Lectures on orbifolds and reflection groups

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These are the notes for my lectures in the Summer School on Transformations Groups and Orbifolds held at the CMS of Zhejiang University in Hangzhou, China from June 30 to July 11, 2008. The notes closely follow the slides which I used to present my lectures.

Most of the material in the first four lectures comes from parts of Bill Thurston's 1976-77 course at Princeton University. Although this material has not been published, it can be found in [13] at the given electronic address.

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1 Lecture 1: transformation groups and orbifolds

1.1 Transformation groups

1.1.1 Definitions

An *action* of a topological group G on a space X is a (continuous) map $G \times X \to X$, denoted by $(g, x) \to gx$, so that

- g(hx)=(gh)x,
- 1x=x.

(Write $G \curvearrowright X$ to mean that G acts on X.)

Given $g \in G$, define $\theta_g : X \to X$ by $x \to gx$. Since $\theta_g \circ \theta_{g^{-1}} = 1_X = \theta_{g^{-1}} \circ \theta_g$, the map θ_g is a homeomorphism and the map $\Theta : G \to \mathrm{Homeo}(X)$ defined by $g \to \theta_g$ is a homomorphism of groups.

Given $x \in X$, $G_x := \{g \in G \mid gx = x\}$ is the *isotropy subgroup*. The action is *free* if $G_x = \{1\}$, for all $x \in X$.

Definitions 1.1. $G(x) := \{gx \in X \mid g \in G\}$ is the *orbit* of x. The action is *transitive* if there is only one orbit. Given $x \in X$, the natural map $G/G_x \to G(x)$ defined by $gG_x \to gx$ is a continuous bijection. The *orbit space* X/G is the set of orbits in X endowed with the quotient topology (with respect to the natural map $X \to X/G$). A map $f : X \to Y$ of G-spaces is *equivariant* (or a *G-map*) if $f(gx) = gf(x)$

Definitions 1.2. Suppose $H \subset G$ is a subgroup and Y is a H-space. Then H acts on $G \times Y$ via $h \cdot (g, x) = (gh^{-1}, hx)$. The orbit space is denoted $G \times_H Y$ and called the *twisted product*. The image of (g, x) in $G \times_H Y$ is denoted $[g, x]$. Note that $G \curvearrowright G \times_H Y$ via $g'[g, x] = [g'g, x]$.

Definition 1.3. A *slice* at a point $x \in X$ is a G_x-stable subset U_x so that the map $G \times_{G_x} U_x \to X$ is an equivariant homeomorphism onto a neighborhood of $G(x)$. If U_x is homeomorphic to a disk, then $G \times_{G_x} U_x$ is an *equivariant tubular neighborhood* of $G(x)$.

Remark 1.4. A neighborhood of the orbit in X/G is homeomorphic to U_x/G_x $(= (G \times_{G_x} U_x)/G)$.

1.1.2 The Differentiable Slice Theorem

The next result is basic in the study of smooth actions of compact Lie groups (including finite groups) on manifolds. For details, see [3].

Theorem 1.5. *Suppose a compact Lie group acts differentiably (= "smoothly") on a manifold M. Then every orbit has a G-invariant tubular neighborhood. More precisely, there is a linear representation of G_x on a vector space S so that that $G \times_{G_x} S$ is a tubular neighborhood of $G(x)$. (The image of S in M is a slice at x.)*

Proof. By integrating over the compact Lie group G we can find a G-invariant Riemannian metric. Then apply the usual proof using the exponential map. $\square$

1.1.3 Proper actions of discrete groups

Suppose Γ a discrete group, X a Hausdorff space and $\Gamma \curvearrowright X$. The Γ-acton is *proper* if given any two points $x, y \in X$, there are open neighborhoods U of x and V of y so that $\gamma U \cap V \neq \emptyset$ for only finitely many γ.

Exercise 1.6. Show that a Γ-action on X is proper iff

- X/Γ is Hausdorff,
- each isotropy subgroup is finite,
- each point $x \in X$ has a slice, i.e., there is Γ_x-stable open neighborhood U_x so that $\gamma U_x \cap U_x = \emptyset, \quad$ for all $\gamma \in \Gamma - \Gamma_x$. (This means that $\Gamma \times_{\Gamma_x} U_x$ maps homeomorphically onto a neighborhood of the orbit of x.)

Actions on manifolds. Suppose a discrete group Γ acts properly on an n-dimensionalt manifold M^n. A slice U_x at $x \in M^n$ is *linear* if there is a linear Γ_x-action on $\mathbf{R}^n$ so that U_x is Γ_x-equivariantly homeomorphic to a Γ_x-stable neighborhood of the origin in $\mathbf{R}^n$. The action is *locally linear* if every point has a linear slice.

Proposition 1.7. *If $\Gamma \curvearrowright M^n$ properly and differentiablly, then action is locally linear.*

Proof. Since Γ_x is finite, we can find a Γ_x-invariant Riemannian metric on M. The exponential map, $exp : T_xM \to M$, is Γ_x-equivariant and takes a small open disk about the origin homeomorphically onto a neighborhood U_x of x. If the disk is small enough, U_x is a slice. $\square$

1.2 Orbifolds

1.2.1 Definitions and terminology

Definition 1.8. An *orbifold chart* on a space X is a 4-tuple $(\widetilde{U}, G, U, \pi)$, where

- U is open subset of X,
- $\widetilde{U}$ is open in $\mathbf{R}^n$ and G is finite group of homeomorphisms of $\widetilde{U}$,

- $\pi : \widetilde{U} \to U$ is a map which can be factored as $\pi = \overline{\pi} \circ p$, where $p : \widetilde{U} \to \widetilde{U}/G$ is the orbit map and $\overline{\pi} : \widetilde{U}/G \to U$ is a homeomorphism.

The chart is *linear* if the G-action on $\mathbf{R}^n$ is linear.

For $i = 1, 2$, suppose $(\widetilde{U}_i, G_i, U_i, \pi_i)$ is an orbifold chart on X. The charts are *compatible* if given points $\tilde{u}_i \in \widetilde{U}_i$ with $\pi_1(\tilde{u}_1) = \pi_2(\tilde{u}_2)$, there is a homeomorphism h from neighborhood of $\tilde{u}_1$ in $\widetilde{U}_1$ onto neighborhood of $\tilde{u}_2$ in $\widetilde{U}_2$ so that $\pi_1 = \pi_2 \circ h$ on this neighborhood.

Definition 1.9. An *orbifold atlas* on X is a collection $\{(\widetilde{U}_i, G_i, U_i, \pi_i)\}_{\imath \in I}$ of compatible orbifold charts which cover X. An orbifold Q consists of an *underlying space* $|Q|$ together with an atlas of charts.

An orbifold is *smooth* if the groups act via diffeomorphisms and the charts are compatible via diffeomorphisms. A *locally linear* orbifold means all charts are equivalent to linear ones. By the Differentiable Slice Theorem a smooth orbifold is locally linear.

From now on, all orbifolds will be locally linear

Exercise 1.10. Suppose Γ acts properly on a manifold M^n. By choosing slices we can cover M/Γ with compatible orbifold charts. Show this gives the underlying space M/Γ the structure of orbifold, which we denote by $M/\!\!/\Gamma$.

Remark 1.11. (*Groupoids*). As Professors Adem and Xu said in their talks, the best way to view an orbifold is as a groupoid. This point was first made by Haefliger [9]. Given an atlas $\{(\widetilde{U}_i, G_i, U_i, \pi_i)\}_{\imath \in I}$ for an orbifold Q one associates a groupoid $\mathcal{G}$ to it as follows. The set of object $\mathcal{G}_0$ is the disjoint union:

$$\mathcal{G}_0 := \coprod_{i \in I} \widetilde{U}_i.$$

The set of morphisms $\mathcal{G}_1$ is defined as follows. Given $\tilde{u}_i \in \widetilde{U}_i$ and $\tilde{u}_j \in \widetilde{U}_j$, a morphism $\tilde{u}_i \to \tilde{u}_j$ is the germ of a local homeomorphism $\widetilde{U} \to \widetilde{V}$ from a neighborhood of $\tilde{u}_i$ to a neighborhood of $\tilde{u}_j$ which commutes with the projections, π_i and π_j. (Note: in the above we can take $i = j$ and f to be the germ of translation by a nontrivial element $\gamma \in G_i$.)

The local group There is more information in an orbifold than just its underlying space. For example, if $q \in |Q|$ and $x \in \pi^{-1}(q)$ is a point in the inverse image of q in some local chart, then the isotropy subgroup G_x is independent of the chart, up to an isomorphism of groups. With this ambiguity, we call it the *local group at q* and denote it by G_q.

A *manifold* is an orbifold in which each local group is trivial.

Strata. In transformation groups, if $G \curvearrowright X$ and $H \subset G$, then

$$X_{(H)} := \{x \in X \mid G_x \text{ is conjugate to } H\}$$

is the set of points of *orbit type* G/H. The image of $X_{(H)}$ in X/G is a *stratum* of X/G.

This image can be described as follows. First, take the fixed set X^H $(:= \{x \in X \mid hx = x, \forall h \in H\})$. Next, remove the points x with $G_x \supsetneq H$ to get $X^H_{(H)}$. Then divide by the free action of $N(H)/H$ to get $X^*_{(H)}$, the *stratum of type* (H) in X/G. In an orbifold, Q, a *stratum of type* (H) is the subspace of $|Q|$ consisting of all points with local group isomorphic to H.

Proposition 1.12. *If Q is a locally linear orbifold, then each stratum is a manifold.*

Proof. Suppose a finite group $G \curvearrowright \mathbf{R}^n$ linearly and $H \subset G$. Then $(\mathbf{R}^n)^H$ is a linear subspace; hence, $(\mathbf{R}^n)^H_{(H)}$ is a manifold. Dividing by the free action of $N(H)/H$, we see that $(\mathbf{R}^n)^*_{(H)}$ is a manifold. □

The origin of the word "orbifold": the true story. Near the beginning of his graduate course in 1976, Bill Thurston wanted to introduce a word to replace Satake's "V-manifold" from [12]. His first choice was "manifolded". This turned out not to work for talking - the word could not be distinguished from "manifold". His next idea was "foldimani". People didn't like this. So Bill said we would have an election after people made various suggestions for a new name for this concept. Chuck Giffen suggested "origam", Dennis Sullivan "spatial dollop" and Bill Browder "orbifold". There were many other suggestions. The election had several rounds with the names having the lowest number of votes being eliminated. Finally, there were only 4 names left, origam, orbifold, foldimani and one other (maybe "V-manifold"). After the next round of voting "orbifold" and the other name were to be eliminated. At this point, I spoke up and said something like "Wait you can't eliminate orbifold because the other two names are ridiculous." So "orbifold" was left on the list. After my impassioned speech, it won easily in the next round of voting.

