{t_i\} \times M$, is transversal to the zero section, and so that $\mathbb{X}^{-1}(0) \cap ([t_{i-1}, t_i] \times M)$ is contained in a simply connected subset for every $1 \leq i \leq k$. The previous paragraph tells us

$$R(Y_{t_i}, g, \omega) - R(Y_{t_{i-1}}, g, \omega) = R(Y_{t_{i-1}}, Y_{t_i}, \omega)$$

for every $1 \leq i \leq k$. Therefore:

$$R(X_2, g, \omega) - R(X_1, g, \omega) = \sum_{i=1}^{k} R(Y_{t_{i-1}}, Y_{t_i}, \omega) = R(X_1, X_2, \omega)$$

It remains to deal with vector fields having degenerate but isolated singularities. Let X be such a vector field and let X' denote a perturbation as used before. Let $\mathcal{X}$ and $\mathcal{X}'$ denote their zero sets, respectively. Choose a function f such that $\omega' := \omega - df$ vanishes on the set B. Recall that B was the union of small balls covering $\mathcal{X}$. Since X and X' agree on $M \setminus B$ we have

$$\begin{aligned} R(X', g, \omega) - R(X, g, \omega) &= \sum_{x\in\mathcal{X}'} \mathrm{IND}_{X'}(x) f(x) - \sum_{x\in\mathcal{X}} \mathrm{IND}_X(x) f(x) \\ &= \int_{\tilde{c}(X,X')} df \\ &= R(X, X', \omega). \end{aligned}$$

This completes the proof of (20). □

Remark 2. A similar definition of $R(X, g, \omega)$ works for any vector field X with arbitrary singularity set $\mathcal{X} := \{x \in M \mid X(x) = 0\}$ provided ω is exact when restricted to a sufficiently small neighborhood of $\mathcal{X}$.

This situation might lead to interesting invariants for holomorphic vector fields on a Kähler manifolds or even more general, for symplectic vector fields on a symplectic manifold equipped with an almost complex structure which tames the symplectic structure.

4. Euler and co-Euler structures

Let (M, x_0) be a base pointed closed connected manifold of dimension n. Let X be a vector field and let $\mathcal{X}$ denote its zero set. Suppose the zeros of X are isolated and define

$$e_X := \sum_{x \in \mathcal{X}} \mathrm{IND}(x) x \in C_0(M; \mathbb{Z}),$$

a singular zero chain. An Euler chain for X is a singular one chain $c \in C_1(M; \mathbb{Z})$ so that

$$\partial c = e_X - \chi(M) x_0.$$

Since $\sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) = \chi(M)$ every vector field with isolated zeros admits Euler chains.

Consider pairs (X, c) where X is a vector field with isolated zeros and c is an Euler chain for X. We call two such pairs (X_1, c_1) and (X_2, c_2) equivalent if

$$c_2 = c_1 + c(X_1, X_2) \in C_1(M; \mathbb{Z}) / \partial(C_2(M; \mathbb{Z})).$$

For the definition of $c(X_1, X_2)$ see section 3. We will write $\mathfrak{Eul}(M, x_0)$ for the set of equivalence classes as above and $[X, c] \in \mathfrak{Eul}(M, x_0)$ for the element represented by the pair (X, c). Elements of $\mathfrak{Eul}(M, x_0)$ are called Euler structures of M based at x_0. There is an obvious $H_1(M; \mathbb{Z})$ action on $\mathfrak{Eul}(M, x_0)$ defined by

$$[X, c] + [\sigma] := [X, c + \sigma],$$

where $[\sigma] \in H_1(M; \mathbb{Z})$ and $[X, c] \in \mathfrak{Eul}(M, x_0)$. Obviously this action is free and transitive. In this sense $\mathfrak{Eul}(M, x_0)$ is an affine version of $H_1(M; \mathbb{Z})$.

Considering Euler chains with real coefficients one obtains in exactly the same way an affine version of $H_1(M; \mathbb{R})$ which we will denote by $\mathfrak{Eul}(M, x_0) \otimes \mathbb{R}$.

Remark 3. There is another way of understanding the $H_1(M; \mathbb{Z})$ action on $\mathfrak{Eul}(M, x_0)$. Suppose $n > 2$ and represent $[\sigma] \in H_1(M; \mathbb{Z})$ by a simple closed curve σ. Choose a tubular neighborhood N of S^1 considered as vector bundle $N \to S^1$. Choose a fiber metric and a linear connection on N. Choose a representative of $[X, c] \in \mathfrak{Eul}(M, x_0)$ such that $X|_N = \frac{\partial}{\partial \theta}$, the horizontal lift of the canonic vector field on S^1.

Choose a function $\lambda : [0, \infty) \to [-1, 1]$, which satisfies $\lambda(r) = -1$ for $r \leq \frac{1}{3}$ and $\lambda(r) = 1$ for $r \geq \frac{2}{3}$. Finally choose a function $\mu : [0, \infty) \to \mathbb{R}$ satisfying $\mu(r) = r$ for $r \leq \frac{1}{3}$, $\mu(r) = 0$ for $r \geq \frac{2}{3}$ and $\mu(r) > 0$ for all $r \in (\frac{1}{3}, \frac{2}{3})$. Now construct a new vector field $\tilde{X}$ on M by setting

$$\tilde{X} := \begin{cases} X & \text{on } M \setminus N \\ \lambda(r)\frac{\partial}{\partial\theta} + \mu(r)\frac{\partial}{\partial r} & \text{on } N, \end{cases}$$

where $r : N \to [0, \infty)$ denotes the radius function determined by the fiber metric on N and $-r\frac{\partial}{\partial r}$ is the Euler vector field of N. This construction is known as Reeb surgery, see e.g. [10]. If the zeros of X are all non-degenerate the homotopy $X_t := (1-t)X + t\tilde{X}$ is a non-degenerate homotopy from $X_0 = X$ to $X_1 = \tilde{X}$ from which one easily deduces that

$$[\tilde{X}, c] = [X, c] + [\sigma].$$

Particularly all the choices that entered the Reeb surgery do not effect the out-coming Euler structure $[\tilde{X}, c]$.

Let us consider a change of base point. Let $x_0, x_1 \in M$ and choose a path σ from x_0 to x_1. Define

$$\mathfrak{Eul}(M, x_0) \to \mathfrak{Eul}(M, x_1), \quad [X, c] \mapsto [X, c - \chi(M)\sigma]. \qquad (21)$$

This is an $H_1(M; \mathbb{Z})$ equivariant bijection but depends on the homology class of σ.

Remark 4. So the identification $\mathfrak{Eul}(M, x_0)$ with $\mathfrak{Eul}(M, x_1)$ does depend on the choice of a homology class of paths from x_0 to x_1. However, different choices will give identifications which differ by the action of an element in $\chi(M)H_1(M; \mathbb{Z})$. So the quotient $\mathfrak{Eul}(M, x_0)/\chi(M)H_1(M; \mathbb{Z})$ does not depend on the base point. Particularly, if $\chi(M) = 0$ then $\mathfrak{Eul}(M, x_0)$ does not depend on the base point.

Let $\mathfrak{X}(M, x_0)$ denote the space of vector fields which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^∞ topology. Let $\pi_0(\mathfrak{X}(M, x_0))$ denote the space of homotopy classes of such vector fields. If $X \in \mathfrak{X}(M, x_0)$ we will write $[X]$ for the corresponding class in $\pi_0(\mathfrak{X}(M, x_0))$. The following proposition (due to Turaev in the case $\chi(M) = 0$) establishes the proof of Theorem 2(i).

Proposition 3. *Suppose $n > 2$. Then there exists a natural bijection*

$$\pi_0(\mathfrak{X}(M, x_0)) = \mathfrak{Eul}(M, x_0), \qquad [X] \mapsto [X, 0]. \qquad (22)$$

Proof. Clearly (22) is well defined. Let us prove that it is onto. So let $[X, c]$ represent an Euler class. Choose an embedded disk $D \subseteq M$

centered at x_0 which contains all zeros of X and the Euler chain c. For this we may have to change c, but without changing the Euler structure $[X, c]$. Choose a vector field X' which equals X on $M \setminus D$ and vanishes just at x_0. Since $H_1(D; \mathbb{Z}) = 0$ we clearly have $[X', 0] = [X, c] \in \mathfrak{Eul}(M, x_0)$ and thus (22) is onto.

Let us prove injectivity of (22). Let $X_1, X_2 \in \mathfrak{X}(M, x_0)$ and suppose $c(X_1, X_2) = 0 \in H_1(M; \mathbb{Z})$. Let $D \subseteq M$ denote an embedded open disk centered at x_0. Consider the vector bundle $p^*TM \to I \times M$ and consider the two vector fields as a nowhere vanishing section of p^*TM defined over the set $\partial I \times \dot{M}$, where $\dot{M} := M \setminus D$. We would like to extend it to a nowhere vanishing section over $I \times \dot{M}$. The first obstruction we meet is an element in

$$\begin{aligned} H^n(I \times \dot{M}, \partial I \times \dot{M}; \{\pi_{n-1}\}) &= H_1(I \times \dot{M}, I \times \partial D; \mathbb{Z}) \\ &= H_1(M, \bar{D}; \mathbb{Z}) \\ &= H_1(M; \mathbb{Z}) \end{aligned}$$

which corresponds to $c(X_1, X_2) = 0$. Here $\{\pi_{n-1}\}$ denotes the system of local coefficients determined by the sphere bundle of p^*TM with $\pi_{n-1} = \pi_{n-1}(S^{n-1})$. Since this obstruction vanishes by hypothesis the next obstruction is defined and is an element in:

$$\begin{aligned} H^{n+1}(I \times \dot{M}, \partial I \times \dot{M}; \{\pi_n\}) &= H_0(I \times \dot{M}, I \times \partial D; \pi_n(S^{n-1})) \\ &= H_0(M, \bar{D}; \pi_n(S^{n-1})) \\ &= 0 \end{aligned}$$

Since there is no other obstructions, obstruction theory, see e.g. [12], tells us that we find a nowhere vanishing section of p^*TM defined over $I \times \dot{M}$, which restricts to X_i on $\{i\} \times \dot{M}$, $i = 1, 2$. Such a section can easily be extended to a globally defined section of $p^*TM \to I \times M$, which restricts to X_i on $\{i\} \times M$, $i = 1, 2$ and whose zero set is precisely $I \times \{x_0\}$. Such a section can be considered as homotopy from X_1 to X_2 showing $[X_1] = [X_2]$. Hence (22) is injective. □

Remark 5. If $n > 2$ Reeb surgery defines an $H_1(M; \mathbb{Z})$ action on $\pi_0(\mathfrak{X}(M, x_0))$ which via (22) corresponds to the $H_1(M; \mathbb{Z})$ action on $\mathfrak{Eul}(M, x_0)$, cf. Remark 3.

