Principally Quasi-Baer Ring Hulls

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Dedicated to Professor S. K. Jain on his seventieth birthday

Abstract. We show the existence of principally (and finitely generated) right FI-extending right ring hulls for semiprime rings. From this result, we prove that right principally quasi-Baer (i.e., right p.q.-Baer) right ring hulls always exist for semiprime rings. This existence of right p.q.-Baer right ring hull for a semiprime ring unifies the result by Burgess and Raphael on the existence of a closely related unique smallest overring for a (von Neumann) regular ring with bounded index and the result of Dobbs and Picavet showing the existence of a weak Baer envelope for a commutative semiprime ring. As applications, we illustrate the transference of certain properties between a semiprime ring and its right p.q.-Baer right ring hull, and we explicitly describe a structure theorem for the right p.q.Baer right ring hull of a semiprime ring with only finitely many minimal prime ideals. The existence of PP right ring hulls for reduced rings is also obtained. Further application to ring extensions such as monoid rings, matrix, and triangular matrix rings are investigated. Moreover, examples and counterexamples are provided.

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Throughout all rings are associative rings with unity. Ideals without the adjectives "right" or "left" mean two-sided ideals.

In this paper, we prove the existence of principally (and finitely generated) right FI-extending right ring hulls for semiprime rings by using the concepts of distinguished extending classes (or $\mathfrak{D}$-$\mathfrak{E}$ classes), pseudo right ring hulls, and techniques studied in [12]. From this result, we obtain the existence of right p.q.-Baer right ring hulls for semiprime rings. Thereby, the existence of right p.q.-Baer right ring hulls for semiprime rings unifies the results on the existence of a closely related unique smallest overring for a (von Neumann) regular ring with bounded index by Burgess and Raphael [16], and that of the weak Baer envelope for a commutative semiprime ring by Dobbs and Picavet [18]. As applications, (i) we investigate the transference of properties between a semiprime ring and its right p.q.-Baer right ring hull; (ii) a structure theorem for the right p.q.-Baer right ring hull of a semiprime ring with only finitely many minimal prime ideals is described; (iii) we establish the existence of PP right ring hulls for reduced rings; and (iii) the existence of right p.q.-Baer right ring hulls of ring extensions such as monoid rings, matrix, and triangular matrix rings are studied. Furthermore, examples and counterexamples are provided.

Recall from [9] that a ring R is called *right p.q.-Baer* (i.e., right principally quasi-Baer) if the right annihilator of a principal ideal of R is generated by an idempotent as a right ideal. Equivalently, R is right p.q.-Baer if R modulo the right annihilator of each principal right ideal is projective. We let $\mathfrak{pq}\mathfrak{B}$ denote the class of right p.q.-Baer rings. Similarly, left p.q.-Baer rings can be defined. If a ring R is both right and left p.q.-Baer, then we say that R is *p.q.-Baer.* A ring R is called *right PP* if the right annihilator of every singleton subset of R is generated by an idempotent as a right ideal. Note that the definition of a right PP ring is equivalent to every principal right ideal of R being projective (these rings are also called right *Rickart* rings). A ring R is called PP if R is both right and left PP.

Recall from [4] that a ring R is called *quasi-Baer* if the right annihilator of every right ideal is generated by an idempotent (see [4], [5], [6], and [8] for more details on quasi-Baer rings). The class of p.q.-Baer rings includes biregular rings, quasi-Baer rings and abelian (i.e., every idempotent is central) PP rings. Also recall that a ring R is called *right* (*FI*)*-extending* if every right ideal (ideal) is essential as a right R-module in an idempotent generated right ideal of R. We let $\mathfrak{E}$ and $\mathfrak{F}\mathfrak{I}$ to denote the class of right extending rings and that of right FI-extending rings, respectively.

We say that a ring R is *principally right FI-extending* (resp., *finitely generated right FI-extending*) if every principal ideal (resp., finitely generated ideal) of R is essential as a right R-module in a right ideal of R generated by an idempotent. We use $\mathfrak{p}\mathfrak{F}\mathfrak{I}$ (resp., $\mathfrak{fg}\mathfrak{F}\mathfrak{I}$) to denote the class of principally (resp., finitely generated) right FI-extending rings.