1.2.2 Covering spaces and π_1^{orb}

Thurston's big improvement over Satake's earlier versionin [12] was to show that the theory of covering spaces and fundamental groups worked for orbifolds. (When I was a graduate student a few years before, this was "well-known" *not* to work.)

The local model for a covering projection between n-dimensional manifolds is the identity map, $id : U \to U$, on an open subset $U \subset \mathbf{R}^n$. Similarly, the local model for an *orbifold covering projection* is the natural map $\mathbf{R}^n/H \to \mathbf{R}^n/G$ where a finite group $G \curvearrowright \mathbf{R}^n$ and $H \subset G$ is a subgroup.

Proposition 1.13. *If* Γ *acts properly on*M *and* $\Gamma' \subset \Gamma$ *is a subgroup, then* $M /\!\!/ \Gamma' \to M /\!\!/ \Gamma$ *is an orbifold covering projection.*

Definition 1.14. An orbifold Q is *developable* if it is covered by a manifold. As we will see, this is equivalent to the condition that Q be the quotient of a discrete group acting properly on a manifold. (In Thurston's terminology, Q is a "good" orbifold.)

Remark 1.15. Not every orbifold is developable (later we will describe the "tear drop," the standard counterexample).

Definition 1.16. Q is *simply connected* if it is connected and does not admit a nontrivial orbifold covering, i.e., if $p : Q' \to Q$ is a covering with $|Q'|$ connected, then p is a homeomorphism.

Fact. Any connected orbifold Q admits a simply connected orbifold covering $\pi : \widetilde{Q} \to Q$. This has the usual universal property: if we pick a "generic" base point $q \in Q$ and $p : Q' \to Q$ is another covering with base points $q' \in Q'$ and $\tilde{q} \in \widetilde{Q}$ lying over q, then π factors through Q' via a covering projection $\widetilde{Q} \to Q'$ taking $\tilde{q}$ to q'. In particular, $\widetilde{Q} \to Q$ is a regular covering in the sense that its group of deck transformations acts simply transitively on $\pi^{-1}(q)$. (A simply transitive action is one which is both free and transitive.)

Definitions of the orbifold fundamental group.

Definition 1.17. (cf. [13]). $\pi_1^{orb}(Q)$ is the group of deck transformations of the universal orbifold cover, $p : \widetilde{Q} \to Q$

There are three other equivalent definitons of $\pi_1^{orb}(Q)$, which we list below. Each involves some technical difficulties.

- In Subsection 1.3, I will give a definition in terms of generators and relations.
- A third definition is in terms of "homotopy classes" of "loops" $[0, 1] \to Q$. The difficulty with this approach is that we must first define what is meant by a "map" from a topological space to Q - it should be a continuous map to $|Q|$ together with a choice of a "local lift" (up to equivalence) for each orbifold chart for Q.
- A fourth definition is in terms of the groupoid. If $\mathcal{G}_Q$ is the groupoid associated to Q and $B\mathcal{G}_Q$ is its classifying space, then $\pi_1^{orb}(Q) := \pi_1(B\mathcal{G}_Q)$, the ordinary fundamental group of the space $B\mathcal{G}_Q$. The only problem with this definition is that one first needs to define the classifying space of a groupoid.

Developability and the local group. For each $x \in |Q|$, let G_x denote the local group at x. (It is a finte subgroup of $GL(n, \mathbf{R})$, well-defined up to conjugation. We can identify G_x with the fundamental group of a neighborhood of the form $\widetilde{U}_x/G_x$ where $\widetilde{U}_x$ is a ball in some linear representation. So, G_x is the "local fundamental group" at x. The inclusion of the neighborhood induces a homomorphism $G_x \to \pi_1^{orb}(Q)$.

Proposition 1.18. *Q is developable $\iff$ each local group injects (i.e., for each $x \in |Q|$, the map $G_x \to \pi_1^{orb}(Q)$ is injective).*

1.2.3 1- and 2-dimensional orbifolds

Dimension 1. The only finite group which acts linearly (and effectively) on $\mathbf{R}^1$ is the cyclic group of order 2, C_2. It acts via the reflection $x \mapsto -x$. The orbit space $\mathbf{R}^1/C_2$ is identified with $[0, \infty)$.

It follows that every 1-dimensional orbifold Q is either a 1-manifold or a 1-manifold with boundary. If Q is compact and connected, then it is either a circle or an interval (say, $[0, 1]$).

The infinite dihedral group, D_∞ is the group generated by 2 distinct affine reflections on $\mathbf{R}^1$ and $\mathbf{R}^1/D_\infty \cong [0, 1]$. (See figure 1.) It follows that the universal orbifold cover of $[0, 1]$ is $\mathbf{R}^1$.

2**-dimensional linear groups**. Suppose a finite group $G \curvearrowright \mathbf{R}^n$ linearly. Then G is conjugate to subgroup of $O(n)$. (Pf: By averaging we get an invariant inner product). Hence, G acts on the unit sphere $S^{n-1} \subset \mathbf{R}^n$.

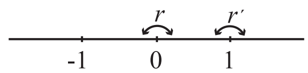
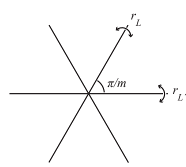


Figure 1: The infinite dihedral group

Suppose $G \subset O(2)$. Then $S^1/\!\!/G = S^1$ or $S^1/\!\!/G = [0,1]$.

- In the first case, $S^1 \to S^1/\!\!/G = S^1$ is an n-fold cover, where $n = |G|$, and G is the cyclic group C_n acting by rotations.
- In the second case, the composition, $\mathbf{R}^1 \to S^1 \to S^1/G = [0,1]$, is the universal orbifold cover with group of deck transformations D_∞. It follows that $G = D_m$ (the dihedral group of order $2m$) or $G = C_2$ $(= D_1)$ acting by reflection across a line.

Theorem 1.19. (Theorem of Leonardo da Vinci, cf. [14, pp. 66, 99].) *Any finite subgroup of $O(2)$ is conjugate to either C_n or D_m.*



Question. *What does $\mathbf{R}^2/\!\!/G$ look like?*

Here are the possibilities:

- $\mathbf{R}^2$ $\quad(G = \{1\})$,
- a cone $\quad(G = C_n)$,
- a half-space $\quad(G = D_1)$,
- a sector $\quad(G = D_m)$.

In the half-space case, a codimension 1 stratum is a *mirror*. In the sector case, a codimension 2 stratum is a *corner reflector*.

2-dimensional orbifolds. Here is the picture: the underlying space of a 2-dimensional orbifold Q is a 2-manifold, possibly with boundary . Certain

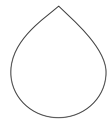


Figure 2: Not developable

points in the interior of the $|Q|$ are "cone points" labeled by an integer n_i specifying that the local group is C_{n_i}. The codimension 1 strata are the mirrors; their closures cover $\partial|Q|$. The closures of two mirrors intersect in a corner reflectors (where local group is D_{m_i}). The picture in Figure 2 is possible; however, it is not developable.

1.2.4 General orbifolds

- If $G \subset O(n)$ and $D^n \subset \mathbf{R}^n$ denotes the unit disk, then $G \curvearrowright D^n$.
- Since $D^n = \text{Cone}(S^{n-1})$, we have $D^n /\!\!/ G = \text{Cone}(S^{n-1} /\!\!/ G)$. Therefore, a point in a general orbifold has a conical neighborhood of this form.

Example 1.20. Suppose $G = C_2$ acting via antipodal map, $x \mapsto -x$. Then $D^n /\!\!/ C_2 = \text{Cone}(\mathbf{R}P^{n-1})$

Suppose Q is an n-dimensional orbifold and $Q_{(2)}$ denotes the complement of the strata of codimension > 2. The description of $Q_{(2)}$ is similar to a 2-dimensional orbifold. $|Q_{(2)}|$ is an n-manifold with boundary ; the boundary is a union of (closures of) mirrors; the codimension 2 strata in the interior are codimension 2 submanifolds labeled by cyclic groups; the codimension 2 strata on the boundary are corner reflectors labeled by dihedral groups.

During one of the problem sessions I was asked the following question.

Question. *When is the underlying space of an orbifold a manifold?*

This question is equivalent to the following.

Question. *For which finite subgroups $G \subset O(n)$ is $\mathbf{R}^n/G$ homeomorphic to $\mathbf{R}^n$.*

One example when this holds is when $G \subset U(n)$ is a finite subgroup generated by "complex reflections." (A *complex reflection* is a linear automorphism of $\mathbf{C}^n$ with only one eigenvalue $\neq 1$ i.e., it is a rotation about a complex hyperplane.) For any complex reflection group G, $\mathbf{C}^n/G \cong \mathbf{C}^n$. (This follows from the famous result that for such a G the ring of invariant polynomials $\mathbf{C}[x_1, \dots, x_n]^G$ is a polynomial ring on n variables.) Identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ we get $\mathbf{R}^{2n}/G \cong \mathbf{R}^{2n}$. Another case where the answer to the question is affirmative is when G is the orientation-preserving subgroup of a finite group W generated by (real) reflections on $\mathbf{R}^n$. We will see in Corollary 3.4 that $\mathbf{R}^n /\!\!/ W$ is a simplicial cone (which is homeomorphic to a half space). The orbifold $\mathbf{R}^n /\!\!/ G$ is the "double" discussed in Example 1.21 below. Hence, in this case we also have $\mathbf{R}^n ? G \cong \mathbf{R}^n$

After making these comments, I made the following conjecture. [1]

Conjecture. $\mathbf{R}^n/G$ *is homeomorphic to* $\mathbf{R}^n$ *if and only if either*

(i) $n = 2m$ *and* G *is complex reflection group on* $\mathbf{C}^m$, *or*

(ii) G *is the orientation-preserving subgroup of a real reflection group on* $\mathbf{R}^n$.

Examples of orbifold coverings. Suppose $X \to |Q|$ is an ordinary covering of topological spaces. Pullback the strata of Q to strata in X to obtain an orbifold Q'. (Here is a specific example: Q is $\mathbf{R}P^2$ with one cone point labeled n. $S^2 \to \mathbf{R}P^2$ is the double cover. The single cone point pulls back to two cone points in S^2 labeled n.)