Let $\mathfrak{X}_0(M)$ denote the space of nowhere vanishing vector fields on M equipped with the C^∞ topology. Let $\pi_0(\mathfrak{X}_0(M))$ denote the set of its connected components. The next proposition is a restatement of Theorem 2(ii).

Proposition 4. *If $n > 2$ then we have a surjection:*

$$\pi_0(\mathfrak{X}_0(M)) \to \mathfrak{Eul}(M, x_0), \qquad [X] \mapsto [X, 0]. \tag{23}$$

Proof. The assignment (23) is certainly well defined. Let us prove surjectivity. Let $[X, c]$ be an Euler structure. Choose an embedded disk $D \subseteq M$ which contains all zeros of X and its Euler chain c, cf. proof of Proposition 3. Since $\chi(M) = 0$ the degree of $X : \partial D \to TD \setminus 0_D$ vanishes. Modifying X only on D we get a nowhere vanishing X' which equals X on $M \setminus D$. Certainly X' has an Euler chain c' which is also contained in D and satisfies $[X, c] = [X', c']$. Since X' has no zeros we get $\partial c' = 0$ and since $H_1(D; \mathbb{Z}) = 0$ we arrive at $[X, c] = [X', c'] = [X', 0]$ which proves that (23) is onto. □

We will now describe another approach to Euler structures which is in some sense Poincaré dual to the other approach. We still consider a closed connected n–dimensional manifold with base point (M, x_0). Consider pairs (g, α) where g is a Riemannian metric on M and $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ with $d\alpha = E(g)$. Here $E(g) \in \Omega^n(M; \mathcal{O}_M)$ denotes the Euler class of g which is a form with values in the orientation bundle $\mathcal{O}_M$. We call two pairs (g_1, α_1) and (g_2, α_2) equivalent if

$$\mathrm{cs}(g_1, g_2) = \alpha_2 - \alpha_1 \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)/d\Omega^{n-2}(M \setminus x_0; \mathcal{O}_M).$$

We will write $\mathfrak{Eul}^*(M, x_0)$ for the set of equivalence classes and $[g, \alpha]$ for the equivalence class represented by the pair (g, α). Elements of $\mathfrak{Eul}^*(M, x_0)$ are called co-Euler structures based at x_0. There is a natural $H^{n-1}(M; \mathcal{O}_M)$ action on $\mathfrak{Eul}^*(M, x_0)$ given by

$$[g, \alpha] + [\beta] := [g, \alpha - \beta]$$

with $[\beta] \in H^{n-1}(M; \mathcal{O}_M)$. Since $H^{n-1}(M; \mathcal{O}_M) = H^{n-1}(M \setminus x_0; \mathcal{O}_M)$ this action is obviously free and transitive.

For a pair (g, α) as above and a closed one form ω we define a regularization of $\int_M \omega \wedge \alpha$ as follows. Choose a function f such that $\omega' := \omega - df$ vanishes locally around the base point x_0 and set:

$$S(g, \alpha, \omega; f) := \int_M \omega' \wedge \alpha - \int_M f E(g) + \chi(M) f(x_0)$$

Lemma 2. *The quantity $S(g, \alpha, \omega; f)$ does not depend on the choice of f and will thus be denoted by $S(g, \alpha, \omega)$. If $[g_1, \alpha_1] = [g_2, \alpha_2] \in \mathfrak{Eul}^*(M, x_0)$ then*

$$S(g_2, \alpha_2, \omega) - S(g_1, \alpha_1, \omega) = \int_M \omega \wedge \mathrm{cs}(g_1, g_2). \qquad (24)$$

Moreover, for a function h we have

$$S(g, \alpha, \omega + dh) - S(g, \alpha, \omega) = -\int_M h E(g) + \chi(M) h(x_0). \qquad (25)$$

Proof. Suppose we have two functions f_1 and f_2 so that both $\omega_1' := \omega - df_1$ and $\omega_2' := \omega - df_2$ vanish locally around x_0. Let B_ϵ denote a ball of radius ϵ around x_0. Then $f_2 - f_1$ will be constant on B_ϵ for ϵ sufficiently small. Using Stokes' theorem, $d\alpha = E(g)$ and $\int_M E(g) = \chi(M)$ we get:

$$\begin{aligned}
&S(g,\alpha,\omega;f_2) - S(g,\alpha,\omega;f_1) = \\
&= -\int_{M\setminus\mathcal{X}} d((f_2-f_1)\wedge\alpha) + \chi(M)(f_2-f_1)(x_0) \\
&= -\lim_{\epsilon\to 0}\int_{\partial(M\setminus B_\epsilon)} (f_2-f_1)\alpha + \chi(M)(f_2-f_1)(x_0) \\
&= -(f_2-f_1)(x_0)\lim_{\epsilon\to 0}\int_{\partial(M\setminus B_\epsilon)} \alpha + \chi(M)(f_2-f_1)(x_0) \\
&= -(f_2-f_1)(x_0)\lim_{\epsilon\to 0}\int_{M\setminus B_\epsilon} E(g) + \chi(M)(f_2-f_1)(x_0) = 0
\end{aligned}$$

The second statement follows immediately from $\alpha_2 - \alpha_1 = \operatorname{cs}(g_1, g_2)$, Stokes' theorem and $d\operatorname{cs}(g_1,g_2) = E(g_2) - E(g_1)$. The last property is obvious. □

In view of (15), (19), (20), (24) and (25) the quantity

$$R(X,g,\omega) - S(g,\alpha,\omega) - \int_c \omega \tag{26}$$

does only depend on $[X,c] \in \mathfrak{Eul}(M,x_0)\otimes\mathbb{R}$, $[g,\alpha]\in\mathfrak{Eul}^*(M,x_0)$ and $[\omega]\in H^1(M;\mathbb{R})$. Thus (26) defines an invariant

$$\mathbb{T} : \big(\mathfrak{Eul}(M,x_0)\otimes\mathbb{R}\big)\times\mathfrak{Eul}^*(M,x_0)\to H_1(M;\mathbb{R}).$$

From the very definition we have

$$\langle[\omega],\mathbb{T}([X,c],[g,\alpha])\rangle = \int_M \omega\wedge(X^*\Psi(g)-\alpha) - \int_c\omega, \tag{27}$$

where ω is any representative of $[\omega]$ which vanishes locally around the zeros of X and vanishes locally around the base point x_0. Moreover we have

$$\mathbb{T}(\mathfrak{e}+\sigma,\mathfrak{e}^*+\beta) = \mathbb{T}(\mathfrak{e},\mathfrak{e}^*) - \sigma + \operatorname{PD}(\beta) \tag{28}$$

for all $\mathfrak{e}\in\mathfrak{Eul}(M,x_0)\otimes\mathbb{R}$, $\mathfrak{e}^*\in\mathfrak{Eul}^*(M,x_0)$, $\sigma\in H_1(M;\mathbb{Z})$ and $\beta\in H^{n-1}(M;\mathcal{O}_M)$. Here PD is the Poincaré duality isomorphism $\operatorname{PD}: H^{n-1}(M;\mathcal{O}_M)\to H_1(M;\mathbb{R})$.

We have the following affine version of Poincaré duality, which establishes the proof of Theorem 2(iii) and (iv).

Proposition 5. *There is a natural bijection:*

$$P : \mathfrak{Eul}^*(M,x_0)\to\mathfrak{Eul}(M,x_0)\otimes\mathbb{R}$$

Moreover, for every $\beta \in H^{n-1}(M; \mathcal{O}_M)$ *and every* $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, x_0)$ *we have*

$$P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta) \tag{29}$$

and $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) = P(\mathfrak{e}^*) - \mathfrak{e}$.

Proof. Given $\mathfrak{e}^* = [g, \alpha] \in \mathfrak{Eul}^*(M, x_0)$ we choose a vector field X with isolated singularities $\mathcal{X}$. Then $X^*\Psi(g) - \alpha$ is closed and thus defines a cohomology class in $H^{n-1}(M \setminus (\mathcal{X} \cup \{x_0\}); \mathcal{O}_M)$. We would like to define $P(\mathfrak{e}^*) := [X, c]$ where c be a representative of its Poincaré dual in $H_1(M, \mathcal{X} \cup \{x_0\}; \mathbb{R})$. That is, we ask

$$\int_c \omega = \int_{M \setminus (\mathcal{X} \cup \{x_0\})} \omega \wedge (X^*\Psi(g) - \alpha)$$

to hold for every closed compactly supported one form ω on $M \setminus (\mathcal{X} \cup \{x_0\})$. In view of (27) this is equivalent to ask for $\mathbb{T}(P(\mathfrak{e}^*), \mathfrak{e}^*) = 0$. So we take the latter one as our definition of P. Because of (28) this has a unique solution. The equivariance property and the last equation follow at once. □

5. Smooth triangulations and extension of Chern–Simons theory

Smooth triangulations. Smooth triangulations provide a remarkable source of vector fields with isolated singularities.

To any smooth triangulation τ of the smooth manifold M one can associate a smooth vector field X_τ called *Euler vector field*, with the following properties:

- P1: The zeros of X_τ are all non-degenerate and are exactly the barycenters x_σ of the simplexes σ.
- P2: The flow of $-X_\tau$ is defined for all $t \in \mathbb{R}$.[4]
- P3: For each zero x_σ the unstable set with respect to $-X_\tau$ is exactly the open simplex σ, consequently the zeros are hyperbolic. The Morse index of $-X_\tau$ at x_σ equals $\dim(\sigma)$ and the (Hopf) index of X_τ at x_σ equals $(-1)^{\dim(\sigma)}$.
- P4: The piecewise differential function $f_\tau : M \to \mathbb{R}$ defined by $f_\tau(x_\sigma) = \dim(\sigma)$ and extended by linearity on M is a Liapunov function for $-X_\tau$, i.e. strictly decreasing on non-constant trajectories of $-X_\tau$.

Such a vector field X_τ is unique up to an homotopy of vector fields which satisfy P1–P4. The convex combination provides the homotopy between any two such vector fields.

[4] This is always the case if M is closed.

To construct such a vector field we begin with a standard simplex Δ_n of vectors $(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ satisfying $0 \leq t_i \leq 1$ and $\sum t_i = 1$.