An overring S of a ring R is said to be a *right ring of quotients* (resp., *right essential overring*) of R if R_R is dense (resp., essential) in S_R. Thus every right ring of quotients of R is a right essential overring of R.

For a right R-module M_R, we use $N_R \leq M_R$, $N_R \trianglelefteq M_R$, $N_R \leq^{\text{ess}} M_R$, and $N_R \leq^{\text{den}} M_R$ to denote that N_R is a submodule of M_R, N_R is a fully invariant submodule of M_R, N_R is an essential submodule of M_R, and N_R is a dense (or rational) submodule of M_R, respectively. We use $\mathbf{I}(R)$, $\mathbf{B}(R)$, $\text{Cen}(R)$, $\text{Mat}_n(R)$, and $T_n(R)$ to denote the set of all idempotents of R, the set of all central idempotents of R, the center of R, the n-by-n matrix ring over R, and the n-by-n upper triangular matrix ring over R, respectively. For a nonempty subset Y of a ring R,

$\langle Y \rangle_R$, $\ell_R(Y)$, and $r_R(Y)$ denote the subring of R generated by Y, the left annihilator of Y in R, and the right annihilator of Y in R, respectively. The notion $I \trianglelefteq R$ means that I is an ideal of a ring R.

We let $Q(R)$, $E(R_R)$, and $\mathcal{E}_R$ denote the maximal right ring of quotients of R, the injective hull of R_R, and the endomorphism ring $\mathrm{End}(E(R_R)_R)$, respectively. Let $\mathcal{Q}_R = \mathrm{End}(_{\mathcal{E}_R}E(R_R))$. Note that $Q(R) = 1 \cdot \mathcal{Q}_R$ (i.e., the canonical image of $\mathcal{Q}_R$ in $E(R_R)$) and that $\mathbf{B}(\mathcal{Q}_R) = \mathbf{B}(\mathcal{E}_R)$ [21, pp.94-96]. Also, $\mathbf{B}(Q(R)) = \{b(1) \mid b \in \mathbf{B}(\mathcal{Q}_R)\}$ [20, p.366]. Thus $R\mathbf{B}(\mathcal{E}_R) = R\mathbf{B}(Q(R))$, the subring of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$. If R is semiprime, then $\mathrm{Cen}(Q(R)) = \mathrm{Cen}(Q^m(R))$ [20, pp.389-390], where $Q^m(R)$ is the Martindale right ring of quotients of R.

Proposition 1. (i) ([5, Proposition 1.8] and [9, Proposition 1.12]) The center of a quasi-Baer (resp., right p.q.-Baer) ring is Baer (resp., PP).

(ii) ([9, Proposition 3.11]) Assume that a ring R is semiprime. Then R is quasi-Baer if and only if R is p.q.-Baer and the center of R is Baer.

(iii) ([26, pp.78-79] and [5, Theorem 3.5]) Let a ring R be (von Neumann) regular (resp., biregular). Then R is Baer (resp., quasi-Baer) if and only if the lattice of principal right ideals (resp., principal ideals) is complete.

(iv) A ring R is biregular if and only if R is right (or left) p.q.-Baer ring and $r_R(\ell_R(RaR)) = RaR$, for all $a \in R$.

Proof. The proof of part (iv) is straightforward. □

Let R be a ring and $e = e^2 \in R$. Recall from [3] that e is called *left* (resp., *right*) *semicentral* if $exe = xe$ (resp., $exe = ex$) for every $x \in R$. Note that $e = e^2 \in R$ is left (resp., right) semicentral if and only if eR (resp., Re) is an ideal of R. We use $\mathbf{S}_\ell(R)$ (resp., $\mathbf{S}_r(R)$) to denote the set of all left (resp., right) semicentral idempotents of R. See [7, Propositions 1.1 and 1.3] for more details on left (or right) semicentral idempotents.

Proposition 2. (i) Let R be a ring, K_i an ideal of R, and $e_i \in \mathbf{S}_\ell(R)$ such that $K_{iR} \leq^{\text{ess}} e_iR_R$ for $i = 1, 2, \ldots, n$. Then there exists $g \in \mathbf{S}_\ell(R)$ such that $(\sum_{i=1}^n K_i)_R \leq^{\text{ess}} gR_R$.

(ii) Let R be a right nonsingular ring. Then R is principally right FI-extending if and only if R is finitely generated right FI-extending.