Example 1.21. Double $|Q|$ along its boundary to get a 2-fold orbifold covering $Q' \to Q$ without codimension 1 strata. For example, if Q is a triangle, then Q' is a 2-sphere with three cone points. As another example, if Q is the nondevelopable orbifold pictured on the previous page (a 2-disk with one corner reflector), then Q' is the tear drop (a 2-sphere with one cone point).

Example 1.22. The n-fold branched cover of Q along a codimension 2 stratum labeled by the cyclic group of order n.

[1] There is an obvious counterexample to this conjecture: let $G \subset SU(2)$ be the binary dodecahedral group of order 120. Then G acts freely on S^3 and S^3/G is Poincaré's homology 3-sphere. If we take the product of this representation with the trivial 1-dimensional representation we obtain a representation on $\mathbf{R}^5$ such that S^4/G is the suspension of S^3/G. It then follows from the Double Suspension Theorem of Cannon that $\mathbf{R}^5/G$ is homeomorphic to $\mathbf{R}^5$. The correct conjecture should be that this is the only counterexample.

1.3 Generators and relations for $\pi_1^{orb}(Q)$

Remark 1.23. $\pi_1^{orb}(Q) = \pi_1^{orb}(Q_{(2)})$. (*Proof*: general position.)

Let $\hat{Q}$ denote the complement in $|Q|$ of the strata of codimension ≥ 2 (retain the mirrors on $\partial|Q|$). Choose a base point x_0 in interior $\hat{Q}$. We are going to construct $\pi_1^{orb}(Q, x_0)$ from $\pi_1(\hat{Q}, x_0)$ by adding generators and relations.

New generators.

- For each component T of a codimension 2 stratum in interior of $|Q|$, choose a loop α_T starting at x_0 which makes a small loop around T. Let $n(T)$ be the order of the cyclic group labeling T.
- Suppose P is a codimension 2 stratum contained in $M \cap N$ (so that P is a corner reflector). Let $m(P)$ be the label on P (so that the dihedral group at P has order $2m(P)$).
- For each mirror M and each homotopy class of paths γ_M from x_0 to M introduce a new generator $\beta_{(M,\gamma_M)}$.

Relations.

- $[\alpha_T]^{n(T)} = 1$,
- $[\beta_{(M,\gamma_M)}]^2 = 1$, and
- $([\beta_{(M,\gamma_M)}][\beta_{(N,\gamma_N)}])^{m(P)} = 1$,

Here P is a component of $\overline{M} \cap \overline{N}$ and γ_M and γ_N are homotopic as paths from x_0 to P.

2 Lecture 2: two-dimensional orbifolds

2.1 Orbifold Euler characteristics

We know what is meant by the Euler characteristic of a closed manifold or finite CW complex (the alternating sum of the number of cells). A key property is that it is multiplicative under finite covers: if $M' \to M$ is an m-fold cover, then

$$\chi(M') = m\,\chi(M).$$

The Euler characteristic of an orbifold should be a rational number with same multiplicative property, i.e., if $M \to Q$ is an m-fold cover and M is a manifold, then we should have $\chi(M) = m\,\chi^{orb}(Q)$, i.e.,

$$\chi^{orb}(Q) = \frac{1}{m}\,\chi(M).$$

("m-fold cover" means $\mathrm{Card}(p^{-1}(\text{generic point})) = m$.)

The Euler characteristic of an orbifold. [2] Suppose Q is an orbifold which is cellulated as a CW complex so that the local group is constant on each open cell c. Let $G(c)$ be the local group on c and $|G(c)|$ denote its order. Define

$$\chi^{orb}(Q) := \sum_{\text{cells } c} \frac{(-1)^{\dim c}}{|G(c)|}.$$

Exercise 2.1. Suppose $\Gamma \curvearrowright M$ properly, cocompactly, locally linearly and $\Gamma' \subset \Gamma$ is a subgroup of index m. Show

$$\chi^{orb}(M/\!\!/\Gamma') = m\,\chi^{orb}(M/\!\!/\Gamma).$$

Alternate formula. Each stratum S of a compact orbifold Q is the interior of a compact manifold with boundary $\hat{S}$. Define $e(S) := \chi(\hat{S}) - \chi(\partial\hat{S})$. Then

$$\chi^{orb}(Q) = \sum_{\text{strata } S} \frac{e(S)}{|G(S)|}$$

Example 2.2. Suppose $|Q| = D^2$ and Q has k mirrors and k corner reflectors labeled $m_1, \dots, m_k$. Then

$$\chi^{orb}(Q) = 1 - \frac{k}{2} + \left(\frac{1}{2m_1} + \cdots + \frac{1}{2m_k}\right) = 1 - \frac{1}{2}\sum_i \left(1 - \frac{1}{m_i}\right).$$

Example 2.3. Suppose $|Q| = S^2$ and Q has l cone points labeled $n_1, \dots, n_l$. Then

$$\chi^{orb}(Q) = 2 - l + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_l}\right) = 2 - \sum_i \left(1 - \frac{1}{n_i}\right)$$

(This is twice the previous example, as it should be.)

[2] In his lectures, Alejandro Adem gave a completely different definition of the "orbifold Euler number, $\chi_{orb}(Q)$". For him, it is a certain integer which is defined using equivariant K-theory. Although this definition has been pushed by string theorists, the rational number which I am using this terminology for goes back to Thurston's 1976 course and before that to Satake.

Example 2.4. (*The general formula*). Suppose $|Q|$ is a surface with boundary and that Q has k corner reflectors labeled $m_1, \dots, m_k$ and l cone points labeled $n_1, \dots, n_l$. Then

$$\chi^{orb}(Q) = \chi(|Q|) - \frac{1}{2}\sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) - \sum_{i=1}^{l}\left(1 - \frac{1}{n_i}\right).$$

Remark 2.5. This formula shows that $\chi^{orb}(Q) \leq \chi(|Q|)$ with equality iff there are no cone points or corner reflectors.

Notation 2.6. If a 2-dimensional orbifold has k corner reflectors which are labeled $m_1, \dots, m_k$ and l cone points labeled $n_1, \dots, n_l$, we will denote this by

$$(n_1, \dots, n_l; m_1, \dots, m_k).$$

If $\partial|Q| = \emptyset$, then there can be no mirrors or corner reflectors and we simply write $(n_1, \dots, n_l)$.

2.2 Classification of 2-dimensional orbifolds

Recall that closed surfaces are classified by orientability and Euler characteristic:

- $\chi(M^2) > 0 \implies M^2 = S^2$ or $\mathbf{R}P^2$ (positive curvature).
- $\chi(M^2) = 0 \implies M^2 = T^2$ or the Klein bottle (flat).
- $\chi(M^2) < 0 \implies$ arbitrary genus > 1 (negative curvature).

The idea is to classify orbifolds Q^2 by their Euler characteristics. Since $\chi^{orb}(\ )$ is multiplicative under finite covers, this will tell us which manifolds can finitely cover a given orbifold. For example, if $Q = S^2 /\!\!/ G$, with G finite, then $\chi^{orb}(S^2 /\!\!/ G) > 0$. Conversely, if Q is developable and $\chi^{orb}(Q) > 0$, then its universal cover is S^2.

Exercise 2.7. List the 2-dimensional orbifolds Q with $\chi^{orb}(Q) \geq 0$. (In fact, I will do this exercise below.)

Sample calculation. Suppose $|Q| = D^2$ with $(\ ; m_1, \dots, m_k)$. Recall

$$\chi^{orb}(Q) = 1 - \frac{1}{2}\sum_{i=1}^{k}(1 - (m_i)^{-1}).$$

Since $1 - (m_i)^{-1} \geq 1/2$, we see that if $k \geq 4$, then $\chi^{orb}(Q) \leq 0$ with equality iff $k = 4$ and all $m_i = 2$. Hence, if $\chi^{orb}(Q) > 0$, then $k \leq 3$.

More calculations. Suppose $|Q| = D^2$ and $k = 3$ (so that Q is a triangle). Then

$$\chi^{orb}(Q) = \frac{1}{2}(-1 + (m_1)^{-1} + (m_2)^{-1} + (m_3)^{-1})$$

So, as $(\pi/m_1 + \pi/m_2 + \pi/m_3)$ is $>$, $=$ or $< \pi$, $\chi^{orb}(Q)$ is, respectively, $>$, $=$ or < 0. For $\chi^{orb} > 0$, we see the only possibilities are: $(\ ; 2, 2, m)$, $(\ ; 2, 3, 3)$, $(\ ; 2, 3, 4)$, $(\ ; 2, 3, 5)$. The last three correspond to the symmetry groups of the Platonic solids. For $\chi^{orb}(Q) = 0$, the only possibilities are: $(\ ; 2, 3, 6)$, $(\ ; 2, 4, 4)$ $(\ ; 3, 3, 3)$.

Making use of Remark 2.5, we do Exercise 2.7 below.

$\boldsymbol{\chi^{orb}(Q) > 0}$**:**

- Nondevelopable orbifolds:
    - $|Q| = D^2$: $(\ ; m)$, $(\ ; m_1, m_2)$ with $m_1 \neq m_2$.
    - $|Q| = S^2$: (n), (n_1, n_2) with $n_1 \neq n_2$.
- Spherical orbifolds:
    - $|Q| = D^2$: $(\ ;\ )$, $(\ ; m, m)$, $(\ ; 2, 2, m)$, $(\ ; 2, 3, 3)$, $(\ ; 2, 3, 4)$, $(\ ; 2, 3, 5)$, $(2; m)$, $(3; 2)$.
    - $|Q| = S^2$: $(\ )$, (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$.
    - $|Q| = \mathbf{R}P^2$: $(\ )$, (n)

Implications for 3-dimensional orbifolds.

- The list of 2-dimensional spherical orbifolds is the list of finite subgroups of $O(3)$.
- Every 3-dimensional orbifold is locally isomorphic to the cone on one of the spherical 2-orbifolds.
- For example, if $|Q| = S^2$ with three cone points, (n_1, n_2, n_3), then Cone(Q) has underlying space an open 3-disk. The three cone points yield three codimension 2 strata labeled m_1, m_2, m_3 and the origin is labeled by the corresponding fintie subgroup of $O(3)$.

Flat orbifolds: $\chi^{orb}(Q) = 0$: the 17 wallpaper groups.

- $|Q| = D^2$: $(\ ; 2, 3, 6)$, $(\ ; 2, 4, 4)$, $(\ ; 3, 3, 3)$, $(\ ; 2, 2, 2, 2)$, $(2; 2, 2)$, $(3; 3)$, $(4; 2)$, $(2, 2;\ )$.
- $|Q| = S^2$: $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$, $(2, 2, 2, 2)$, $()$, $()$, .
- $|Q| = \mathbf{R}P^2$: $(2, 2)$,
- $|Q| = T^2$: $(\ )$.
- $|Q| =$ Klein bottle: $(\ )$.
- $|Q| =$ annulus: $(\ ;\ )$.
- $|Q| =$ Möbius band: $(\ ;\ )$.