(i) Let E_n denote the Euler vector field of the corresponding affine space ($\sum t_i = 1$) centered at the barycenter O (of coordinates $(1/(n+1), \ldots, 1/(n+1))$ and restricted to Δ_n.
(ii) Let $e : \Delta_n \to [0,1]$ denote the function which is 1 on the barycenter O and zero on all vertices.
(iii) Let $r : \Delta_n \setminus \{O\} \to \partial\Delta_n$ denote the radial retraction to the boundary.

Set $X'_n := e \cdot E_n$, which is a vector field on Δ_n.

By induction we will construct a canonical vector field X_n on Δ_n which at any point $x \in \Delta_n$ is tangent to the open face the point belongs to and vanishes only at the barycenter of each face. We proceed as follows:

Suppose we have constructed such canonical vector fields on all $\Delta(k)$, $k \leq n-1$. Using the canonical vector fields X_{n-1} we define the vector field X_n on the boundary $\partial\Delta_n$ and extend it to the vector field X''_n by taking at each point $x \in \Delta_n$ the vector parallel to $X_n(r(x))$ multiplied by the function $(1-e)$ and at O the vector zero. Clearly such vector field vanishes on the radii $\overline{OP}$ (P the barycenter of any face). We finally put

$$X_n := X'_n + X''_n.$$

The vector field X_n is continuous and piecewise differential (actually Lipschitz) and has a well defined continuous flow.

Note that one can replace Δ_n by any convex polyhedron P_n and by a similar construction produce inductively the vector field X_n with similar properties.

Putting together the vector fields X_n on all simplexes (cells) we provide a piecewise differential (and Lipschitz) vector field X on any simplicial (cellular) complex or polyhedron and in particular on any smoothly triangulated manifold. The vector field X has a flow and f_τ is a Liapunov function for $-X$. The vector field X is not necessary smooth but by a small (Lipschitz) perturbation we can approximate it by a smooth vector field X_τ which satisfies P1–P4. Any of the resulting vector fields is referred to as the Euler vector field of a smooth triangulation τ.

Extension of Chern-Simons theory. Let M be a closed manifold of dimension n. We equip $\Omega^k(M;\mathbb{R})$ with the C^∞ topology. The continuous linear functionals on $\Omega^k(M;\mathbb{R})$ are called k currents. We denote the space of all k currents by $\mathcal{D}_k(M)$. Let $\delta : \mathcal{D}_k(M) \to \mathcal{D}_{k-1}(M)$ be given by $(\delta\varphi)(\alpha) := \varphi(d\alpha)$. Clearly $\delta^2 = 0$.[5]

We have a morphism of chain complexes

$$C_*(M;\mathbb{R}) \to \mathcal{D}_*(M), \quad \sigma \mapsto \hat{\sigma}, \quad \hat{\sigma}(\alpha) := \int_\sigma \alpha.$$

Her $C_*(M;\mathbb{R})$ denotes the space of singular chains with real coefficients. Moreover we have a morphism of chain complexes

$$\Omega^{n-*}(M;\mathcal{O}_M) \to \mathcal{D}_*(M), \quad \beta \mapsto \hat{\beta}, \quad \hat{\beta}(\alpha) := (-1)^{\frac{1}{2}|\alpha|(|\alpha|+1)} \int_M \alpha \wedge \beta.$$

Here $|\alpha|$ denotes the degree of α. The weird sign is necessary so that this mappings actually intertwines the two differentials d and δ.

Every vector field with isolated singularities X gives rise to a zero chain e_X, cf. section 4. Via the first morphism we get a zero current $\hat{E}(X)$. More explicitly $(\hat{E}(X))(h) = \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x)$ for a function $h \in \Omega^0(M;\mathbb{R})$.

A Riemannian metric g has an Euler form $E(g) \in \Omega^n(M;\mathcal{O}_M)$. Via the second morphism we get a zero current $\hat{E}(g)$. More explicitly $(\hat{E}(g))(h) = \int_M hE(g)$ for a function $h \in \Omega^0(M;\mathbb{R})$.

Let $\mathcal{Z}^k(M;\mathbb{R}) \subseteq \Omega^k(M;\mathbb{R})$ denote the space of closed k forms on M equipped with the C^∞ topology. The continuous linear functionals on $\mathcal{Z}^k(M;\mathbb{R})$ are referred to as *k currents rel. boundary* and identify to $\mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M))$. The two chain morphisms provide mappings

$$C_k(M;\mathbb{R})/\partial(C_{k+1}(M;\mathbb{R})) \to \mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M)) \tag{30}$$

and

$$\Omega^{n-k}(M;\mathcal{O}_M)/d(\Omega^{n-k-1}(M;\mathcal{O}_M)) \to \mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M)) \tag{31}$$

For two vector fields with isolated zeros X_1 and X_2 we have constructed $c(X_1,X_2) \in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z}))$, cf. section 3. This gives rise to $c(X_1,X_2) \in C_1(M;\mathbb{R})/\partial(C_2(M;\mathbb{R}))$, and via (30) we get a one current rel. boundary which we will denote by $\hat{\mathrm{cs}}(X_1,X_2)$. More precisely, $(\hat{\mathrm{cs}}(X_1,X_2))(\omega) = \int_{c(X_1,X_2)} \omega$ for a closed one form $\omega \in \mathcal{Z}^1(M;\mathbb{R})$. Recall that $c(X_2,X_1) = -c(X_1,X_2)$, $c(X_1,X_3) = c(X_1,X_2) + c(X_2,X_3)$, $\partial c(X_1,X_2) = e_{X_2} - e_{X_1}$ and thus

$$\begin{aligned}
\hat{\mathrm{cs}}(X_2,X_1) &= -\hat{\mathrm{cs}}(X_1,X_2) \\
\hat{\mathrm{cs}}(X_1,X_3) &= \hat{\mathrm{cs}}(X_1,X_2) + \hat{\mathrm{cs}}(X_2,X_3) \\
\delta\hat{\mathrm{cs}}(X_1,X_2) &= \hat{E}(X_2) - \hat{E}(X_1).
\end{aligned}$$

[5]The chain complex $(\mathcal{D}_*(M),\delta)$ computes the homology of M with real coefficients.

For two Riemannian metrics g_1 and g_2 we have the Chern–Simons form $\operatorname{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$. Via (31) we get a one current rel. boundary which we will denote by $\hat{\operatorname{cs}}(g_1, g_2)$. More precisely $(\hat{\operatorname{cs}}(g_1, g_2))(\omega) = -\int_M \omega \wedge \operatorname{cs}(g_1, g_2)$ for a closed one form $\omega \in \mathcal{Z}^1(M; \mathbb{R})$. Recall that $\operatorname{cs}(g_2, g_1) = -\operatorname{cs}(g_1, g_2)$, $\operatorname{cs}(g_1, g_3) = \operatorname{cs}(g_1, g_2) + \operatorname{cs}(g_2, g_3)$, $d\operatorname{cs}(g_1, g_2) = E(g_2) - E(g_1)$ and thus

$$\begin{aligned}
\hat{\operatorname{cs}}(g_2, g_1) &= -\hat{\operatorname{cs}}(g_1, g_2) \\
\hat{\operatorname{cs}}(g_1, g_3) &= \hat{\operatorname{cs}}(g_1, g_2) + \hat{\operatorname{cs}}(g_2, g_3) \\
\delta\hat{\operatorname{cs}}(g_1, g_2) &= \hat{E}(g_2) - \hat{E}(g_1)
\end{aligned}$$

Suppose X is a vector field with isolated zeros and g is a Riemannian metric. We define one currents rel. boundary by $(\hat{\operatorname{cs}}(g, X))(\omega) := R(X, g, \omega)$ and $\hat{\operatorname{cs}}(X, g) := -\hat{\operatorname{cs}}(g, X)$. Proposition 1 and Proposition 2 tell that

$$\begin{aligned}
\delta\hat{\operatorname{cs}}(g, X) &= \hat{E}(X) - \hat{E}(g) \\
\hat{\operatorname{cs}}(g_1, X) &= \hat{\operatorname{cs}}(g_1, g_2) + \hat{\operatorname{cs}}(g_2, X) \\
\hat{\operatorname{cs}}(g, X_2) &= \hat{\operatorname{cs}}(g, X_1) + \hat{\operatorname{cs}}(X_1, X_2)
\end{aligned}$$

We summarize these observations in

Proposition 6. *Let any of the symbols x, y, z denote either a Riemannian metric g or a vector field with isolated zeros. Then one has:*

(i) $\hat{\operatorname{cs}}(y, x) = -\hat{\operatorname{cs}}(x, y)$
(ii) $\hat{\operatorname{cs}}(x, z) = \hat{\operatorname{cs}}(x, y) + \hat{\operatorname{cs}}(y, z)$
(iii) $\delta\hat{\operatorname{cs}}(x, y) = \hat{E}(y) - \hat{E}(x)$.

Suppose τ is a smooth triangulation. We define its Euler current by $\hat{E}(\tau) := \hat{E}(X_\tau)$, where X_τ is the Euler vector field. Similarly for two triangulations τ_1 and τ_2 we define a one current rel. boundary by $\hat{\operatorname{cs}}(\tau_1, \tau_2) := \hat{\operatorname{cs}}(X_{\tau_1}, X_{\tau_2})$.

Corollary 1. *Let any of the symbols x, y, z denote either a Riemannian metric g or a smooth triangulation. Then one has:*

(i) $\hat{\operatorname{cs}}(y, x) = -\hat{\operatorname{cs}}(x, y)$
(ii) $\hat{\operatorname{cs}}(x, z) = \hat{\operatorname{cs}}(x, y) + \hat{\operatorname{cs}}(y, z)$
(iii) $\delta\hat{\operatorname{cs}}(x, y) = \hat{E}(y) - \hat{E}(x)$.

More about putting together Riemannian metrics and smooth triangulations will be addressed in forthcoming work.

6. Theorem of Bismut–Zhang

Let (M, x_0) be a closed connected manifold with base point. Let $\mathbb{K}$ be a field and suppose F is a flat $\mathbb{K}$ vector bundle over M, that is F is equipped with a flat connection ∇. Let F_{x_0} denote the fiber over the base point x_0. Holonomy at the base point provides a right $\pi_1(M, x_0)$ action on F_{x_0} and when composed with the inversion in $\mathrm{GL}(F_{x_0})$ a representation $\rho_F : \pi_1(M, x_0) \to \mathrm{GL}(F_{x_0})$. So we get a homomorphism $\det \circ \rho_F : \pi_1(M, x_0) \to \mathbb{K}^*$ which descends to a homomorphism $H_1(M; \mathbb{Z}) \to \mathbb{K}^*$ and thus determines a cohomology class $\Theta_F \in H^1(M; \mathbb{K}^*)$.