Proof. (i) We will first prove the result for $n = 2$. Let $A = K_1$, $B = K_2$, $e = e_1$, and $f = e_2$. Then $A_R \leq^{\text{ess}} eR_R$, $B_R \leq^{\text{ess}} fR_R$, and $e, f \in \mathbf{S}_\ell(R)$. Since $A+B$ is an ideal of R, we have that $A + B = [(A + B) \cap eR] \oplus [(A + B) \cap (1 - e)R]$. Note that $(A+B)\cap(1-e)R = B\cap(1-e)R$. Thus $A+B = [(A+B)\cap eR]\oplus[B\cap(1-e)R]$. Now $[(A+B)\cap eR]_R \leq^{\text{ess}} eR_R$. Also $[B\cap(1-e)R]_R \leq^{\text{ess}} fR_R\cap(1-e)R_R = (1-e)fR_R$ because $B_R \leq^{\text{ess}} fR_R$ and $fR \cap (1 - e)R = (1 - e)fR$. So $(A + B)_R \leq^{\text{ess}} (eR + (1 - e)fR)_R = (e + f - ef)R_R$. In this case, we see that $e + f - ef \in \mathbf{S}_\ell(R)$. Now an induction argument can be used to complete the proof.

(ii) This part follows from part (i) and [10, Proposition 1.10]. □

We include the following result from [9], for the convenience of the reader, which shows the connections between the right p.q.-Baer condition and some "finitely generated" right FI-extending conditions for semiprime rings.

Lemma 3. ([9, Corollary 1.11]) Let R be a semiprime ring. Then the following conditions are equivalent.

(i) R is right p.q.-Baer.

(ii) R is principally right FI-extending.

(iii) R is finitely generated right FI-extending.

Definition 4. (cf. [12, Definition 2.1]) Let $\mathfrak{K}$ denote a class of rings. For a ring R, $\widehat{Q}_{\mathfrak{K}}(R)$ denotes the smallest right ring of quotients of R which is in $\mathfrak{K}$. Further, let $Q_{\mathfrak{K}}(R)$ be the smallest right essential overring of R which is in $\mathfrak{K}$. We say that $Q_{\mathfrak{K}}(R)$ is the *absolute* $\mathfrak{K}$ *right ring hull* of R. Note that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. In this paper, we call $\widehat{Q}_{\mathfrak{K}}(R)$ the $\mathfrak{K}$ *right ring hull* of R.

Since our interest is primarily in classes of rings which are defined by properties on the set of right ideals of the rings in the classes, we recall the following definition.

Definition 5. ([12, Definition 1.6]) Let $\mathfrak{R}$ be a class of rings, $\mathfrak{K}$ a subclass of $\mathfrak{R}$, and $\mathfrak{X}$ a class containing all subsets of every ring. We say that $\mathfrak{K}$ is a class *determined by a property on right ideals* if there exist an assignment $\mathfrak{D}_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\mathfrak{D}_{\mathfrak{K}}(R) \subseteq \{\text{right ideals of } R\}$ and a property P such that each element of $\mathfrak{D}_{\mathfrak{K}}(R)$ has P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is a class determined by the particular property P such that a right ideal is essential in an idempotent generated right ideal, then we say that $\mathfrak{K}$ is a $\mathfrak{D}$-$\mathfrak{E}$ *class* and use $\mathfrak{C}$ to designate a $\mathfrak{D}$-$\mathfrak{E}$ class. Note that every $\mathfrak{D}$-$\mathfrak{E}$ class contains the class $\mathfrak{E}$ of right extending (hence right self-injective) rings. Recall from [10] that a ring R is right FI-extending if every ideal is essential in an idempotent generated right ideal. Thus the class $\mathfrak{FI}$ of right FI-extending rings is a $\mathfrak{D}$-$\mathfrak{E}$ class. Furthermore, from their definitions, we see that $\mathfrak{pFI}$ and $\mathfrak{fgFI}$ are $\mathfrak{D}$-$\mathfrak{E}$ classes.

Some examples illustrating Definition 5 are (see [12]):

(1) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{\text{right ideals of } R\}$, and P is the property that a right ideal is finitely generated.

(2) $\mathfrak{K}$ is the class of (von Neumann) regular rings, $\mathfrak{D}_{\mathfrak{K}}(R) = \{$principal right ideals of $R\}$, and P is the property that a right ideal is generated by an idempotent.