Remark. In [14, pp. 103-115], Weyl emphasized the fact that there are exactly 17 discrete, cocompact subgroups of $\mathrm{Isom}(\mathbb{E}^2)$ up to conjugation in the group of affine automorphisms. These 17 "wallpaper groups" are exactly the orbifold fundamental groups of the orbifolds listed above.

$\chi^{orb}(Q) < 0$: It turns out that all remaining 2-dimensional orbifolds are developable and can be given a hyperbolic structure.

The triangular orbifolds, i.e., $|Q| = D^2$; $(\ ; m_1, m_2, m_3)$, with $(m_1)^{-1} + (m_2)^{-1} + (m_3)^{-1} < 1$, have a unique hyperbolic structure (because hyperbolic triangles are determined, up to congruence, by their angles). The others have a positive-dimensional moduli space.

2.3 Spaces of constant curvature

In each dimension n, there are three simply connected spaces of constant curvature: $\mathbb{S}^n$ (the sphere), $\mathbb{E}^n$ (Euclidean space) and $\mathbb{H}^n$ (hyperbolic space).

Definition 2.8. (*Minkowski space*). Let $\mathbf{R}^{n,1}$ denote $\mathbf{R}^{n+1}$ equipped with the indefinite symmetric bilinear form:

$$\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$

Definition 2.9. The hypersurface defined by $\langle x, x\rangle = -1$ is a hyperboloid of two sheets. The component with $x_{n+1} > 0$ is $\mathbb{H}^n$.

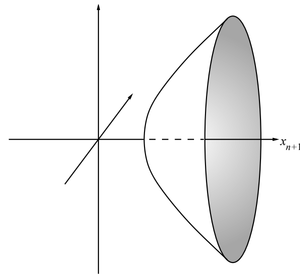


Figure 3: The quadratic form model of the hyperbolic plane

Definition 2.10. (*The Riemannian metric on* $\mathbb{H}^n$). As in the case of a sphere, given $x \in \mathbb{H}^n$, $T_x\mathbb{H}^n = x^\perp$. Since $\langle x, x\rangle < 0$, the restriction of $\langle\ ,\ \rangle$ to $T_x\mathbb{H}^n$ is positive definite. So, this defines a Riemannian metric on $\mathbb{H}^n$. It turns out this metric has constant secional curvature -1.

Geometric structures on orbifolds. Suppose G is a group of isometries acting real analytically on a manifold $\mathbb{X}$. (The only examples we will be concerned with are $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ and G the full isometry group.) By a $(G, \mathbb{X})$-*structure* we mean that each of the charts $(\widetilde{U}, H, U, \pi)$ has $\widetilde{U} \subset \mathbb{X}$, that H is a finite subgp of G and the overlap maps (= compatibility maps) are required to be restrictions of isometries in G.

Convex polytopes in $\mathbb{X}^n$. A *hyperplane* or *half-space* in $\mathbb{S}^n$ or $\mathbb{H}^n$ is the intersection of a linear hyperplane or half-space with the hypersurface. The *unit normal vector* u to a hyperplane means that the hyperplane is the orthogonal complement, $u^\perp$, of u (orthogonal wiith respect to the standard bilinear form, in the case of $\mathbb{S}^n$, or the form $\langle\ ,\ \rangle$, in the case of $\mathbb{H}^n$). A *half-space* in $\mathbb{H}^n$ bounded by the hyperplane $u^\perp$ is a set of the form $\{x \in \mathbb{H}^n \mid \langle u, x\rangle \geq 0\}$ and similarly, for $\mathbb{S}^n$. A convex polytope in $\mathbb{S}^n$ or $\mathbb{H}^n$ is a compact intersection of a finite number of half-spaces.

Reflections in $\mathbb{S}^n$ **and** $\mathbb{H}^n$. Suppose u is unit vector in $\mathbf{R}^{n+1}$. Reflection across the hyperplane $u^\perp$ (either in $\mathbf{R}^{n+1}$ or $\mathbb{S}^n$) is given by

$$x \mapsto x - 2(x \cdot u)u.$$

Similarly, suppose $u \in \mathbf{R}^{n,1}$ satisfies $\langle u, u\rangle = 1$. Reflection across the hyperplane $u^\perp$ in $\mathbb{H}^n$ is given by $x \mapsto x - 2\langle x, u\rangle u$.

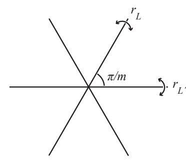


Figure 4: The dihedral group of order 6

3 Lecture 3: reflection groups

3.1 Geometric reflection groups

Suppose K is a convex polytope in $\mathbb{X}^n$ ($= \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$) such that if two codimension 1 faces have nonempty intersection, then the dihedral angle between them has form π/m for some integer $m \geq 2$. (This condition is familiar: it means that each codimension 2 face has the structure of a codimension 2 corner reflector.) Let W be the subgroup of $\mathrm{Isom}(\mathbb{X}^n)$ generated by reflections across the codimension 1 faces of K.

Some basic facts:

- W is discrete and acts properly on $\mathbb{X}^n$.
- K is a strict fundamental domain in the sense that the restriction to K of the orbit map, $p : \mathbb{X}^n \to \mathbb{X}^n/W$, is a homeomorphism. It follows that $\mathbb{X}^n /\!/ W \cong K$ and hence, K can be given the structure of an orbifold with an $\mathbb{X}^n$-structure.

(Neither fact is obvious.)

Example 3.1. A *dihedral group* is any group which is generated by two involutions, call them s, t. It is determined up to isomorphism by the order m of st (m is an integer ≥ 2 or the symbol ∞). Let $\mathbf{D}_m$ denote the dihedral group corresponding to m.

Example 3.2. For $m \neq \infty$, $\mathbf{D}_m$ can be represented as the subgroup of $O(2)$ which is generated by reflections across lines L, L', making an angle of π/m. (See Figure 4.)

History and properties. [3]

[3]In this paragraph I have relied on the Historical Note of [2, pp. 249-257].

- In 1852 Möbius determined the finite subgroups of $O(3)$ generated by isometric reflections on the 2-sphere.

- The fundamental domain for such a group on the 2-sphere was a spherical triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, with p, q, r integers ≥ 2.

- Since the sum of the angles is $> \pi$, we have $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$.

- For $p \geq q \geq r$, the only possibilities are: $(p, 2, 2)$, for $p \geq 2$, and $(p, 3, 2)$ with $p = 3$, 4 or 5. (The last three cases are the symmetry groups of the Platonic solids.)

- Later work by Riemann and Schwarz showed there were discrete groups of isometries of $\mathbb{E}^2$ or $\mathbb{H}^2$ generated by reflections across the edges of triangles with angles integral submultiples of π. Poincaré and Klein proved similar results for polygons with more than three sides in $\mathbb{H}^2$.

In 2^{nd} half of the 19^{th} century work began on finite reflection groups on $\mathbb{S}^n$, $n > 2$, generalizing Möbius' results for $n = 2$. It developed along two lines.

- Around 1850, Schläfli classified regular polytopes in $\mathbf{R}^{n+1}$, $n > 2$. The symmetry group of such a polytope was a finite group generated by reflections and as in Möbius' case, the projection of a fundamental domain to S^n was a spherical simplex with dihedral angles integral submultiples of π.

- Around 1890, Killing and E. Cartan classified complex semisimple Lie algebras in terms of their root systems. In 1925, Weyl showed the symmetry group of such a root system was a finite reflection group.

- These two lines were united by Coxeter [4] in the 1930's. He classified discrete groups reflection groups on $\mathbb{S}^n$ or $\mathbb{E}^n$.

Let K be a fundamental polytope for a geometric reflection group. For $\mathbb{S}^n$, K is a simplex. For $\mathbb{E}^n$, K is a product of simplices. For $\mathbb{H}^n$ there are other possibilities, e.g., a right-angled pentagon in $\mathbb{H}^2$ (see Figure [**?**]) or a right-angled dodecahedron in $\mathbb{H}^3$.

- Conversely, given a convex polytope K in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ so that all dihedral angles have form π/integer, there is a discrete group W generated by isometric reflections across the codimension 1 faces of K.

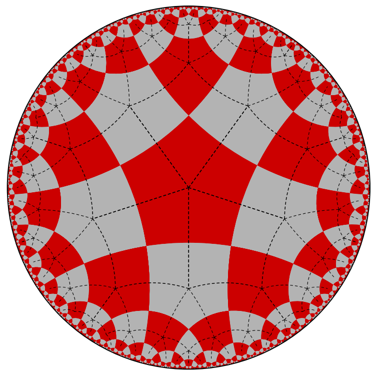


Figure 5: Tessellation of hyperbolic plane by right-angled pentagons

- Let S be the set of reflections across the codimension 1 faces of K. For $s, t \in S$, let $m(s,t)$ be the order of st. Then S generates W, the faces corresponding to s and t intersect in a codimension 2 face iff $m(s,t) \neq \infty$, and for $s \neq t$, the dihedral angle along that face is $\pi/m(s,t)$. Moreover,
- If $m(,t) = 1$ for $s = t$ and is as defined above for $s \neq t$, then

$$\langle S \mid (st)^{m(s,t)}, \quad \text{where } (s,t) \in S \times S \rangle$$

is a presentation for W.

Polytopes with nonobtuse dihedral angles.

Lemma 3.3. (Coxeter, [4]). *Suppose $K \subset \mathbb{S}^n$ is an n-dimenional convex polytope which is "proper" (meaning that it does not contain any pair of antipodal points). Further suppose that whenever two codimension 1 faces intersect along a codimension 2 face, the dihedral angle is $\leq \pi/2$. Then K is a simplex.*

A similar result holds for a polytope $K \subset \mathbb{E}^n$ which is not a product.

Corollary 3.4. *The fundamental polytope for a spherical reflection group is a simplex.*

Proof. For m an integer ≥ 2, we have $\pi/m \leq \pi/2$. $\square$

3.2 Simplicial Coxeter groups

3.2.1 The Gram matrix of a simplex in $\mathbb{X}^n$

Suppose σ^n is a simplex in $\mathbb{X}^n$. Let $u_0, \dots u_n$ be its inward pointing unit normal vectors. (The u_i lie in $\mathbf{R}^{n+1}$, $\mathbf{R}^n$ or $\mathbf{R}^{n,1}$ as $\mathbb{X}^n = \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.) The *Gram matrix*, G, of σ is the symmetric $(n+1) \times (n+1)$ matrix (g_{ij}) defined by $g_{ij} = u_i \cdot u_j$. $G > 0$ means the symmetric matrix G is positive definite.