Suppose we have a smooth triangulation τ of M. It gives rise to a cellular complex $C^*_\tau(M; F)$ which computes the cohomology $H^*(M; F)$.[6] Let $\mathcal{X}_\tau$ denote the set of barycenters of τ. For a cell σ of τ we let x_σ denote the barycenter of σ. Let X_τ denote the Euler vector field of τ, cf. section 5. Then $\mathcal{X}_\tau$ is the zero set of X_τ. Moreover for a cell σ we have $\mathrm{IND}_{X_\tau}(\sigma_x) = (-1)^{\dim \sigma}$. As a graded vector space we have $C^k_\tau(M; F) = \bigoplus_{\dim \sigma = k} F_{x_\sigma}$. So we get a canonical isomorphism of $\mathbb{K}$ vector spaces:

$$\det C^*_\tau(M; F) = \det H(C^*_\tau(M; F)) = \det H^*(M; F) \qquad (32)$$

Recall that the determinant line of a vector space W is by definition $\det W := \Lambda^{\dim W} W$. For a $\mathbb{Z}$ graded vector space V^* one sets $V^{\mathrm{even}} := \bigoplus_{k \text{ even}} V^k$, $V^{\mathrm{odd}} := \bigoplus_{k \text{ odd}} V^k$ and defines its determinant line by $\det V^* := \det V^{\mathrm{even}} \otimes (\det V^{\mathrm{odd}})^*$.

Suppose we also have given an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M, x_0)$. For every $x \in \mathcal{X}_\tau$ choose a path π_x from x_0 to x, so that with $c := \sum_{x \in \mathcal{X}_\tau} \mathrm{IND}_{X_\tau}(x_\sigma) \pi_x$ we have $\mathfrak{e} = [X_\tau, c]$.[7] Let f_0 be a non-zero element in $\det F_{x_0}$. Note that a frame (basis) in F_{x_0} determines such an element in $\det F_{x_0}$. Using parallel transport along π_x we get a non-zero element in every $\det F_{x_\sigma}$. If the barycenters x_σ where ordered we would get a well defined non-zero element in $\det C^*_\tau(M; F)$.

Suppose $\mathfrak{o}$ is a cohomology orientation of M, that is an orientation of $\det H^*(M; \mathbb{R})$. We say an ordering of the zeros x_σ is compatible with $\mathfrak{o}$ if the non-zero element in $\det C^*_\tau(M; \mathbb{R})$ provided by this ordered base is compatible with the orientation $\mathfrak{o}$ via the canonic isomorphism

$$\det C^*_\tau(M; \mathbb{R}) = \det H(C^*_\tau(M; \mathbb{R})) = \det H^*(M; \mathbb{R}).$$

So an Euler structure $\mathfrak{e}$, a cohomology orientation $\mathfrak{o}$ and an element $f_0 \in \det F_{x_0}$ provide a non-zero element in $\det C^*_\tau(M; F)$ which corresponds to a non-zero element in $\det H^*(M; F)$ via (32). We thus get a mapping

$$\det F_{x_0} \setminus 0 \to \det H^*(M; F) \setminus 0. \qquad (33)$$

[6] $H^*(M; F)$ can be thought of as singular, deRham or sheaf cohomology, they are all canonically isomorphic.

[7] Such a representative for the Euler structure is called spray or Turaev spider.

This mapping is obviously homogeneous of degree $\chi(M)$. A straight forward calculation shows that it does not depend on the choice of π_x. As a matter of fact this mapping does not depend on τ either, only on the Euler structure $\mathfrak{e}$ and the cohomology orientation $\mathfrak{o}$. This is a non-trivial fact, and its proof is contained in [9], for acyclic case and implicit in the existing literature cf. [2]. We define the combinatorial torsion to be the element

$$\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}} \in \det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$$

corresponding to the homogeneous mapping (33). Note that we also have

$$\tau^{\text{comb}}_{F,\mathfrak{e}+\sigma,\mathfrak{o}} = \tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}} \cdot \langle \Theta_F, \sigma\rangle^{-1},$$

for all $\sigma \in H_1(M;\mathbb{Z})$. Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing of homology with integer coefficients and cohomology with coefficients in an arbitrary Abelian group, in our case $\mathbb{K}^*$. Moreover

$$\tau^{\text{comb}}_{F,\mathfrak{e},-\mathfrak{o}} = (-1)^{\operatorname{rank} F} \tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}.$$

Clearly, if $\chi(M) = 0$ then $\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}} \in \det H^*(M;F)$.

Consider the case $\mathbb{K} = \mathbb{R}$. So F is a flat real vector bundle over M. Let τ be a triangulation as above. Let μ be a Hermitian structure, i.e. a fiber metric, on F. This Hermitian structure induces a metric on $\det C^*_\tau(M;F)$ and via (32) a metric $||\cdot||^{\mathcal{M}}_{F,\tau,\mu}$ on the line $\det H^*(M;F)$. This is exactly what is called Milnor metric in [1]. The Hermitian structure μ also defines a metric on $(\det F_{x_0})^{-\chi(M)}$ which we will denote by $||\cdot||_{\mu_{x_0}}$. The Hermitian structure gives rise to a closed one form $\omega(\nabla,\mu)$, cf. [1] and section 2, where ∇ is the flat connection of F. For its cohomology class we have $[\omega(\nabla,\mu)] = (\log|\cdot|^{-2})_*\Theta_F$, where $(\log|\cdot|^{-2})_* : H^1(M;\mathbb{R}^*) \to H^1(M;\mathbb{R})$. Suppose we also have an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M,x_0)$ and choose an Euler chain c so that $[X_\tau, c] = \mathfrak{e}$. We define a metric on $\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$ by:

$$||\cdot||^{\text{comb}}_{F,\mathfrak{e}} := ||\cdot||^{\mathcal{M}}_{F,\tau,\mu} \otimes ||\cdot||_{\mu_{x_0}} \cdot e^{-\frac{1}{2}\int_c \omega(\nabla,\mu)} \tag{34}$$

As already indicated this is independent of μ and does only depend on the Euler structure $\mathfrak{e} = [X, c]$. This follows from known anomaly formulas for the Milnor torsion, implicit in [1], or can be seen as a consequence of (35) below. Note that

$$||\cdot||^{\text{comb}}_{F,\mathfrak{e}+\sigma} = ||\cdot||^{\text{comb}}_{F,\mathfrak{e}} \cdot \big|\langle \Theta_F, \sigma\rangle\big|,$$

for all $\sigma \in H_1(M;\mathbb{Z})$ and

$$||\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}||^{\text{comb}}_{F,\mathfrak{e}} = 1. \tag{35}$$

Now let g be a Riemannian metric on M. Then we also have the Ray–Singer metric $||\cdot||^{\mathrm{RS}}_{F,g,\mu}$ on $\det H^*(M;F)$, cf. [1]. Let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M,x_0)$ and suppose $[g,\alpha] = \mathfrak{e}^*$, i.e. $d\alpha = E(g)$. Define a metric on $\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$ by:

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} := ||\cdot||^{\mathrm{RS}}_{F,g,\mu} \otimes ||\cdot||_{\mu_{x_0}} \cdot e^{\frac{1}{2}S(g,\alpha,\omega(F,\mu))} \tag{36}$$

The known anomaly formulas for the Ray–Singer torsion, see [1], imply that this is independent of μ and only depends on the co-Euler structure $\mathfrak{e}^*$. Note that

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*+\beta} = ||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} \cdot \big|\langle \Theta_F, \mathrm{PD}(\beta)\rangle\big| \tag{37}$$

for all $\beta \in H^{n-1}(M;\mathcal{O}_M)$. The main theorem of Bismut–Zhang, see [1], can now be reformulated as follows:

Theorem 5 (Bismut–Zhang)**.** *Suppose (M,x_0) is a closed connected manifold with base point and F a flat real vector bundle over M. Let $\mathfrak{e} \in \mathfrak{Eul}(M,x_0)$ be an Euler structure and let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M,x_0)$ be a co-Euler structure, both based at x_0. Then one has:*

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} = ||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}} \cdot \big|\langle \Theta_F, \exp_*(\mathbb{T}(\mathfrak{e},\mathfrak{e}^*))\rangle\big|$$

Particularly, if $\mathfrak{e} = P(\mathfrak{e}^*)$ then $||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} = ||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}}$.

Recall that the homomorphism of groups $\exp : \mathbb{R} \to \mathbb{R}^*$ induces $\exp_* : H_1(M;\mathbb{R}) \to H_1(M;\mathbb{R}^*)$, and thus $\exp_*(\mathbb{T}(\mathfrak{e},\mathfrak{e}^*)) \in H_1(M;\mathbb{R}^*)$. The pairing $\langle\cdot,\cdot\rangle$ in the theorem is the natural pairing of $H_1(M;\mathbb{R}^*)$ with $H^1(M;\mathbb{R}^*)$.

For an alternative proof of the (original) Bismut–Zhang theorem see also [3].

7. Proof of Theorem 3

Let N be a closed connected manifold and $\varphi : N \to N$ a diffeomorphism. Set

$$P^k_\varphi(z) := \det\Big(z\varphi^* - \mathrm{Id} : H^k(N;\mathbb{R}) \to H^k(N;\mathbb{R}\Big)$$

and let

$$\zeta_\varphi(z) := \frac{\prod_{k \text{ even}} P^k_\varphi(z)}{\prod_{k \text{ odd}} P^k_\varphi(z)}$$

denote its Lefschetz zeta function.

Let M denote the mapping torus. More precisely M is obtained from $I \times N$ by identifying $(1,x)$ with $(0,\varphi(x))$ for all $x \in N$. Let $p : M \to S^1 = I/\{0,1\}$ denote the projection induced by the projection $I \times N \to I$and $\omega := p^*ds \in \Omega^1(M;\mathbb{R})$. Here $ds \in \Omega^1(S^1;\mathbb{R})$ denotes the volume element on S^1.