(3) $\mathfrak{K} = \mathfrak{pq}\mathfrak{B}, \mathfrak{D}_{\mathfrak{pq}\mathfrak{B}}(R) = \{r_R(xR) \mid x \in R\}$, and P is the property that a right ideal is generated by an idempotent.

(4) $\mathfrak{C} = \mathfrak{E}$ (resp., $\mathfrak{C} = \mathfrak{FI}$), $\mathfrak{D}_{\mathfrak{E}}(R) = \{I \mid I_R \leq R_R\}$ (resp., $\mathfrak{D}_{\mathfrak{FI}}(R) = \{I \mid I \trianglelefteq R\}$). (Recall that $\mathfrak{E}$ is the class of right extending rings and $\mathfrak{F}$ is the class of right FI-extending rings.)

(5) $\mathfrak{C} = \mathfrak{pFI}$, $\mathfrak{D}_{\mathfrak{pFI}}(R) = \{\text{principal ideal of } R\}$.

(6) $\mathfrak{C} = \mathfrak{fgFI}$, $\mathfrak{D}_{\mathfrak{fgFI}}(R) = \{\text{finitely generated ideal of } R\}$.

Next, we consider generating a right essential overring in a class $\mathfrak{K}$ from a base ring R and some subset of $\mathcal{E}_R$. By using equivalence relations, in [12] we reduce the size of the subsets of $\mathcal{E}_R$ needed to generate a right essential overring of R in a $\mathfrak{D}$-$\mathfrak{E}$ class of rings $\mathfrak{C}$. Also in [12], to develop the theory of pseudo right ring hulls for $\mathfrak{D}$-$\mathfrak{E}$ classes $\mathfrak{C}$, we fix $\mathfrak{D}_{\mathfrak{C}}(R)$ for each ring R and define

$$\delta_{\mathfrak{C}}(R) = \{e \in \mathbf{I}(\mathcal{E}_R) \mid V_R \leq^{\text{ess}} eE(R_R) \text{ for some } V \in \mathfrak{D}_{\mathfrak{C}}(R)\}.$$

We set $\delta_{\mathfrak{C}}(R)(1) = \{e(1) \mid e \in \delta_{\mathfrak{C}}(R)\}$.

Definition 6. (cf. [12, Definition 2.2]) Let S be a right essential overring of R. If $\delta_{\mathfrak{C}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S \in \mathfrak{C}$, then we call $\langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_S$ the *pseudo right ring hull of R with respect to S* and denote it by $R(\mathfrak{C}, S)$. If $S = R(\mathfrak{C}, S)$, then we say that S is a $\mathfrak{C}$ *pseudo right ring hull of R.*

To find a right essential overring S of R such that $S \in \mathfrak{C}$, one might naturally look for a right essential overring T of R with $\delta_{\mathfrak{C}}(R)(1) \subseteq T$ and take $S = \langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_T$. Indeed, under some mild conditions, this choice of S can be in $\mathfrak{C}$. However, in order to obtain a right essential overring with some hull-like behavior, we need to determine subsets Λ of $\delta_{\mathfrak{C}}(R)(1)$ for which $\langle R \cup \Lambda\rangle_T \in \mathfrak{C}$ in some minimal sense. Moreover, to facilitate the transfer of information between R and $\langle R \cup \Lambda\rangle_T$, one would want to include in Λ enough of $\delta_{\mathfrak{C}}(R)(1)$ so that for all (or almost all) $V \in \mathfrak{D}_{\mathfrak{C}}(R)$ there is $e \in \delta_{\mathfrak{C}}(R)$ with $V_R \leq^{\text{ess}} e(1) \cdot (\langle R \cup \Lambda\rangle_T)_R$ and $e(1) \in \Lambda$.

Lemma 7. Let $\{e_1, \ldots, e_n\} \subseteq \mathbf{B}(T)$, where T is an overring of a ring R. Then there exists a set of orthogonal idempotents $\{f_1, \ldots, f_m\} \subseteq \mathbf{B}(T)$ such that $\sum_{i=1}^n e_i R \subseteq \sum_{i=1}^m f_i R$.