Definition 3.5. A symmetric matrix G with 1's on the diagonal is type

(1) if $G > 0$,

(0) if G is positive semidefinite with 1-dimensional kernel, each principal submatrix is > 0, and there is a vector $v \in \operatorname{Ker} G$ with all its coordinates > 0,

(-1) if G has signature $(n, 1)$ and each principal submatrix is > 0.

Linear algebra fact. The extra condition in type 0 (that $\operatorname{Ker} G$ is spanned by a vector with positive coordinates) is automatic when G is indecomposable and has $g_{ij} \leq 0$, for all $i \neq j$, i.e., when all dihedral angles are nonobtuse. (See [6, Lemma 6.3.7].)

Theorem 3.6. *Suppose G is a symmetric $(n+1) \times (n+1)$ matrix with 1's on the diagonal. Let $\varepsilon \in \{+1, 0, -1\}$. Then G is the Gram matrix of a simplex $\sigma^n \subset \mathbb{X}^n_\varepsilon \iff G$ is type ε.*

Let $\mathbb{X}^n_\varepsilon$ is $\mathbb{S}^n$, $\mathbb{E}^n$, $\mathbb{H}^n$ as $\varepsilon = +1,\ 0,\ -1$.

Proof. For $\mathbb{S}^n$: we can find basis vectors $u_0, \dots u_n$ in $\mathbf{R}^{n+1}$, well-defined up to isometry, so that $(u_i \cdot u_j) = G$. (This is because $G > 0$.) Since the u_i form a basis, the half-spaces, $u_i \cdot x \geq 0$, intersect in a simplicial cone and the intersection of this with $\mathbb{S}^n$ is σ^n.

The proof for $\mathbb{H}^n$ is similar. The argument for $\mathbb{E}^n$ has additional complications. □

Suppose $\sigma^n \subset \mathbb{X}^n$ is a fundamental simplex for a geometric reflecton group. Let $\{u_0, \dots u_n\}$ be the set of inward-pointing unit normal vectors. Then

$$u_i \cdot u_j = -\cos(\pi/m_{ij})$$

where (m_{ij}) is a symmetric matrix of posiive integers with 1's on the diagonal and all off-diagonal entries ≥ 2. (The matrix (m_{ij}) is called the *Coxeter matrix* while the matrix $(\cos(\pi/m_{ij}))$ is the associated *cosine matrix*.) The formula above says:

$$\text{Gram matrix} = \text{cosine matrix}.$$

Suppose $M = (m_{ij})$ is a Coxeter matrix, i.e., a symmetric $(n+1) \times (n+1)$ matrix with 1's on the diagonal and with off-diagonals ≥ 2 (sometimes we allow the off-diagonal m_{ij} to $= \infty$, but not here).

Theorem 3.7. *Let M be a Coxeter matrix as above and C its associated cosine matrix (i.e., $c_{ij} = -\cos(\pi/m_{ij})$). Then there is a geometric refllection group with fundamental simplex $\sigma^n \subset \mathbb{X}^n_\varepsilon \iff C$ is type ε.*

So, the problem of determining the geometric reflection groups with fundamental polytope a simplex in $\mathbb{X}^n_\varepsilon$ becomes the problem of determining the Coxeter matrices M whose cosine matrix is type ε. This was done by Coxeter, [4], for $\varepsilon = 1$ or 0 and by Lannér, [10], for $\varepsilon = -1$. The information in a Coxeter matrix is best encoded by its "Coxeter diagram."

3.2.2 Coxeter diagrams

Associated to (W, S), there is a labeled graph Γ called its "Coxeter diagram." Put $\mathrm{Vert}(\Gamma) := S$. Connect distinct elements s, t by an edge iff $m(s,t) \neq 2$. Label the edge by $m(s,t)$ if this is > 3 or $= \infty$ and leave it unlabeled if it is $= 3$.

(W, S) is *irreducible* if Γ is connected. (The components of Γ give the irreducible factors of W.)

Figure 6 shows Coxeter's classification from [4] of the irreducible spherical and cocompact Euclidean reflection groups. Figure 7 shows Lannér's classification from [10] of the hyperbolic reflection groups with fundamental polytope a simplex in $\mathbb{H}^n$.

Exercise 3.8. Derive Lannér's list in Figure 7 from Coxeter's lists in Figure 6.

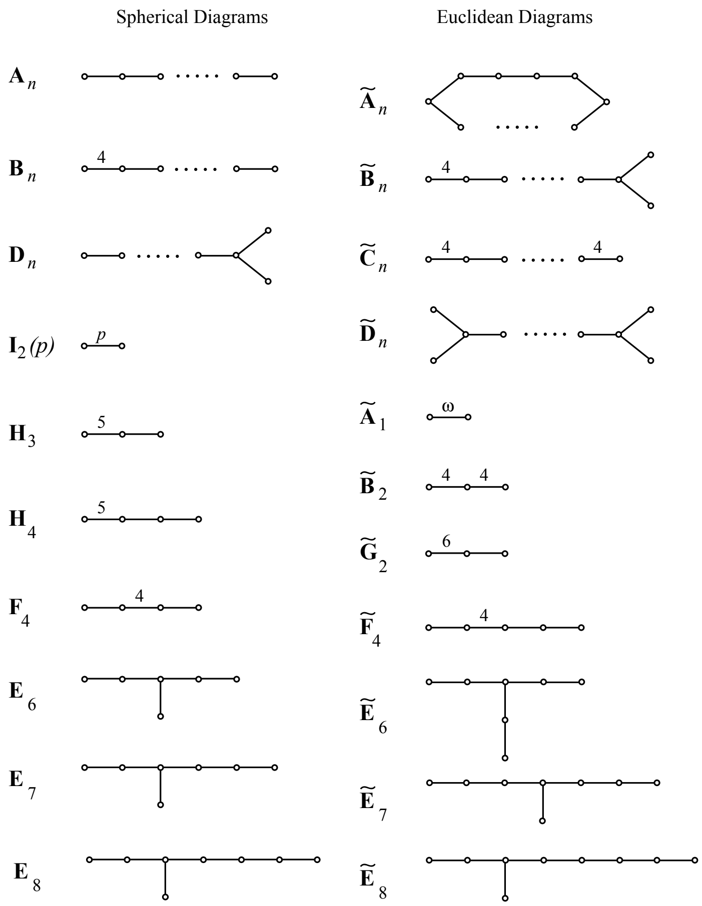


Figure 6: Coxeter diagrams

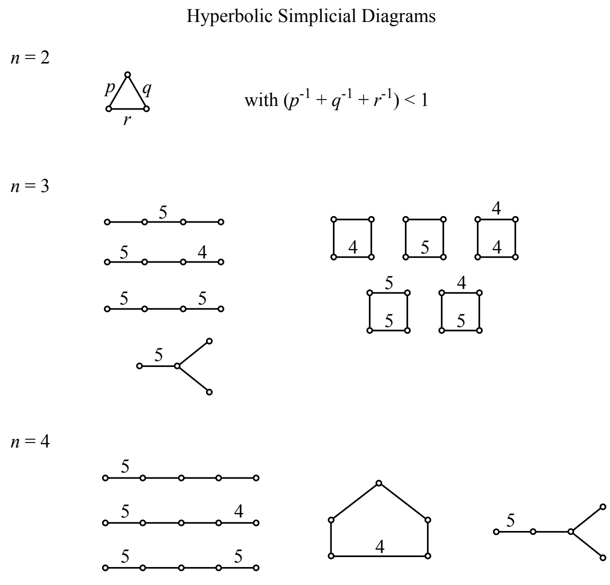


Figure 7: Hyperboloic simplicial diagrams

3.3 More reflection groups

Recall $\mathbb{X}^n$ stands for $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. Let $K \subset \mathbb{X}^n$ be a convex polytope with dihedral angles between codimension 1 faces of the form π/m, where m is an integer ≥ 2 or the symbol ∞ (where π/∞ means the faces do not intersect). W the group generated by reflections across the codimension 1 faces of K.

Goal: Show W is discrete, acts properly on $\mathbb{X}^n$ and that K is an orbifold with geometric structure of an $\mathbb{X}^n$-orbifold.

3.3.1 Generalities on abstract reflection groups

Suppose W is a group and S a set of involutions which generate it. For each s, $t \in S$, let $m(s,t)$ denote the order of st. (W, S) is a *Coxeter system* (and W is a *Coxeter group*) if the group defined by the presentation,

$$\{\text{generators}\} = S$$
$$\{\text{relations}\} = \{(st)^{m(s,t)}\}, \quad \text{where } (s,t) \in S \times S,\ m(s,t) \neq \infty,$$

is isomorphic to W (via the natural map).

For each $T \subset S$, let W_T denote the subgroup generated by T.

Definition 3.9. A *mirror structure* on a space X, indexed by a set S, is a family of closed subspaces $\{X_s\}_{s\in S}$. The X_s are called *mirrors*. For each $x \in X$, put $S(x) := \{s \in S \mid x \in X_s\}$.

Example 3.10. Suppose K is a convex polytope with its codimension 1 faces indexed by S. For each $s \in S$, let K_s denote the face corresponding to s. This defines a mirror structure on K. $S(x)$ is the set of faces which contain x. (In particular, if x is in the interior of K, then $S(x) = \emptyset$.)

The basic construction. Starting with a Coxeter system (W, S) and a mirror structure $\{X_s\}_{s\in S}$ we are going to define a new space $\mathcal{U}(W, X)$ with W-action. The idea is to paste together copies of X, one for each element of W. Each copy of X will be a fundamental domain and will be called a "chamber."

Define an equivalence relation $\sim$ on $W \times X$ by

$$(w, x) \sim (w', x') \iff x = x' \text{ and } wW_{S(x)} = w'W_{S(x)}.$$

Here W has the discrete topology. (Recall that $S(x)$ indexes the set of mirrors which contain x.) Put

$$\mathcal{U}(W, X) = (W \times X)/\sim .$$

To simplify notation, write $\mathcal{U}$ for $\mathcal{U}(W, X)$. Denote the image of (w, x) in $\mathcal{U}$ by $[w, x]$.

Some properties of the construction.