Let g be a Riemannian metric on M. Denote by $d_\omega(t)\alpha := d\alpha + t\omega\wedge\alpha$ the Witten deformed differential on $\Omega^*(M;\mathbb{R})$, by $\delta_\omega(t)$ its adjoint, by

$$\Delta_\omega(t) := d_\omega(t)\delta_\omega(t) + \delta_\omega(t)d_\omega(t)$$

the Laplacian and by $\Delta^k_\omega(t)$ the Laplacian on k–forms. Introduce the Ray–Singer torsion:

$$T_{\mathrm{an}}(\omega,g)(t) := \exp\Big(\frac{1}{2}\sum_k(-1)^{k+1}k\log\det{}'\Delta^k_\omega(t)\Big)$$

Here $\det'\Delta$ denotes the regularized determinant of the nonnegative selfadjoint elliptic operator Δ obtained by ignoring the zero modes.

Theorem 6 (J. Marcsik)**.** [8] *Let N be a closed manifold, $\varphi : N \to N$ a diffeomorphism, M the mapping torus of φ, $p : M \to S^1$ the projection and $\omega := p^*ds$, where ds denotes the volume form on S^1. Suppose X is a vector field on M with $\omega(X) < 0$ and suppose g is a Riemannian metric on M. Then*

$$|\zeta_\varphi(e^t)| = T_{\mathrm{an}}(\omega,g)(t)e^{-tR(X,g,\omega)}.$$

for all $t\in\mathbb{R}$ which satisfy $P^k_\varphi(e^t)\neq 0$, for all k.

The rest of this section is devoted to the derivation of this theorem from the Bismut–Zhang theorem cf 7.

We think of $N\subseteq M$ via $x\mapsto(0,x)$. Pick a base point $x_0\in N\subseteq M$. For every $t\in\mathbb{R}$ the cohomology class $[t\omega]$ provides a representation

$$\rho_t : \pi_1(M,x_0)\to H_1(M;\mathbb{Z}) \xrightarrow{\langle[t\omega],\cdot\rangle} \mathbb{R} \xrightarrow{\exp} \mathbb{R}_+\subseteq\mathbb{R}^*$$

on the vector space $\mathbb{R}$. Let F_{ρ_t} denote the assotiated flat line bundle. Obviously we have $\Theta_{F_{\rho_t}} = \exp_*[t\omega]\in H^1(M;\mathbb{R}^*)$.

Let $\pi : (\tilde M,\tilde x_0)\to(M,x_0)$ denote the universal covering. Let $r_t : \tilde M\to\mathbb{R}$ denote the unique function such that $dr_t = \pi^*t\omega$ and $r(\tilde x_0) = 0$. Then $r_t(\sigma\tilde x) = r_t(\tilde x) + \langle[t\omega],\sigma\rangle$ and thus $\sigma^*e^{r_t} = \rho_t(\sigma)e^{r_t}$, for all $\sigma\in\pi_1(M,x_0)$. So e^{r_t} is a nowhere vanishing ρ_t–equivariant function on $\tilde M$, hence a section of F_{ρ_t}. We use this section to trivialize F_{ρ_t}. So we also have the identification

$$\Omega^*(M;\mathbb{R}) = \Omega^*(\tilde M;\mathbb{R})^{\rho_t} = \Omega^*(M;F_{\rho_t}), \qquad \alpha\mapsto e^{r_t}\pi^*\alpha,$$

where $\alpha\in\Omega^*(\tilde M;\mathbb{R})^{\rho_t}$ if $\sigma^*\alpha = \rho_t(\sigma)\alpha$, for all $\sigma\in\pi_1(M,x_0)$. Via this identification the deRham differential for forms with values in F_{ρ_t} is $d_\omega(t)$.

Let μ_t denote the Hermitian structure on F_{ρ_t} which corresponds to the trivial Hermitian structure via our trivialization $F_{\rho_t} = M\times\mathbb{R}$.

[8] this statement is sligtly more precise than the one formulated in [7]

Then $\nabla^{\rho_t}_Y \mu_t = -2t\omega(Y)\mu_t$ and thus $\omega(F_{\rho_t}, \mu_t) = -2t\omega$. Moreover we obviously have

$$T_{\mathrm{an}}(\omega, g)(t) = T_{\mathrm{an}}(F_{\rho_t}, g, \mu_t). \tag{38}$$

Let $\mathfrak{e}$ denote the Euler structure determined by X, i.e. $\mathfrak{e} = [X, 0]$.

Proposition 7. *Suppose $H^*(M; F_{\rho_t})$ vanishes. Then*

$$||1||^{\mathrm{comb}}_{F_{\rho_t},\mathfrak{e}} = T_{\mathrm{an}}(\omega, g)(t)e^{-tR(X,g,\omega)}, \tag{39}$$

where $1 \in \mathbb{R} = \det H^*(M; F_{\rho_t})$.

Proof. Choose α so that $\mathfrak{e}^* := [g, \alpha]$ is a co-Euler structure. From the Bismut–Zhang theorem cf we get

$$|| \cdot ||^{\mathrm{an}}_{F_{\rho_t},\mathfrak{e}^*} = || \cdot ||^{\mathrm{comb}}_{F_{\rho_t},\mathfrak{e}} \cdot \left|\langle \Theta_{F_{\rho_t}}, \exp_*(\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*))\rangle\right| \tag{40}$$

as metrics on the line $\det H^*(M; F_{\rho_t})$. Note that the Euler characteristics of M always vanishes. Using $\omega(F_{\rho_t}, \mu_t) = -2t\omega$ we get

$$\begin{aligned} \langle \Theta_{F_{\rho_t}}, \exp_*(\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*))\rangle &= \exp\big(\langle [t\omega], \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*)\rangle\big) \\ &= \exp\big(R(X, g, t\omega) - S(\alpha, g, t\omega)\big) \\ &= \exp\big(tR(X, g, \omega) + \tfrac{1}{2}S(\alpha, g, \omega(F_{\rho_t}, \mu_t))\big) \end{aligned}$$

Thus (40) combined with (36) and with

$$|| \cdot ||^{\mathrm{RS}}_{F_{\rho_t},g,\mu_t} = || \cdot ||^{\mathrm{Hodge}}_{F_{\rho_t},g,\mu_t} \cdot T_{\mathrm{an}}(F_{\rho_t}, g, \mu_t)$$

implies

$$|| \cdot ||^{\mathrm{comb}}_{F_{\rho_t},\mathfrak{e}} = || \cdot ||^{\mathrm{Hodge}}_{F_{\rho_t},g,\mu_t} \cdot T_{\mathrm{an}}(F_{\rho_t}, g, \mu_t) \cdot e^{-tR(X,g,\omega)}.$$

(Recall that for deRham complex $\Omega(M; F)$ F a flat bundle with scalar products induced by the Riemannian metric g and Hermitian structure μ the metric $||\cdot||^{\mathrm{Hodge}}_{F,g,\mu}$ on $\det H^*(M; F)$ is obtained by the identification of $H^*(M; F)$ with the space of harmonic forms.) Since $H^*(M; F_{\rho_t})$ vanishes we certainly have $||1||^{\mathrm{Hodge}}_{F_{\rho_t},g,\mu_t} = 1$, which together with (38) now implies (39). □

Now choose a triangulation τ of M such that $N \subseteq M$ is a subcomplex. For every simplex σ of τ we choose a path π_σ from x_0 to the baricenter x_σ of σ in such a way that $0 \leq \int_{\pi_\sigma} \omega < 1$. Set $c_\tau := \sum_\sigma (-1)^{\dim \sigma} \pi_\sigma$. Then c_τ is an Euler chain for $-X_\tau$, where X_τ denotes the Euler vector field of τ, see section 5. Let $\tilde{\mathfrak{e}} = [-X_\tau, c_\tau]$ denote the corresponding Euler structure.

Proposition 8. *We have $\langle \Theta_{F_{\rho_t}}, \exp_*(\tilde{\mathfrak{e}} - \mathfrak{e})\rangle = 1$ and thus $|| \cdot ||^{\mathrm{comb}}_{F_{\rho_t},\mathfrak{e}} = || \cdot ||^{\mathrm{comb}}_{F_{\rho_t},\tilde{\mathfrak{e}}}$ on the line $\det H^*(M; F_{\rho_t})$.*

Proof. Consider $N_\epsilon \subseteq M$, $x \mapsto (x, 1-\epsilon)$, where $\epsilon > 0$ is sufficiently small such that $\omega(-X_\tau) < 0$ on N_ϵ and such that no zeros of $-X_\tau$ are contained in $(1-\epsilon, 1) \times N$. This is possible since N is a subcomplex, hence the Euler vector field X_τ is tangential to N and points towards N in a small neighborhood of N. We thus find a homotopy $\mathbb{X}_\lambda$ from X to $-X_\tau$ which has the property that $\omega(\mathbb{X}_\lambda) < 0$ on N_ϵ for all $\lambda \in I$. We conclude that $c(X, -X_\tau)$ is contained in $M \setminus N_\epsilon$. Since $\tilde{\mathfrak{e}} - \mathfrak{e} \in H_1(M;\mathbb{Z})$ is represented by $c_\tau - c(X, -X_\tau)$ we conclude $\langle [\omega], \tilde{\mathfrak{e}} - \mathfrak{e}\rangle = 0$ and therefore $\langle \Theta_{F_{\rho_t}}, \exp_*(\tilde{\mathfrak{e}} - \mathfrak{e})\rangle = \exp\big(\langle [t\omega], \tilde{\mathfrak{e}} - \mathfrak{e}\rangle\big) = 1$. □

Consider the short exact sequence of complexes:

$$0 \to C^*_\tau(M, N; F_{\rho_t}) \to C^*_\tau(M; F_{\rho_t}) \to C^*_\tau(N; F_{\rho_t}) \to 0 \qquad (41)$$

Clearly we have canonic isomorphisms

$$H(C^*_\tau(N; F_{\rho_t})) = H(C^*_\tau(N;\mathbb{R})) = H^*(N;\mathbb{R}).$$

Moreover we have isomorphisms of chain complexes

$$C^*_\tau(M, N; F_{\rho_t}) = C^*_\tau(I \times N, \partial I \times N; F_{\rho_t}) \xrightarrow{e^{ts}} C^*_\tau(I \times N, \partial I \times N; \mathbb{R}),$$

where $s : I \times N \to I \subseteq \mathbb{R}$ is the projection. This provides an isomorphism

$$H(C^*_\tau(M, N; F_{\rho_t})) = H^*(I \times N, \partial I \times N; \mathbb{R}) = H^{*-1}(N;\mathbb{R}) \qquad (42)$$

Proposition 9. *We have a commutative diagram*

$$\begin{array}{ccc} H^k(C^*_\tau(N; F_{\rho_t})) & \xrightarrow{\delta} & H^{k+1}(C^*_\tau(M, N; F_{\rho_t})) \\ \| & & \| \\ H^k(N;\mathbb{R}) & \xrightarrow{e^t\varphi^* - \mathrm{Id}} & H^k(N;\mathbb{R}) \end{array}$$

where the upper horizontal mapping is the connecting homomorphism of (41) *and the right vertical isomorphism is* (42). *Particularly* $H^*(M; F_{\rho_t})$ *vanishes iff* $P^k_\varphi(e^t) \neq 0$ *for all* k.