Proof. The proof is similar to that of [23, Lemma 3.2]. □

For a semiprime ring R, the concepts of (right) FI-extending and quasi-Baer coincide by [10, Theorem 4.7]. Recall that the existence of the quasi-Baer right ring hull and that of right FI-extending right ring hull of a semiprime ring were shown in [14, Theorem 3.3]. It was also proved in [14, Theorem 3.3] that the quasi-Baer right ring hull is precisely the same as its right FI-extending right ring hull for a semiprime ring. In view of this result, it is natural to ask: *Do the right principally quasi-Baer right ring hull and the principally right FI-extending right ring hull exist for a semiprime ring and if they do, are they equal?* In our next result, we provide affirmative answers to these two questions.

Burgess and Raphael [16] study ring extensions of (von Neumann) regular rings with bounded index. In particular for a (von Neumann) regular ring R with bounded index, they obtain a closely related unique smallest overring, $R^{\#}$, which is "almost biregular" (see [16, p.76 and Theorem 1.7]). The next result shows that their ring $R^{\#}$ is precisely our principally right FI-extending pseudo right ring hull of a (von Neumann) regular ring R with bounded index (see also [14, Theorem

3.8]). When R is a commutative semiprime ring, the "weak Baer envelope" defined in [18] is exactly the right p.q.-Baer right ring hull $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$.

Theorem 8. Let R be a semiprime ring. Then we have the following.

(i) $\langle R \cup \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) = R(\mathfrak{p}\mathfrak{F}\mathfrak{I}, Q(R))$.

(ii) $\langle R \cup \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$.

(iii) $\langle R \cup \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R) = R(\mathfrak{fg}\mathfrak{F}\mathfrak{I}, Q(R))$.

Proof. (i) Let $\mathbf{B}_p(Q(R)) = \{c \in \mathbf{B}(Q(R)) \mid$ there exists $x \in R$ with $RxR_R \leq^{\text{ess}} cR_R\}$. We first *claim* that

$$\mathbf{B}_p(Q(R)) = \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1).$$

For this claim, note that by [1, Theorem 7], $\delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) \subseteq \mathbf{B}(\mathcal{E}_R)$. Thus $\delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1) \subseteq \mathbf{B}(Q(R))$. To prove the claim, let $e(1) \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$ with $e \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$. Then there exists $x \in R$ such that $RxR_R \leq^{\text{ess}} eE(R_R)$. Thus $RxR = eRxR = e(1)RxR \subseteq e(1)R = eR$. So $RxR_R \leq^{\text{ess}} eR_R = e(1)R_R$. Hence $e(1) \in \mathbf{B}_p(Q(R))$ because $e(1) \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1) \subseteq \mathbf{B}(Q(R))$. Conversely, let $c \in \mathbf{B}_p(Q(R))$. Then there exists $b \in \mathbf{B}(\mathcal{E}_R)$ such that $c = b(1)$. Also there is $x \in R$ such that $RxR_R \leq^{\text{ess}} cR_R = b(1)R_R = bR_R$. Thus $RxR_R \leq^{\text{ess}} bE(R_R)$. So $b \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$. Hence $c = b(1) \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$. Therefore $\mathbf{B}_p(Q(R)) = \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$.

Let $S = \langle R \cup \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)\rangle_{Q(R)}$. Take $0 \neq s \in S$. From Lemma 7, $s = \sum r_i b_i$, where each $r_i \in R$ and the b_i are mutually orthogonal idempotents in $\mathbf{B}(S)$. There exists $c_i \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$ such that $Rr_iR_R \leq^{\text{ess}} c_iR_R$ for each i. Hence $s = \sum r_i e_i$, where $e_i = b_i c_i$ for each i. Observe that the e_i are mutually orthogonal idempotents in $\mathbf{B}(S)$ since $c_i \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1) = \mathbf{B}_p(Q(R))$ and $SsS \subseteq D = \bigoplus e_i S$. Now we *claim* that $SsS_S \leq^{\text{ess}} D_S$. Let $0 \neq y \in D$. There exist $y_i \in S$ such that $y = \sum e_i y_i$. In this case, there is $e_j y_j \neq 0$ for some j and $v \in R$ with $0 \neq e_j y_j v \in R$. Since $ye_jv = e_jy_jv = b_jc_jy_jv \in c_jR$ and $Rr_jR_R \leq^{\text{ess}} c_jR_R$, there exists $w \in R$ such that $0 \neq ye_jvw \in Rr_jR$. Hence $0 \neq e_jy_jvw \in Rr_je_jR = Rse_jR \subseteq SsS$ because $se_j = r_je_j$ and $e_j = b_jc_j \in S$. Since $e = \sum e_i \in \mathbf{B}(S)$ and $SsS_S \leq^{\text{ess}} D_S = \bigoplus e_iS_S = eS_S$, it follows that $S \in \mathfrak{p}\mathfrak{F}\mathfrak{I}$. Hence $S = R(\mathfrak{p}\mathfrak{F}\mathfrak{I}, Q(R))$.