- $W \curvearrowright \mathcal{U}$ via $u[w, x] = [uw, x]$. The isotropy subgroup at $[w, x]$ is $wW_{S(x)}w^{-1}$.
- We can identify X with the image of $1 \times X$ in $\mathcal{U}$. X is a strict fundamental domain for the W-action in the sense that the restriction of the orbit map $\mathcal{U} \to \mathcal{U}/W$ to X is a homeomorphism (i.e., $\mathcal{U}/W = X$).
- $W \curvearrowright \mathcal{U}$ properly $\iff$ X is Hausdorff and each $W_{S(x)}$ is finite (i.e., $\bigcap_{s\in T} X_s = \emptyset$, whenever $|W_T| = \infty$).

Universal property. Suppose $W \curvearrowright Z$ and $f : X \to Z$ is a map so that for all $s \in S$, $f(X_s) \subset Z^s$. (Z^s denotes the fixed set of s on Z.) Then there is a unique extension to a W-equivariant map $\tilde{f} : \mathcal{U}(W, X) \to Z$. (In fact, $\tilde{f}$ is defined by $\tilde{f}([w, x]) = wf(x)$.)

Exercise 3.11. Prove the above properties hold.

3.3.2 Geometric reflection groups, again

The set up:

- K is a convex polytope in $\mathbb{X}^n$ ($= \mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$). S is the set of reflections across the codimension 1 faces of K. The face corresponding to s is denoted by K_s.
- If $K_s \cap K_t \neq \emptyset$, then it is a codimension 2 stratum and the dihedral angle is $\pi/m(s, t)$, where $m(s, t)$ is some integer ≥ 2. (We know this implies K is a simple polytope.) If $K_s \cap K_t = \emptyset$, then put $m(s, t) = \infty$.
- Let $\overline{W} \subset \mathrm{Isom}(\mathbb{X}^n)$ be the subgroup generated by S.
- Let W be the group defined by the presentation corresponding to the $(S \times S)$ Coxeter matrix, $(m(s, t))$. It turns out that (W, S) is a Coxeter system. (There is something to prove here, namely, that the order of st is $= m(s, t)$ rather than that it just divides $m(s, t)$.) Let $p : W \to \overline{W}$ be the natural surjection.

By the universal property, the inclusion $\iota : K \hookrightarrow \mathbb{X}^n$ induces a W-equivariant map $\tilde{\iota} : \mathcal{U}(W, X) \to \mathbb{X}^n$.

Theorem 3.12. *$\tilde{\iota} : \mathcal{U}(W, K) \to \mathbb{X}^n$ is a W-equivariant homeomorphism.*

Some consequences:

- $p : W \to \overline{W}$ is an isomorphism
- $\overline{W}$ is discrete and acts properly on $\mathbb{X}^n$
- K is a strict fundamental domain for the action on $\mathbb{X}^n$ (i.e., $\mathbb{X}^n/W = K$).
- $\mathcal{U}(W, K)$ is a manifold (because $\mathbb{X}^n$ is a manifold).

- K is an $\mathbb{X}^n$-orbifold (because it is identified with $\mathbb{X}^n/W$).
- If $W' \curvearrowright \mathbf{R}^n$ as a finite linear group, then $\mathbf{R}^n /\!/ W$ is isomorphic to the fundamental simplicial cone.

Sketch of proof of the theorem. The proof is by induction on the dimension n. A neighborhood of a point in K looks like the cone over the suspension, σ, of a spherical simplex. By induction, $\mathcal{U}(W_T, \sigma) = \mathbb{S}^{n-1}$ (where W_T is the finite Coxeter group corresponding to σ). Since a neighorhood in K is an open $\mathbb{X}^n$-cone over σ, it follows that $\tilde{\iota} : \mathcal{U}(W, K) \to \mathbb{X}^n$ is a local homeomorphism and a covering projection and that $\mathcal{U}(W, K)$ has the structure of an $\mathbb{X}^n$-manifold. Since $\mathbb{X}^n$ is simply connected, the covering projection $\tilde{\iota}$ ust be a homeomorphism. (The case $\mathbb{X}^n = \mathbb{S}^1$ is handled separately.) □

4 Lecture 4: 3-dimensional hyperbolic reflection groups

4.1 Andreev's Theorem

A geometric reflection group on $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is determined by its fundamental polytope. In the spherical case the fundamental polytope must be a simplex and in the Euclidean case it must be a a product of simplices. Furthermore, all the possibilities for these simplices are listed in Figure 6. So, there is nothing more to said in the spherical and Euclidean cases.

In the hyperbolic case we know what happens in dimension 2: the fundamental polygon can be an k-gon for any $k \geq 3$ and almost any assignment of angles can be realized by a hyperbolic polygon (there are a few exceptions when $k = 3$ or 4). What happens in dimension 3?

There is a beautiful theorem due to Andreev, which gives a complete answer. Roughly, it says given a simple polytope K, for it to be the fund polytope of a hyperbolic reflection group,

- there is no restriction on its combinatorial type
- subject to the condition that the group at each vertex be finite, almost any assignment of dihedral angles to the edges of K can be realized (provided a few simple inequalities hold).

In contrast to dimension 2, the 3-dimensional hyperbolic polytope is uniquely determined, up to isometry, by its dihedral angles – the moduli space is a point.

Remark. By a theorem of Vinberg, hyperbolic examples do not exist in dimensions ≥ 30.

Theorem 4.1 (Thurston's Conjecture, Perelman's Theorem)**.** *A closed, developable* 3*-orbifold* Q^3 *with infinite* π_1^{orb} *admits a hyperbolic structure iff it satisfies the following two conditions:*

(i) *Every embedded* 2*-dimensional spherical suborbifold bounds a quotient of a* 3*-ball in* Q^3*. (This condition implies* Q^3 *is aspherical.)*

(ii) $\mathbb{Z} \times \mathbb{Z} \not\subset \pi_1^{orb}(Q^3)$.

A 2-dimensional suborbifold of Q^3 is *incompressible* if the inclusion into Q^3 induces an injection on $\pi_1^{orb}(\ )$. The orbifold Q^3 is *Haken* if it does not contain any nondevelopable 2-dimensional suborbifolds, if every spherical 2-dimensional suborbifold bounds the quotient of a 3-ball by a finite linear group and if it contains an incompressible 2-dimensional Euclidean or hyperbolic orbififold.

Proposition 4.2. ([13, Prop. 13.5]). *An orbifold with underlying space a* 3*-disk and with no singular points in its interior (called a "reflectofold" in Subsection 5.1) is Haken iff it is neither a tetrahedron nor the product of a triangular spherical orbifold with* $[0,1]$ *(i.e., a triangular prism).*

In the late 1970's Thurston proved his conjecture for Haken manifolds or orbifolds. This can be stated as follows.

Theorem 4.3. (Thurston $\sim$ 1977). *A* 3*-dimensional Haken orbifold* Q^3 *admits a hyperbolic structure iff it has no incompressible* 2*-dimensional Euclidean suborbifolds (i.e.,* Q^3 *is "atoroidal").*

Combining this with Proposition 4.2 we get Corollary **??** below as a special case. This had been proved several years earlier by Andreev as a corollary to the following theorem about convex polytopes in $\mathbb{H}^3$.

Theorem 4.4. (Andreev $\sim$ 1967, see [1, 11]). *Suppose* K *is (the combinatorial type of) a simple* 3*-dimensional polytope, different from a tetrahedron. Let* E *be its edge set and* $\theta : E \to (0, \pi/2]$ *any function. Then* (K, θ) *can be realized as a convex polytope in* $\mathbb{H}^3$ *with dihedral angles as prescribed by* θ *if and only if the following conditions hold:*

(i) *At each vertex, the angles at the three edges e_1, e_2, e_3 which meet there satisfy $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.*

(ii) *If three faces intersect pairwise but do not have a common vertex, then the angles at the three edges of intersection satisfy $\theta(e_1)+\theta(e_2)+\theta(e_3) < \pi$.*

(iii) *Four faces cannot intersect cyclically with all four angles $= \pi/2$ unless two of the opposite faces also intersect.*

(iv) *If K is a triangular prism the angles along base and top cannot all be $\pi/2$.*

Moreover, when (K, θ) is realizable, it is unique up to an isometry of $\mathbb{H}^3$.

Corollary 4.5. *Suppose K is (the combinatorial type of) a simple 3-polytope, different from a tetrahedron, that $\{F_s\}_{s\in S}$ is its set of codimension 1 faces and that e_{st} is the edge $F_s \cap F_t$ (when $F_s \cap F_t \neq \emptyset$). Given an angle assignment $\theta : E \to (0, \pi/2]$, with $\theta(e_{st}) = \pi/m(s,t)$ and $m(s,t)$ an integer ≥ 2, then (K, θ) is a hyperbolic orbifold iff the $\theta(e_{st})$ satisfy Andreev's Conditions. Moreover, the geometric reflection group W is unique up to conjugation in* $\mathrm{Isom}(\mathbb{H}^3)$.

Remark. The condition that K is not a tetrahedron and Andreev's Condition (iv) deal with the case when the orbifold K is not Haken.

Examples 4.6. Here are some hyperbolic orbifolds:

- K is a dodecahedron with all dihedral angles equal to $\pi/2$.
- K is a cube with disjoint edges in different directions labeled by integers > 2 and all other edges labeled 2.

Exercise 4.7. Make up your own examples.

The dual form of Andreev's Theorem. Let L be the triangulation of S^2 dual to ∂K.

$$\begin{aligned} \mathrm{Vert}(L) &\longleftrightarrow \mathrm{Face}(K) \\ \mathrm{Edge}(L) &\longleftrightarrow \mathrm{Edge}(K) \\ \{\text{2-simplices in } L\} &\longleftrightarrow \mathrm{Vert}(K) \end{aligned}$$

Input data. Suppose we are given $\theta : \mathrm{Edge}(L) \to (0, \pi/2]$. The condition that K has a spherical link at each vertex is that if e_1, e_2, e_3 are the edges of a triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.

Theorem 4.8. (Dual form of Andreev's Theorem). *Suppose L is a triangulation of S^2 and $L \neq \partial\Delta^3$. Let $\theta : \mathrm{Edge}(L) \to (0, \pi/2]$ be any function. Then the dual polytope K can be realized as convex polytope in$\mathbb{H}^3$ with prescribed dihedral angles iff the following conditions hold:*

(i) *If e_1, e_2, e_3 are the edges of any triangle, then $\theta(e_1) + \theta(e_2) + \theta(e_3) > \pi$.*

(ii) *If e_1, e_2, e_3 are the edges of a 3-circuit $\neq \partial\Delta^2$, then $\theta(e_1)+\theta(e_2)+\theta(e_3) < \pi$.*

(iii) *If e_1, e_2, e_3, e_4 are the edges of a 4-circuit which is $\neq$ to boundary of union of two adjacent triangles, then all four $\theta(e_i)$ cannot $= \pi/2$.*

(iv) *If L is suspension of $\partial\Delta^2$, then all "vertical" edges cannot have $\theta(e_i) = \pi/2$.*

A dimension count. Given a convex 3-dimensional polytope K, Andreev's Theorem asserts that a certain map Θ from the space $C(K)$ of isometry classes convex polyhedra of the same combinatorial type as K to a certain subset $A(K) \subset \mathbf{R}^E$ is a homeomorphism (where $E := \mathrm{Edge}(K)$ and where $A(K)$ is the convex subset defined by Andreev's inequalities).