Elementray linear algebra provides the following

Lemma 3. *Suppose* $0 \to C^*_2 \to C^*_1 \to C^*_3 \to 0$ *is short exact sequence of chain compelxes. Then the following diagram of canonical isomorphisms commutes:*

$$\begin{array}{ccc} \det C^*_2 \otimes \det C^*_3 & = & \det C^*_1 \\ \| & & \| \\ \det H(C^*_2) \otimes \det H(C^*_3) & = & \det H(C^*_1) \end{array}$$

Here the left vertical isomorphism is the tensor product of the two canonical identifications and the lower horizontal isomorphism comes from the long exact sequence in cohomology.

Applying Lemma 3 to (41) and using Proposition 9 we get a commutative diagram:

$$\begin{array}{ccc}
\det C^*_\tau(M,N;F_{\rho_t}) \otimes \det C^*_\tau(N;F_{\rho_t}) & = & \det C^*_\tau(M;F_{\rho_t}) \\
\| & & \| \\
\det H(C^*_\tau(M,N;F_{\rho_t})) \otimes \det H(C^*_\tau(N;F_{\rho_t})) & = & \det H(C^*_\tau(M;F_{\rho_t})) \\
\| & & \| \\
\det H^{*-1}(N;\mathbb{R}) \otimes \det H^*(N;\mathbb{R}) & & \det H^*(M;F_{\rho_t}) \\
\| & & \| \\
\mathbb{R} & \xleftarrow{\zeta_\varphi(e^t)} & \mathbb{R}
\end{array}$$

provided $H^*(M;F_{\rho_t})$ vanishes, equivalently $P^k_\varphi(e^t) \neq 0$, for all k.
Let $|\cdot|$ denote the standard metric on $\mathbb{R}$, i.e. $|1| = 1$.

Lemma 4. *Via the canonic isomorphisms in the diagram above the metric* $|\cdot|$ *on the lower left* $\mathbb{R}$ *corresponds to the metric* $||\cdot||^{\text{comb}}_{F_{\rho_t},\tilde{\mathfrak{e}}}$ *on* $\det H^*(M;F_{\rho_t})$.

Proof. Let $||\cdot||_{\mathbb{Z}}$ denote the metric on $\det H^*(N;\mathbb{R})$ provided by integral cohomology. A simple inspection of the isomorphisms shows that the metric $||\cdot||_{\mathbb{Z}} \otimes ||\cdot||_{\mathbb{Z}}$ on $\det H^{*-1}(N;\mathbb{R}) \otimes \det H^*(N;\mathbb{R})$ corresponds to $||\cdot||^{\text{comb}}_{F_{\rho_t},\tilde{\mathfrak{e}}}$ on $\det H^*(M;F_{\rho_t})$. On the other hand $||\cdot||_{\mathbb{Z}} \otimes ||\cdot||_{\mathbb{Z}}$ clearly corresponds to $|\cdot|$ on $\mathbb{R}$. $\square$

Using Lemma 4, Proposition 8 and (39) we finally get

$$|\zeta_\varphi(e^t)| = ||1||^{\text{comb}}_{F_{\rho_t},\tilde{\mathfrak{e}}} = ||1||^{\text{comb}}_{F_{\rho_t},\mathfrak{e}} = T_{\text{an}}(\omega,g)(t)e^{-tR(X,g,\omega)}$$

which finishes the proof of Marcsik's theorem.

NOTE: The function $\zeta_\varphi(e^t)$ can be interpretd as the Laplace transform of the counting function for closed trajectories of the vector field $-\operatorname{grad}_g \omega$, cf [4]. In [4] one expresses this Laplace transform for vector fields of the form $X = -\operatorname{grad}_g \omega$ on an arbitrary manifold M when ω is a closed one form with Morse zeros. A generic closed ome form is of this type. This is a considerable extension of Marcsik theorem and the begining of a link between dynamics and spectral geometry.

8. Complex representations and the proof of Theorem 4

Complex representations. Let Γ be a finitely presented group with generators $g_1, \dots, g_r$ and relations $R_i(g_1, g_2, \dots, g_r) = e$, $i = 1, \dots, p$ and V be a complex vector space of dimension N. Let $\mathrm{Rep}(\Gamma; V)$ be the set of linear representations of Γ on V, i.e. group homomorphisms $\rho : \Gamma \to \mathrm{GL}_{\mathbb{C}}(V)$. By identifying V to $\mathbb{C}^N$ this set is in a natural way an affine algebraic variety inside the space $\mathbb{C}^{rN^2+1}$ given by $pN^2 + 1$ equations. Precisely if $A_1, \dots, A_r, z$ represent the coordinates in $\mathbb{C}^{rN^2+1}$ with $A := ||a^{ij}||$, $a^{ij} \in \mathbb{C}$ so $A \in \mathbb{C}^{N^2}$ and $z \in \mathbb{C}$, then the equations defining $\mathrm{Rep}(\Gamma; V)$ are

$$\begin{aligned} z \cdot \det(A_1) \cdot \det(A_2) \cdots \det(A_r) &= 1 \\ R_i(A_1, \dots, A_r) &= \mathrm{Id}, \qquad i = 1, \dots, p \end{aligned}$$

with each of the equalities R_i representing N^2 polynomial equations.

Define the function $d^M : \mathrm{Rep}(\Gamma; V) \to \mathbb{Z}$ by

$$d^M(\rho) := \sum_i \dim H^i(M; \rho)$$

and call a representation ρ generic if the function d^M is constant near ρ. The set of non-generic representations is a closed subset and a complex analytic subspace $\Sigma(M)$ of $\mathrm{Rep}(\Gamma; V)$. Moreover, for any connected component $\mathrm{Rep}_\alpha(\Gamma; V)$ of $\mathrm{Rep}(\Gamma; V)$, the set $\mathrm{Rep}_\alpha(\Gamma; V) \setminus \Sigma(M)$ is open and dense in $\mathrm{Rep}_\alpha(\Gamma; V)$. We denote by $\mathrm{Rep}^M(\Gamma; V)$ the union of connected components $\mathrm{Rep}_\alpha(\Gamma; V)$ which contain generic representations ρ with $d^M(\rho) = 0$. This is a complex analytic space which can be empty (for example if $\chi(M) \neq 0$) but in some interesting cases can be the full space $\mathrm{Rep}(\Gamma; V)$.

The following observations are supposed to be well known and tacitly used.

(i) Suppose (M, x_0) is a basepointed manifold and $\Gamma := \pi_1(M, x_0)$. Every representation $\rho \in \mathrm{Rep}(\Gamma; V)$ induces a vector bundle F_ρ equipped with a flat connection ∇_ρ. The fiber of this vector bundle above x_0 is canonically identified to V. They are obtained from the trivial bundle $\tilde{M} \times V \to \tilde{M}$ and the trivial connection by passing to the Γ quotient spaces. Here $\tilde{M}$ is the canonical universal covering provided by the base point x_0. The Γ action is the diagonal action of deck transformations on $\tilde{M}$ and the action ρ on V. The fiber of F_ρ over x_0 identifies canonically with V. The holonomy representation determines a right Γ action on the fiber of F_ρ over x_0, i.e. an anti homomorphism $\Gamma \to \mathrm{GL}(V)$. When composed with the inversion in $\mathrm{GL}(V)$ we get back the representation ρ.

(ii) Two representations in the same connected component of $\mathrm{Rep}(\Gamma; V)$ induce topologically isomorphic bundles, however not isomorphic as bundles with connection.

(iii) If ρ_0 is a representation in the connected component $\mathrm{Rep}_\alpha(\Gamma; V)$ one can identify $\mathrm{Rep}_\alpha(\Gamma; V)$ to the connected component of ∇_{ρ_0} in the complex analytic space of flat connections of the bundle F_{ρ_0} modulo the group of bundle isomorphisms of F_{ρ_0} which restrict to the identity on the fiber above x_0.

Proof of Theorem 4. Recall that the Ray–Singer torsion is a positive real number $T_{\mathrm{an}}(g, \nabla, \mu)$ associated to a Riemannian manifold (M, g) and a bundle $\xi : F \to M$ equipped with a flat connection ∇ and a Hermitian structure μ.

The connection ∇ gives rise to the deRham complex $(\Omega^*(M; F), d^*_\nabla)$ while the Riemannian metric g and the Hermitian structure μ give rise to a scalar product in $\Omega^*(M; F)$ which, in turn, provides formal adjoints $(d^q_\nabla)^\sharp$ for the differential operators d^q_∇. The Laplace–Beltrami operators $\Delta_q : \Omega^q(M; F) \to \Omega^q(M; F)$ are defined by $\Delta_q := d^{q-1}_\nabla \cdot (d^{q-1}_\nabla)^\sharp + (d^q_\nabla)^\sharp \cdot d^q_\nabla$. They are non-negative selfadjoint elliptic operators and therefore have regularized determinants $\det'\Delta_q$.[9] The analytic torsion $T_{\mathrm{an}}(g, \nabla, \mu)$ is defined by:

$$\log T_{\mathrm{an}}(\nabla, g, \mu) := \frac{1}{2} \sum_i (-1)^{i+1} i \log \det' \Delta_i \tag{43}$$

If $\mathfrak{e}^* \in \mathfrak{Eul}^*(M, x)$ is a co-Euler structure represented by (g, α) then the quantity

$$\log T^*_{\mathrm{an}}(\nabla, g, \alpha, \mu) := \log T_{\mathrm{an}}(\nabla, g, \mu) - \frac{1}{2} S(g, \alpha, \omega(\nabla, \mu)) \tag{44}$$

depends only on $\rho = \rho_\nabla$, the holonomy representation in the fiber F_{x_0}, and the co-Euler structure $\mathfrak{e}^*$, provided that $d^M(\rho) = 0$. This was already explained in section 6.