Next we assume that T is a right ring of quotients of R and $T \in \mathfrak{p}\mathfrak{F}\mathfrak{I}$. Take $e \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$. Then by the above claim, $e(1) \in \mathbf{B}_p(Q(R))$. So there is $x \in R$ such that $RxR_R \leq^{\text{ess}} e(1)R_R$. Hence $RxR_R \leq^{\text{ess}} e(1)Q(R)_R$. Note that $TxT = T(RxR)T \subseteq T(e(1)Q(R))T = e(1)Q(R)$. Thus $TxT_R \leq^{\text{ess}} e(1)Q(R)_R$, so $TxT_R \leq^{\text{ess}} e(1)Q(R)_R$. Hence $TxT_T \leq e(1)Q(R)_T$ from [12, Lemma 1.4(i)] because $R_R \leq^{\text{den}} T_R$. Therefore $TxT_T \leq^{\text{ess}} e(1)T_T$. On the other hand, since $T \in \mathfrak{p}\mathfrak{F}\mathfrak{I}$, there exists $c = c^2 \in T$ such that $TxT_T \leq^{\text{ess}} cT_T$. Thus $e(1) = c$ because $e(1) \in \mathbf{B}(Q(R))$. Hence $e(1) \in T$ for each $e(1) \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$. So S is a subring of T. Therefore $S = \widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$.

(ii) It is a direct consequence of part (i) and Lemma 3.

(iii) As in the proof of part (i), we can verify that $\delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R)(1) = \{e \in \mathbf{B}(Q(R)) \mid$ there is a finitely generated ideal I of R with $I_R \leq^{\text{ess}} eR_R\}$. A proof similar to that used in part (i) yields that

$$\langle R \cup \delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R)(1)\rangle_{Q(R)} = R(\mathfrak{fg}\mathfrak{F}\mathfrak{I}, Q(R)) = \widehat{Q}_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R).$$

Since $\delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1) \subseteq \delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R)(1)$, $\widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) \subseteq \widehat{Q}_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R)$. By Lemma 3, $\widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) \in \mathfrak{fg}\mathfrak{F}\mathfrak{I}$, so $\widehat{Q}_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R) \subseteq \widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$. Thus $\widehat{Q}_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R) = \widehat{Q}_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)$. □

Recall that a ring R is *left π-regular* if for each $a \in R$ there exist $b \in R$ and a positive integer n such that $a^n = ba^{n+1}$. Note from [17] that the class of special radicals includes most well known radicals (e.g., the prime radical, the Jacobson radical, the Brown-McCoy radical, the nil radical, the generalized nil radical, etc.). For a ring R, the *classical Krull dimension* $\mathrm{kdim}(R)$ is the supremum of all lengths of chains of prime ideals of R.

By Theorem 8, if R is a semiprime ring, then $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R) = R\mathbf{B}_p(Q(R))$, the subring of $Q(R)$ generated by R and $\mathbf{B}_p(Q(R))$. Thus we have the following corollaries which show the transference of certain properties between R and $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$. We use LO, GU, and INC for "lying over", "going up", and "incomparability", respectively (see [25, p.292]).

Corollary 9. Let R be a semiprime ring.

(i) If K is a prime ideal of $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$, then $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)/K \cong R/(K \cap R)$.

(ii) LO, GU, and INC hold between R and $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$.

Proof. The proof follows from Theorem 8 and [14, Lemma 2.1]. □

Corollary 10. Assume that R is a semiprime ring. Then:

(i) $\varrho(R) = \varrho(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)) \cap R$, where $\varrho(-)$ is a special radical of a ring.

(ii) R is π-regular if and only if $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$ is π-regular.