Let's compute $\dim C(K)$. For each $F \in \mathrm{Face}(K)$, let $u_F \in \mathbb{S}^{2,1}$ be the inward-pointing unit normal vector to F. (Here $\mathbb{S}^{2,1} := \{x \in \mathbf{R}^{3,1} \mid \langle x, x\rangle = 1\}$.) The $(u_F)_{F\in\mathrm{Face}(K)}$ determine K (since K is the intersection of the half-spaces determined by the u_F). The assumption that K is simple means that the hyperbolic hyperplanes normal to the u_F intersect in general position. So, a slight perturbation of the u_F will not change the combinatorial type of K. That is to say, the subset of Face(K)-tuples (u_F) which define a polytope combinatorially isomorphic to K is an open subset Y of $(\mathbb{S}^{2,1})^{\mathrm{Face}(K)}$.

- Let $f = \mathrm{Card}(\mathrm{Face}(K))$, $e = \mathrm{Card}(\mathrm{Edge}(K)$, $v = \mathrm{Card}(\mathrm{Vert}(K)$.
- $\mathrm{Isom}(\mathbb{H}^3) = O(3,1)$, $\dim(O(3,1) = 6$, and $\dim \mathbb{S}^{2,1} = 3$.
- So, $\dim C(K) = 3f - 6$.

Since $f - e + v = 2$, we have $3f - 6 = 3e - 3v$. Since three edges meet at each vertex, we have $3v = 2e$. Hence, $3f - 6 = 3e - 3v = e$. So, $\Theta : C(K) \to A(K) \subset \mathbf{R}^E$ is a map between manifolds with boundary of the same dimension.

4.2 3-dimensional orbifolds

Recall the list of 2-dimensional spherical orbifolds from Subsection 2.2:

- $|Q^2| = D^2$: $(\,;\,)$, $(\,;m,m)$, $(\,;2,2,m)$, $(\,;2,3,3)$, $(\,;2,3,4)$, $(\,;2,3,5)$, $(2;m)$, $(3;2)$.
- $|Q^2| = S^2$: $(\,)$, (n,n), $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$.
- $|Q^2| = \mathbf{R}P^2$: $(\,)$, (n)

The local models for 3-dimensional orbifolds are cones on any one of the above.

For example, if $|Q^2| = S^2$ with (n,n), then the 3-dimensional model is D^3 with an interval of cone points labeled n. Quotients of n-fold branched covers of knots or links in S^3 (or any other 3-manifold) have this form.

Example 4.9. (A flat orbifold). Consider the 3 families of lines in $\mathbb{E}^3$ of the form $(t,n,m+\frac{1}{2})$, $(m+\frac{1}{2},t,n)$ and $(n,m+\frac{1}{2},t)$, where $t \in \mathbf{R}$ and $n,m \in \mathbb{Z}$. Let Γ be the subgroup of $\mathrm{Isom}(\mathbb{E}^3)$ generated by rotation by π about each of these lines. A fundamental domain is the unit cube. The orbifold $\mathbb{E}^3/\!\!/\Gamma$ is obtained by "folding up" the cube to get the 3-sphere. The image of the lines (= the singular set) are 3 circles in S^3 each labeled by 2 (meaning C_2, the cyclic group of order 2). These 3 circles form the Borromean rings. (See [13] for pictures of the folding up process.)

Example 4.10. Suppose Q is an orbifold with underlying space S^3, with singular set the Borromean rings and with the components of the singular set labeled by cyclic groups of order p, q and r. I showed in my lecture how to use Andreev's theorem to show that this orbifold admits a hyperbolic structure iff all three integers are > 2. The proof uses the second example in Examples 4.6.

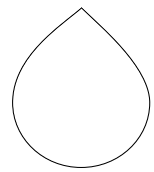


Figure 8: A nondevelopable reflectofold

5 Lecture 5: aspherical orbifolds

5.1 Reflectofolds

Definition 5.1. An n-dimensional orbifold Q is a *reflectofold* [4] if is locally modeled on finite linear reflection groups acting on $\mathbf{R}^n$.

If W acts on $\mathbf{R}^n$ as a finite reflection group, then $\mathbf{R}^n/W$ is a simplicial cone, i.e., up to linear isomorphism it looks like $[0,\infty)^n$. It follows that the underlying space of a reflectofold Q is a manifold with corners. Conversely, to give a manifold with corners the structure of a reflectofold, essentially all we need to do is label its codimension 2 strata by integers ≥ 2 in such a way that the strata of higher codimension correspond to *finite* Coxeter groups (which are listed in Figure 6).

It follows from the description of $\pi_1^{orb}(Q)$ in Subsection 1.3 that $\pi_1^{orb}(Q)$ is generated by reflections if and only if $\pi_1(|Q|) = 1$. (Here "reflection" means an involution with codimension 1 fixed set.) Henceforth, let's assume this (that $|Q|$ is simply connected).

If Q is developable, then any codimension 2 stratum is contained in the closures of two distinct codimension 1 strata. Otherwise, we would have a nondevelopable suborbifold pictured in Figure 8. Similarly, developability implies that if intersection of two codimension 1 strata contains two distinct codimension 1 strata, then they must be labeled by the same integer.

[4] When I introduced this term in my lecture I suggested that, as in Thurson's class, we should have an election to name the concept. Lizhen Ji was enthusiastic about this idea; however, in the end I didn't implement it.

5.2 Asphericity

Definition 5.2. An orbifold is *aspherical* if its universal cover is a contractible manifold.

One might ask why, in the above definition, we require the universal cover to be a contractible manifold rather than just a contractible orbifold. (A contractible orbifold is a simply connected orbifold all of whose higher homotopy groups also vanish. This definiton does not automatically imply that the orbifold is developable.) In fact, in the next questin we ask if it makes any difference which condition is required.

Question. *Is it true that every contractible orbifold is developable?*

Remark. I think the question has an affirmative answer, but I have never seen it written down.

Remark. A 2-dimensional orbifold Q^2 is aspherical $\iff \chi^{orb}(Q^2) \leq 0$.

My favorite conjecture.

Conjecture. (Hopf, Chern, Thurston). *Suppose Q^{2n} is a closed aspherical orbifold. Then $(-1)^n \chi^{orb}(Q^{2n}) \geq 0$.*

Hopf and Chern made this conjecture for nonpositively curve manifolds (I believe they thought it might follow from the Gauss-Bonnet Theorem) and Thurston extended it to aspherical manifolds (at least in the 4-dimensional case). For much more about this conjecture in the case of aspherical reflectofolds, see [6].

The set up. Let Q be a reflectofold. Denote the underlying space by K (instead of $|Q|$). Let S index the set of mirrors ($=$ {codimension 1 strata}). K_s denotes the closed mirror corresponding to s. Let $m(s,t)$ be the label on the codimension 2 strata in $K_s \cap K_t$. Put $m(s,t) = \infty$ if $K_s \cap K_t = \emptyset$. Let (W, S) be the Coxeter system defined by the presentation (i.e., $W = \pi_1^{orb}(Q)$). For each $T \subset S$, let W_T denote the subgroup generated by T. The subset T is *spherical* if W_T is finite. Let $\mathcal{S}$ be the set of spherical subsets of S, partially ordered by inclusion. (N.B. $\emptyset \in \mathcal{S}$.) Put

$$K_T = \bigcap_{s \in T} K_s.$$

Since Q is an orbifold, whenever $K_T \neq \emptyset$, we must have $W_T \in \mathcal{S}$.

Theorem 5.3. *The reflectofold Q is aspherical iff the following conditions hold:*

(i) $K_T \neq \emptyset \iff T \in \mathcal{S}$ *(i.e., when W_T is finite).*

(ii) *For each $T \in \mathcal{S}$, K_T is acyclic (i.e., $\overline{H}_*(K_T) = 0$).*

(Notes: $K_\emptyset = K$; also, when K is simply connected and acyclic, it is contractible.)

The first condition means that the combinatorics of intersections of mirrors is determined by (W, S). It is the analog of Andreev's Conditions (without the atoroidal condition), cf. Theorem 4.8. The second condition says that the manifold with corners K "looks like" a convex polytope up to homology. We elucidate these points below.

Definition 5.4. The *nerve* of the mirror structure $\{K_s\}_{s \in S}$ on K is an abstract simplicial complex L' defined as follows: its vertex set, $\mathrm{Vert}(L')$, is S and a nonempty subset T of S is the vertex set of a simplex in L' iff $K_T \neq \emptyset$.

If L'' is any simplicial complex with $\mathrm{Vert}(L'') = S$, write $\mathcal{S}(L'')$ for the poset of vertex sets of simplices in L''. If σ is a simplex of L'' with vertex set T, let $\mathrm{Lk}(\sigma, L'')$ denote the abstract simplicial complex corresponding to the poset $\mathcal{S}(L'')_{>T}$. ($\mathrm{Lk}(\sigma, L'')$ is called the *link* of σ in L''.) Given two topological spaces X and Y, write $X \sim Y$ to mean that $H_*(X; \mathbb{Z}) \cong H_*(Y; \mathbb{Z})$. If $\dim K = n$, then, by standard arguments in algebraic topology, condition (ii) of Theorem 5.3 means that

$$L' \sim S^{n-1} \quad \text{and} \quad \mathrm{Lk}(\sigma, L') \sim S^{n-1-\dim \sigma}. \tag{5.1}$$

for all simplices σ in L' (cf., [5] or [6, §8.2]). In the case where K is a convex polytope, L' is the boundary of the dual polytope, i.e., L' is dual to ∂K (cf. the last part of Subsection 4.1).

Definition 5.5. Suppose (W, S) is a Coxeter system. The elements of $\mathcal{S}$ which are $\neq \emptyset$ are the simplices of an abstract simplicial complex, denoted by $L(W, S)$ (or more simply, by L) and called the *nerve* of (W, S). More precisely, $\mathrm{Vert}(L) = S$ and a nonempty subset T of S is the vertex set of a simplex in L iff T is spherical.