Indeed, by passing from (F, ∇, μ) to (F', ∇', μ') by a gauge transformation the right side of (44) does not change. In view of the Hermitian and metric anomalies for $\log T_{\mathrm{an}}(\nabla, g, \mu)$, cf. [1] or [3], and of the definition of $S(g, \alpha, \omega(\nabla, \mu))$, the change of μ, g and α (with $\mathfrak{e}^* = [g, \alpha]$) also leaves $T^*_{\mathrm{an}}(\nabla, g, \alpha, \mu)$ unchanged. Therefore $T^*_{\mathrm{an}}(\nabla, g, \alpha, \mu)$ defines

[9] $\det'$ denotes regularized determinant with zero modes ignored.

a smooth (actually analytic) real valued positive function

$$T_{\mathrm{an}}(M,\mathfrak{e}^*):\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)\to\mathbb{R}$$

referred to as the corrected Ray–Singer torsion.

We like to show that this function is the absolute value of a holomorphic function which admits a meromorphic extension to to $\mathrm{Rep}^M(\Gamma;V)$ and whose zeros and poles are contained in $\Sigma(M)$. This will prove Theorem 4.

Recall that a smooth triangulation τ of M, and an Euler structure $\mathfrak{e}=[X_\tau,c]$, where c is a collection of paths $\{\pi_{x_\sigma}\}$ from x_0 to x_σ, provide a cochain complex of finite dimensional vector spaces with scalar products $(C^*(\tau,\rho_\nabla),\delta_{\rho_\nabla})$. The q component of this complex is $\bigoplus_{\dim\sigma=q}F_{x_\sigma}$ where F_{x_σ} is equipped with the scalar product μ_{x_σ} obtained by parallel transport along π_{x_σ} from a fixed scalar product μ_{x_0} on F_{x_0}. For any representation ρ one obtains a real valued number $\log T_{\mathrm{comb}}(\rho,X_\tau,c,\mu_{x_0})$ by a similar formula as (43). This number depends only on ρ and $\mathfrak{e}$ provided $d^M(\rho)=0$, as it has already been indicated in section 6, cf. also [5]. Therefore one obtains a smooth (actually analytic) real valued positive function

$$T_{\mathrm{comb}}(M,\mathfrak{e}):\mathrm{Rep}^M(\Gamma,\mathbb{C}^N)\setminus\Sigma(M)\to\mathbb{R}.$$

If in addition $P(\mathfrak{e}^*)=\mathfrak{e}$ then Bismut–Zhang theorem (cf. Theorem 7 above) implies that the two functions are the same. Therefore it suffices to produce the holomorphic map and its meromorphic extension for the real valued function $T_{\mathrm{comb}}(M,\mathfrak{e})$ instead of $T_{\mathrm{an}}(M,\mathfrak{e}^*)$.

Remark 6. One can refine the definition of $T_{\mathrm{comb}}(M,\mathfrak{e})(\rho)$ for ρ with $H^*(M;\rho)=0$ in the following way.

One chooses a frame (basis) ϵ_{x_0} of F_{x_0}. One uses the parallel transport along π_{x_σ} (and obtain a frame in each F_{x_σ}) and an ordering $o_{\mathcal{X}}$ of the cells of τ, to provide a base for the acyclic complex of $\mathbb{C}$ vector spaces, $(C^*(\tau,\rho_\nabla),\delta_{\rho_\nabla})$. One uses the Milnor construction of the torsion, cf. [9], and obtains a complex number $\mathcal{T}^*_{\mathrm{comb}}(\rho,-X_\tau,c,\epsilon_{x_0},o_{\mathcal{X}})$. The absolute value of this number is the real number $T_{\mathrm{comb}}(M,\mathfrak{e})(\rho)$. The linear algebra considerations for this conclusion are detailed in the Appendix. By changing the frame in F_{x_0} this number does not change since the Euler characteristic of M vanishes.[10] As before one can argue that this complex numbers depends only on $\rho,\mathfrak{e}$ and $o_{\mathcal{X}}$, and therefore it will be denoted by $\mathcal{T}^*_{\mathrm{comb}}(\rho,\mathfrak{e},o_{\mathcal{X}})$.

[10]This follows from the fact that $H^*(M;\rho)=0$.

A change in the ordering $o_{\mathcal{X}}$ might change $\mathbb{T}_{\mathrm{comb}}(\rho, \mathfrak{e}, o_{\mathcal{X}})$, but only up to multiplication by ± 1 product of determinants of permutations. Let as also observe that:

(i) The graded vector space $C^*(\tau, \rho_\nabla)$ depends on ρ_∇ only via the underlying vector space V, hence can be written as $C^*(\tau, \rho_\nabla)$.

(ii) An ordering $o_{\mathcal{X}}$ and the frames F_{x_σ} define a base for the graded vector space $C^*(\tau; V)$ and then an element in the determinant line of this graded vector space. Orderings $o_{\mathcal{X}}$ which induce colinear elements in this determinant line provide the same complex number $\mathbb{T}^*_{\mathrm{comb}}(\rho, \mathfrak{e}, o_{\mathcal{X}})$.

(iii) A choice of an orientation o_H in $\det H^*(M; \mathbb{R})$ defines (many) orderings $o_{\mathcal{X}}$ of the cells of τ, and all these orderings provide colinear elements in that determinant line in view of the canonical identification of this determinant line with $\det H^*(M; \mathbb{R}) \otimes (\det V)^{-\chi(M)}$.

Let X be a connected component $\mathrm{Rep}^M_\alpha(\Gamma; V)$ of $\mathrm{Rep}^M(\Gamma; V)$ and $X' = \mathrm{Rep}^M_\alpha(\Gamma; V) \setminus \Sigma(M)$. Denote by $\mathcal{O}(X)$ and $\mathcal{O}(X')$ resp. by $\mathcal{M}(X)$ and $\mathcal{M}(X')$, the $\mathbb{C}$ algebra of holomorphic resp. meromorphic functions on X and X'.

We have the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{i_X} & \mathcal{M}(X) \\ \Big\downarrow r^{\mathcal{O}} & & r^{\mathcal{M}} \Big\downarrow \\ \mathcal{O}(X') & \xrightarrow{i_{X'}} & \mathcal{M}(X') \end{array}$$

where i_X resp. $i_{X'}$ denote the inclusions of the algebra of holomorphic functions in the field of meromorphic functions and $r^{\mathcal{O}}$ resp. $r^{\mathcal{M}}$ denote the restriction of the algebras of holomorphic/meromorphic functions on X to X'.

By choosing a frame in F_{x_0} and an ordering $o_{\mathcal{X}}$ consistent with the orientation o_H in $\det H^*(M; \mathbb{R})$, we obtain a base in the finite dimensional vector space $C^*(\tau; \rho)$ which varies holomorphically in ρ. Recall that the graded vector space $C^*(\tau; \rho)$ depends on ρ only via its underlying space V which is the same for all $\rho \in X$. Note also that when ρ varies in X one obtains a holomorphic family of based cochain complexes[11] of $\mathbb{C}$ vector spaces, which can be interpreted as a based cochain complex of $\mathcal{O}(X)$ modules. Unfortunately this cochain complex is not necessary acyclic. Our hypothesis implies that when tensored by $\mathcal{O}(X')$

[11]A based cochain complex of A modules is a cochain complex of free modules with a specified base for each component

and therefore also by $\mathcal{M}(X)$, our based cochain complex becomes acyclic therefore, has a well defined Milnor torsion, an element in $\mathcal{O}(X')$ resp. in $\mathcal{M}(X)$. The element in $\mathcal{M}(X)$ is the solution to our problem and is a meromorphic function $\mathcal{T}_{\text{comb}}(\mathfrak{e}, o_H) = \mathcal{T}_{\text{an}}(P^{-1}(\mathfrak{e}), o_H)$. The element in $\mathcal{O}(X')$ is a nonzero holomorphic function whose absolute value is $T_{\text{comb}}(M, \mathfrak{e})$. The commutativity of the diagram above implies that this holomorphic function is the restriction to X' of $\mathcal{T}_{\text{comb}}(\mathfrak{e}, o_H)$.

Note that by changing o_H one might change the meromorphic function $\mathcal{T}_{\text{an}}(\mathfrak{e}^*, o_H)$ but only up to sign. q.e.d.

Appendix A. Complex versus real torsion

Suppose V is a finite dimensional complex vector space. Let $V_{\mathbb{R}}$ denote the vector space V considered as real vector space. We have a mapping

$$\begin{aligned} \theta_V : \det V &\to \det(V_{\mathbb{R}}) \\ v_1 \wedge v_2 \wedge \cdots \wedge v_n &\mapsto v_1 \wedge iv_1 \wedge v_2 \wedge iv_2 \wedge \cdots \wedge v_n \wedge iv_n. \end{aligned}$$

It has the property

$$\theta_V(z\alpha) = |z|^2 \theta_V(\alpha),$$

for all $z \in \mathbb{C}$ and $\alpha \in \det V$. If $f : V \to W$ is a complex linear mapping then the following diagram commutes:

$$\begin{array}{ccc} \det V & \xrightarrow{\theta_V} & \det(V_{\mathbb{R}}) \\ {\scriptstyle \det f}\downarrow & & \downarrow{\scriptstyle \det(f_{\mathbb{R}})} \\ \det W & \xrightarrow{\theta_W} & \det(W_{\mathbb{R}}) \end{array}$$

After identifying $\det \mathbb{C} = \mathbb{C}$ and $\det(\mathbb{C}_{\mathbb{R}}) = \Lambda^2 \mathbb{R}^2 = \mathbb{R}$ we have

$$\theta_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}, \qquad \theta_{\mathbb{C}}(z) = |z|^2.$$

Suppose L is a complex line, R a real line and $\theta : L \to R$ a mapping which satisfies

$$\theta(z\lambda) = |z|^2 \theta(\lambda), \tag{45}$$

for all $z \in \mathbb{C}$ and all $\lambda \in L$. If L' is another complex line, R' another real line and $\theta' : L' \to R'$ another mapping which satisfies (45) we can define

$$\theta \otimes \theta' : L \otimes L' \to R \otimes R', \qquad (\theta \otimes \theta')(\lambda \otimes \lambda') := \theta(\lambda) \otimes \theta'(\lambda')$$

which again satisfies (45) Note that

$$\begin{array}{ccc} L\otimes\mathbb{C} & = & L \\ {\scriptstyle \theta\otimes\theta_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle \theta} \\ R\otimes\mathbb{R} & = & R \end{array}$$

commutes. If $0\to V\to W\to U\to 0$ is a short exact sequence of complex vector spaces we have a commutative diagram:

$$\begin{array}{ccc} \det V\otimes\det U & = & \det W \\ {\scriptstyle \theta_V\otimes\theta_U}\downarrow & & \downarrow{\scriptstyle \theta_W} \\ \det(V_{\mathbb{R}})\otimes\det(U_{\mathbb{R}}) & = & \det(W_{\mathbb{R}}) \end{array}$$