(iii) $\mathrm{kdim}\,(R) = \mathrm{kdim}\,(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$.

Proof. Theorem 8 and [14, Theorem 2.2] yield this result. □

Corollary 11. Let R be a semiprime ring. Then:

(i) R is (von Neumann) regular if and only if $Q_{\mathfrak{pq}\mathfrak{B}}(R)$ is (von Neumann) regular.

(ii) R is strongly regular if and only if $Q_{\mathfrak{pq}\mathfrak{B}}(R)$ is strongly regular.

(iii) R has bounded index at most n if and only if $Q_{\mathfrak{pq}\mathfrak{B}}(R)$ has bounded index at most n.

Proof. This can be verified by Theorem 8 and similar arguments as used in the proof of [14, Corollary 3.6 and Theorem 3.8]. □

Let $\mathfrak{q}\mathfrak{B}$ be the class of quasi-Baer rings. In [14, Theorem 3.3], it is shown that there exist $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ and $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R)$ for each semiprime ring R.

Theorem 12. (cf. [14, Theorem 3.3]) Let R be a semiprime ring. Then $\widehat{Q}_{\mathfrak{F}\mathfrak{I}}(R) = R\mathbf{B}(Q(R)) = R(\mathfrak{F}\mathfrak{I}, Q(R))$.

From Theorem 12 and [5, Theorem 3.5], one can see that for a semiprime ring R, $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ is the smallest right ring of quotients of R which is right p.q.-Baer and has a complete lattice of annihilator ideals. However, in general, $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ is a proper subring of $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ as in the next example.

Example 13. (i) Let F be a field and let $F_n = F$ for all positive integer n. Put

$$R = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid a_n \text{ is eventually constant} \right\},$$

which is a subring of $\prod_{n=1}^{\infty} F_n$. Then $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R) = R$, but $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) = \prod_{n=1}^{\infty} F_n$.

(ii) Let R be a biregular ring (i.e., every principal ideal of R is generated by a central idempotent). Then $R = \widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ and if its lattice of principal ideals is not complete then $R \neq \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ (see [5, Theorem 3.5]). In fact, let $R = \{(d_n) \in \prod_{n=1}^{\infty} D_n \mid d_n \text{ is eventually constant}\}$, a subring of $\prod_{n=1}^{\infty} D_n$ where $D_n = D$ is a division ring for all n. Then R is biregular, so $R = \widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$, but $R \neq \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ by Theorem 8 because $\mathbf{B}(Q(R)) \not\subseteq R$ or by [5, Theorem 3.5].

Despite Example 13, we have the following result in which $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ does coincide with $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$. Recall that the *extended centroid* of R is $\text{Cen}(Q(R))$.

Theorem 14. Assume that R is a semiprime ring with only finitely many minimal prime ideals, say $P_1, \ldots, P_n$. Then $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ and $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R) \cong R/P_1 \oplus \cdots \oplus R/P_n$.

Proof. Since R has exactly n minimal prime ideals, the extended centroid $\text{Cen}(Q(R))$ of R has a complete set of primitive idempotents with n elements by [1, Theorem 11]. Note that the extended centroid of R is equal to that of $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$. Thus $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ also has exactly n minimal prime ideals by [1, Theorem 11]. By [11, Theorem 3.4] and [9, Theorem 3.7], $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ is quasi-Baer and so $\widehat{Q}_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R) = \widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$. The rest of the proof follows from [15, Theorem 3.15]. □

Theorem 15. Let R be a reduced ring. Then $Q_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$ exists and is the PP absolute right ring hull of R.

Proof. Note that since R is reduced, then $Q(R) = E(R_R)$; and so $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$ for any class $\mathfrak{K}$ of rings. By Theorem 8, $Q_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) = Q_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$. Let $S = Q_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R) = Q_{\mathfrak{p}\mathfrak{q}\mathfrak{B}}(R)$. From [9, Corollary 1.15], S is right (and left) PP.