The fact that K is the underlying space of an orbifold means that $L' \subset L$ (i.e., all local groups are finite). Condition (i) of Theorem 5.3 is that $L' = L$. By Condition (ii), L satisfies (5.1).

Before sketching the proof of Theorem 5.3 we discuss the following two questions:

(A) How do you produce a large number of examples of Coxeter systems (W, S) with $L(W, S)$ satisfying (5.1)?

(B) How do you recover K from L?

More generally, how do we find Coxeter system (W, S) with nerve a given finite simplicial complex J? We should start as follows. Put $S = \mathrm{Vert}(J)$. Label each edge $\{s, t\}$ by an integer $m(s, t) \geq 2$. This defines the Coxeter system (W, S). The condition that we need to get an orbifold is that whenever T is the vertex set of a simplex of J, then $T \in \mathcal{S}$. Condition (i) of Theorem 5.3 (an analog of Andreev's Theorem) is the converse: whenever $T \in \mathcal{S}$, then T is the vertex set of a simplex in J. We will see below that when all the $m(s, t)$'s are 2 or ∞ these conditions are easy to decide.

Definition 5.6. A simplicial complex J is a *flag complex* if T is any finite, nonempty collection of vertices which are pairwise connected by edges, then T spans a simplex of J

Remark 5.7. In [8] Gromov uses the terminology that J satisfies the "no Δ condition" for this concept. I once used the terminology that J is "determined by its 1-skeleton" for the same notion. Combinatorialists call such a J a "clique complex".

Examples 5.8.

- If J is a k-gon (i.e., a triangulation of S^1 into k edges), then J is a flag complex iff $k > 3$.
- The barycentric subdivision of any simplicial complex (or, in fact, of any cell complex) is a flag complex.

The second of Examples 5.8 shows that the condition of being a flag complex does not restrict the topological type of J – it can be any polyhedron.

Definition 5.9. A Coxeter system (W, S) is *right-angled* if for each $s \neq t$, $m(s, t)$ is either 2 or ∞.

Since the nerve of any right-angled (W, S) is obviously a flag complex, the second of Examples 5.8 yields the following answer to Question (A).

Proposition 5.10. *The barycentric subdivision of any finite cell complex occurs as the nerve of a right-angled Coxeter system.*

Reconstructing $\boldsymbol{K}$. Now suppose that L $(= L')$ is a PL triangulation of S^{n-1}. Let $K = D^n$ $(= \mathrm{Cone}(S^{n-1})$ and identify ∂K with L. We want to find a mirror structure on K dual to L. The construction is the usual one for defining the dual cell structure on a manifold. For each vertex s of L, let K_s be the closed star of s in the barycentric subdivision, bL. Thus, $K_s = \mathrm{Cone}(b\,\mathrm{Lk}(s, L))$. For each $T \in \mathcal{S}$, we then have $K_T = \bigcap_{s \in T} K_s = \mathrm{Cone}\, b(\mathrm{Lk}(\sigma_T, L))$, where σ_T is the simplex in L corresponding to T. The assumption that the triangulation is *PL* means that each $\mathrm{Lk}(\sigma_T, L)$ is a sphere $(= S^{n-1-\dim \sigma_T})$; so, each K_T is a cell.

Exactly the same construction works when L is a PL triangulation of a homology sphere (that is, a closed PL manifold with the same homology as S^{n-1}), except that instead of being a disk, K is a compact contractible manifold with boundary L. (This uses the fact that any homology sphere L is topolgically the boundary of a contractible 4-manifold. This fact follows from surgery theory when $\dim L > 3$ and and is due to Freedman when $\dim L = 3$.) In general, when L is only required to satisfy (5.1), one must repeatedly apply this step of replacing $\mathrm{Cone}(b\,\mathrm{Lk}(\sigma, L))$ by a contractible manifold bounded by a contractible manifold (see [5]).

5.3 Proof of the asphericity theorem

For each $w \in W$, define the following subset of S:

$$\mathrm{In}(w) := \{s \in S \mid l(ws) < l(w)\}.$$

($l(w)$ is the word length of w with respect to the generating set S.)

The following lemma in the theory of Coxeter groups is key to the proof of Theorem 5.3.

Lemma 5.11. (See [6, Lemma 4.7.2]). *For each $w \in W$,* $\mathrm{In}(W)$ *is a spherical subset of S.*

Sketch of proof of Theorem 5.3. The universal cover of the reflectofold Q is the manifold $\mathcal{U}(W, K)$, which we denote simply by $\mathcal{U}$. The manifold $\mathcal{U}$ is contractible if and only if it is simply connected and acyclic. We will derive necessary and sufficient conditions for this to hold.

Order the elemnts of W: $w_1, w_2, \dots w_k \dots$, in any fashion so that $l(w_k) \leq l(w_{k+1})$. Let P_k denote the union of the first k "chambers" in $\mathcal{U}$, i.e.,

$$P_k := w_1 K \cup \cdots \cup w_k K.$$

We propose to study the exact sequence of the pair (P_{k+1}, P_k) in homology. To simplify notation, put $w = w_{k+1}$. By excision,

$$H_*(P_{k+1}, P_k) = H_*(wK, wK^{\mathrm{In}(w)}) = H^*(K, K^{\mathrm{In}(w)}),$$

where for any subset $T \subset S$,

$$K^T := \bigcup_{s \in T} K_s.$$

So, the long exact sequence of the pair becomes

$$\cdots \to H_*(P_k) \to H_*(P_{k+1}) \to H_*(K, K^{\mathrm{In}(w)}) \to \dots.$$

It is not hard to see that there is a splitting, $H_*(K, K^{\mathrm{In}(w)}) \to H_*(P_{k+1})$, of the right hand map defined by multiplication by $wh_{\in(w)}w^{-1} \in \mathbb{Z}W$, where for any $T \in \mathcal{S}$, h_T is the element in the group ring $\mathbb{Z}W_T$ defined by $h_T := \sum_{u \in W_T} (-1)^{l(u)} u$. Hence,

$$H_*(P_{k+1}) \cong H_*(P_k) \oplus H_*(K, K^{\mathrm{In}(w)}) \quad \text{and therefore,}$$

$$H_*(\mathcal{U}) \cong \bigoplus_{k=1}^{\infty} H_*(K, K^{\mathrm{In}(w_k)})$$

If A^T denotes the free abelian group on $\{w \in W \mid \mathrm{In}(w) = T\}$, then the above formula can be rewritten as

$$H_*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H_*(K, K^T) \otimes A^T. \tag{5.2}$$

From (5.2) we see that $\overline{H}_*(\mathcal{U}) = 0$ iff $\overline{H}_*(K, K^T) = 0$ for all $T \in \mathcal{S}$. Standard arguments using Mayer-Vietoris sequences (or the Mayer-Vietoris spectral sequence) show that these terms all vanish iff for all $T \in \mathcal{S}$, the intersection K_T is acyclic (this includes the statement that K is acyclic. (See [6, §8.2].)

A similar argument using van Kampen's Theorem applied to $P_{k+1} = P_k \cup K$ shows that $\mathcal{U}$ is simply connected iff K is simply connected, each K_s is connected and for each $\{s,t\} \in \mathcal{S}$, $K_{\{s,t\}} \neq \emptyset$. $\square$

5.4 Aspherical orbifolds not covered by Euclidean space

A noncompact space X is *simply connected at infinity* if given any compact subset C there is a larger compact subset D so that any loop in $X - D$ is null homotopic in $X - C$. In favorable circumstances the inverse system of fundamental groups $\{\pi_1(X - C)\}$, where C ranges over all compact subsets, has a well-defined inverse limit, $\pi_1^\infty(X)$, called the *fundamental group at infinity.* If X is simply connected at infinity, then $\pi_1^\infty(X)$ is trivial. (See [7] for the basic facts about the concepts in this paragraph.)

Theorem 5.12. (Stallings, Freedman, Perelman). *A contractible n-manifold is homeomorphic to $\mathbf{R}^n$ iff it is simply connected at infinity.*

(Stallings proved this in dimensions ≥ 5, Freedman in dimension 4 and in dimension 3, I believe it follows from Perelman's proof of the Poincaré Conjecture.)

For some time it was an open problem if the universal cover of a closed, aspherical manifold had to be homeomorphic to Euclidean space. Of course, the issue was not the existence of exotic (i.e., not simply connected at infinity) contractible manifolds but whether such an exotic contractible manifold could admit a cocompact transformation group. This was resolved in [5] by using the techniques of this section.

Let L be a triangulation of a homology $(n-1)$-sphere as a flag complex. Label its edges by 2 and let (W, S) be the associated right-angled Coxeter group with nerve L. Let K be a contractible manifold with $\partial K = L$. As explained above, we can put the dual cell structure on ∂K to give K the structure of a manifold with corners and hence, the structure of a reflectofold Q. The claim is that if $n > 2$ and L^{n-1} is not simply connected, then the contractible manifold $\mathcal{U}(W, K)$ is not simply connected at infinity. As before, let P_k be the union of the first k chambers and let $\overset{\circ}{P}_k$ be its interior. The argument goes as follows.

- Since P_k is obtained by gluing on a copy of K to P_{k-1} along an $(n-1)$-disk in its boundary, it follows that P_k is a contractible manifold with boundary and that its boundary is the connected sum of k copies of $\pi_1(\partial K)$. Hence, $\pi_1(\partial(P_k))$ is the free product of k copies of $\pi_1(\partial k)$.

- For a similar reason one $\mathcal{U} - \overset{\circ}{P}_k$ is homotopy equivalent to ∂P_k.

- Hence, $\pi_1^\infty(\mathcal{U})$ is the inverse limit, $\varinjlim(\pi_1(L) * \cdots * \pi_1(L))$. In other words, it is the "projective free product" of copies of $\pi_1(L)$. In particular, it is nontrivial whenever $\pi_1(L) \neq 1$.

The above is a sketch of the proof of the following result.

Theorem 5.13. ([5]) *For each $n \geq 4$ there are closed, aspherical n-dimensional orbifolds with universal cover not homeomorphic to $\mathbf{R}^n$.*

Since Coxeter groups have faithful linear representations (cf. [2]), Selberg's Lemma implies that they are virtually torsion-free. So, there is a torsion-free subgroup $\Gamma \subset W$ which then necessarily acts freely on $\mathcal{U}$. Hence, $M = \mathcal{U}/\Gamma$ is a closed, aspherical manifold. Thus, the previous theorem has the following corollary.

Corollary 5.14. ([5]) *For each $n \geq 4$, there are closed, aspherical n-dimensional manifolds with universal cover not homeomorphic to $\mathbf{R}^n$.*

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