Note that for a complex vector space V we have a canonic isomorphism

$$(V^*)_{\mathbb{R}}=(V_{\mathbb{R}})^*,\qquad \varphi\mapsto\Re\circ\varphi.$$

Using this identification we get a commutative diagram:

$$\begin{array}{ccccc} \det V\otimes\det(V^*) & = & \det V\otimes(\det V)^* & = & \mathbb{C} \\ {\scriptstyle \theta_V\otimes\theta_{V^*}}\downarrow & & & & \downarrow{\scriptstyle \theta_{\mathbb{C}}} \\ \det(V_{\mathbb{R}})\otimes\det((V^*)_{\mathbb{R}}) & = & \det(V_{\mathbb{R}})\otimes(\det(V_{\mathbb{R}}))^* & = & \mathbb{R} \end{array}$$

Putting all this together we obtain

Proposition 10. *Let C^* be a finite dimensional chain complex over $\mathbb{C}$. Let $C^*_{\mathbb{R}}$ denote the same chain complex viewed as chain complex over $\mathbb{R}$. Clearly $H(C^*_{\mathbb{R}})=H(C^*)_{\mathbb{R}}$ and we have a commuatiove diagram:*

$$\begin{array}{ccc} \det C^* & = & \det H(C^*) \\ {\scriptstyle \theta_{C^*}}\downarrow & & \downarrow{\scriptstyle \theta_{H(C^*)}} \\ \det(C^*_{\mathbb{R}}) & = & \det H(C^*_{\mathbb{R}}) \end{array}$$

Now suppose F is a flat complex vector bundle over a closed manifold (M,x_0) with base point. Let $F_{\mathbb{R}}$ denote the vetor bundle F considered as real bundle. Recall the mappings (33) from section 6. Clearly $H^*(M;F)_{\mathbb{R}}=H^*(M;F_{\mathbb{R}})$. Let $A:=\theta_{H^*(M;F)}\otimes(\theta_{F_{x_0}})^{-\chi(M)}$ denote the canonical mapping:

$$\det H^*(M;F)\otimes(\det F_{x_0})^{-\chi(M)}\xrightarrow{A}\det H^*(M;F_{\mathbb{R}})\otimes(\det(F_{\mathbb{R}})_{x_0})^{-\chi(M)}$$

Let $\mathfrak{e}\in\mathfrak{Eul}(M,x_0)$ be an Euler structure and $\mathfrak{o}$ a cohomology orientation.

Proposition 11. *In this situation we have $A(\tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}})=\tau^{\mathrm{comb}}_{F_{\mathbb{R}},\mathfrak{e},\mathfrak{o}}$.*

Remark 7. Note that $\Theta_F \in H^1(M;\mathbb{C}^*)$ and $\Theta_{F_\mathbb{R}} \in H^1(M;\mathbb{R}^*)$ are related by $(|\cdot|^2)_*\Theta_F = \Theta_{F_\mathbb{R}}$, where $(|\cdot|^2)_* : H^1(M;\mathbb{C}^*) \to H^1(M;\mathbb{R}^*)$. Thus Proposition 11 implies

$$\begin{aligned}
\tau^{\text{comb}}_{F_\mathbb{R},\mathfrak{e}+\sigma,\mathfrak{o}} &= A(\tau^{\text{comb}}_{F,\mathfrak{e}+\sigma,\mathfrak{o}}) \\
&= A(\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}} \cdot \langle\Theta_F,\sigma\rangle^{-1}) \\
&= A(\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}) \cdot |\langle\Theta_F,\sigma\rangle|^{-2} \\
&= A(\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}) \cdot \langle\Theta_{F_\mathbb{R}},\sigma\rangle^{-1} \\
&= \tau^{\text{comb}}_{F_\mathbb{R},\mathfrak{e},\mathfrak{o}} \cdot \langle\Theta_{F_\mathbb{R}},\sigma\rangle^{-1}
\end{aligned}$$

as should. Also note that Proposition 11 implies

$$\tau^{\text{comb}}_{F_\mathbb{R},\mathfrak{e},-\mathfrak{o}} = A(\tau^{\text{comb}}_{F,\mathfrak{e},-\mathfrak{o}}) = A(\pm\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}) = A(\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}) = \tau^{\text{comb}}_{F_\mathbb{R},\mathfrak{e},\mathfrak{o}}$$

as should be since $F_\mathbb{R}$ has even rank.

Still assuming F is a flat complex vector bundle over (M, x_0). Let μ be a Hermitian structure on F and let $[g,\alpha] = \mathfrak{e}^*$ be a co-Euler structure. Using μ to construct adjoints, Laplacians, etc. we get a Hermitian scalar product $||\cdot||^{\text{an}}_{F,\mathfrak{e}^*}$ on the complex line

$$\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)},$$

which satisfies

$$||\cdot||^{\text{an}}_{F,\mathfrak{e}^*+\beta} = ||\cdot||^{\text{an}}_{F,\mathfrak{e}^*} \cdot |\langle\Theta_F, \text{PD}(\beta)\rangle|,$$

for all $\beta \in H^{n-1}(M;\mathcal{O}_M)$. The real part of μ defines a fiber metric $\mu\mathbb{R}$ on the flat real bundle $F_\mathbb{R}$. Using this fiber metric to compute $||\cdot||^{\text{an}}_{F_\mathbb{R},\mathfrak{e}^*}$ and using $\omega(F,\mu) = \omega(F_\mathbb{R},\mu_\mathbb{R})$ we immediatly get

Proposition 12. *In this situation* $||\cdot||^{\text{an}}_{F_\mathbb{R},\mathfrak{e}^*} \circ A = (||\cdot||^{\text{an}}_{F,\mathfrak{e}^*})^2$.

Remark 8. Note that Proposition 12 implies

$$\begin{aligned}
(||\cdot||^{\text{an}}_{F,\mathfrak{e}^*+\beta})^2 &= ||\cdot||^{\text{an}}_{F_\mathbb{R},\mathfrak{e}^*+\beta} \circ A \\
&= (||\cdot||^{\text{an}}_{F_\mathbb{R},\mathfrak{e}^*} \circ A) \cdot |\langle\Theta_{F_\mathbb{R}}, \text{PD}(\beta)\rangle| \\
&= (||\cdot||^{\text{an}}_{F_\mathbb{R},\mathfrak{e}^*} \circ A) \cdot |\langle\Theta_F, \text{PD}(\beta)\rangle|^2 \\
&= (||\cdot||^{\text{an}}_{F,\mathfrak{e}^*} \cdot |\langle\Theta_F, \text{PD}(\beta)\rangle|)^2
\end{aligned}$$

and thus $||\cdot||^{\text{an}}_{F,\mathfrak{e}^*+\beta} = ||\cdot||^{\text{an}}_{F,\mathfrak{e}^*} \cdot |\langle\Theta_F, \text{PD}(\beta)\rangle|$, as should be.

Together with the Bismut–Zhang theorem we obatain

Corollary 2. *In this situation we have:*

$$||\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}||^{\text{an}}_{F,\mathfrak{e}^*} = |\langle\Theta_F, \exp_*(\mathbb{T}(\mathfrak{e},\mathfrak{e}^*))\rangle|$$

Remark 9. It is tempting to look for an analytic defintion of an element $\tau^{\mathrm{an}}_{F,\mathfrak{e}^*,\mathfrak{o}}$ in the complex line $\det H^*(M;F)\otimes(\det F_{x_0})^{-\chi(M)}$ which satisfies

$$\tau^{\mathrm{an}}_{F,\mathfrak{e}^*+\beta,\mathfrak{o}} = \tau^{\mathrm{an}}_{F,\mathfrak{e}^*,\mathfrak{o}}\cdot\langle\Theta_F,\mathrm{PD}(\beta)\rangle^{-1}$$

and

$$\tau^{\mathrm{an}}_{F,\mathfrak{e}^*,-\mathfrak{o}} = (-1)^{\mathrm{rank}\,F}\tau^{\mathrm{an}}_{F,\mathfrak{e}^*,\mathfrak{o}}$$

and the Bismut–Zhang property

$$\tau^{\mathrm{an}}_{F,\mathfrak{e}^*,\mathfrak{o}} = \tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}}\cdot\langle\Theta_F,\exp_*(\mathbb{T}(\mathfrak{e},\mathfrak{e}^*))\rangle^{-1}.$$

References

[1] J.–M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque. **205**, Société Mathématique de France, 1992.
[2] D. Burghelea, *Removing Metric Anomalies from Ray–Singer Torsion*, Lett. Math. Phys. **47**(1999), 149–158.
[3] D. Burghelea, L. Friedlander and T. Kappeler, *Relative Torsion*, Commun. Contemp. Math. **3**(2001), 15–85.
[4] D. Burghelea and S. Haller, *Laplace transform, topology and spectral geometry of a closed one form. II*, ...
[5] M. Farber and V. Turaev, *Poincaré–Reidemeister metric, Euler structures and torsion*, J. Reine Angew. Math. **520**(2000), 195–225.
[6] M. Farber, *Singularities of the analytic torsion*, J. Differential Geom. **41**(1995), 528–572.
[7] J. Marcsik, ..., Thesis OSU **8**(1998).
[8] V. Mathai and D. Quillen, *Superconnections, Thom Classes, and Equivariant differential forms*, Topology **25**(1986), 85–110.
[9] J. Milnor, *Whitehead trosion*, Bull. Amer. Math. Soc. **72**(1966), 358–426.
[10] L. Nicolaescu, *The Reidemeister Torsion of 3–manifolds*, de Gruyter Studies in Mathematics **30**, Walter de Gruyter & Co., Berlin, 2003.
[11] V. Turaev, *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Math. USSR–Izv. **34**(1990), 627–662.
[12] G.W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics **61**, Springer Verlag, New York–Berlin, 1978.

Dan Burghelea, Dept. of Mathematics, The Ohio State University, 231 West Avenue, Columbus, OH 43210, USA.
E-mail address: burghele@mps.ohio-state.edu

Stefan Haller, Institute of Mathematics, University of Vienna, Strudlhofgasse 4, A-1090, Vienna, Austria.
E-mail address: Stefan.Haller@univie.ac.at