Suppose A is a right ring of quotients of R which is right PP. Let $e \in \delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1)$. (Note that $\delta_{\mathfrak{p}\mathfrak{F}\mathfrak{I}}(R)(1) = \mathbf{B}_p(Q(R)) \subseteq \mathbf{B}(Q(R))$ as in the proof of Theorem 8.) Then there exists $x \in R$ such that $RxR_R \leq^{\text{ess}} eR_R$. So we have that $SxS_S \leq^{\text{ess}} eS_S$. Since S is semiprime and e is a central idempotent in S, it follows that $\ell_{eS}(SxS) = r_{eS}(SxS) = 0$ by noting that the ring S is semiprime. Therefore $r_S(SxS) = (1-e)S$. Moreover, since $Q_{\mathfrak{q}\mathfrak{B}}(R)$ is reduced by [14, Theorem 3.8], so is S $(\subseteq Q_{\mathfrak{q}\mathfrak{B}}(R))$. Thus $r_S(x) = r_S(SxS) = (1-e)S$. Since A is right PP, there exists $f \in \mathbf{I}(A)$ such that $r_A(x) = fA$. Then $r_R(x) = (1-e)S \cap R$ and $r_R(x) = r_A(x) \cap R$. Hence $r_R(x)_R \leq^{\text{ess}} (1-e)S_R \leq^{\text{ess}} (1-e)Q(R)_R$ and $r_R(x)_R \leq^{\text{ess}} fA_R \leq^{\text{ess}} fQ(R)_R$. Therefore

$$r_R(x)_R \leq^{\text{ess}} ((1-e)Q(R) \cap fQ(R))_R = f(1-e)Q(R)_R$$

because $1-e$ is central. Thus $(1-e)Q(R) = f(1-e)Q(R) = fQ(R)$, so $1-e = f$. Therefore $e = 1-f \in A$, hence $Q_{\mathfrak{pq}\mathfrak{B}}(R) = S \subseteq A$ by Theorem 8. □

Note that Theorem 15 shows that when R is a commutative semiprime ring, $Q_{\mathfrak{pq}\mathfrak{B}}(R)$ is related to the *Baer extension* considered in [19]. Also note that the generalized nil radical, $\mathbf{N}_g$ [17], is the radical whose semisimple class is the class of reduced rings. Hence for every ring R such that $R \neq \mathbf{N}_g(R)$, R has a nontrivial homomorphic image, $R/\mathbf{N}_g(R)$, which has a Baer absolute right ring hull and a right PP absolute right ring hull.

A monoid G is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets A, $B \subseteq G$ there exists an element $x \in G$ uniquely presented in the form ab, where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [24] and [22]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid is cancellative, and every u.p.-group is torsion-free.

Theorem 16. Let $R[G]$ be a semiprime monoid ring of a monoid G over a ring R. Then:

(i) $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[G] \subseteq \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[G])$.

(ii) If G is a u.p.-monoid, then $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[G]) = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[G]$.

Proof. (i) To show that $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[G] \subseteq \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[G])$, we *claim* that $\mathbf{B}_p(Q(R)) \subseteq \mathbf{B}_p(Q(R[G]))$. To prove the claim, let $e \in \mathbf{B}_p(Q(R))$. Then there exists $a \in R$ such that $RaR_R \leq^{\text{ess}} eR_R$. Since $R[G]$ is a free right R-module, a routine argument shows that $(RaR)[G]_R \leq^{\text{ess}} eR[G]_R$. Thus $(RaR)[G]_{R[G]} \leq^{\text{ess}} eR[G]_{R[G]}$. Since $\mathbf{B}_p(Q(R)) \subseteq \mathbf{B}(Q(R[G]))$ from the proof of part (i), $e \in \mathbf{B}(Q(R[G]))$. So $e \in \mathbf{B}_p(Q(R[G]))$ because $(RaR)[G] = R[G]aR[G]$. Hence $\mathbf{B}_p(Q(R)) \subseteq \mathbf{B}_p(Q(R[G]))$. Theorem 8 shows that $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[G] \subseteq \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[G])$.

(ii) This is a consequence of part (i) and [11, Theorem 1.2]. □

Corollary 17. Let R be a semiprime ring. Then $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[x, x^{-1}]) = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[x, x^{-1}]$ and $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[X]) = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[X]$, where X a nonempty set of not necessarily commuting indeterminates.

Proof. Note that $R[x, x^{-1}] \cong R[C_\infty]$, which is semiprime, where C_∞ is the infinite cyclic group. Since R is semiprime, so is $R[X]$. Thus $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[x, x^{-1}]) = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[x, x^{-1}]$ and $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[X]) = \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[X]$ follow from Theorem 16. □

Example 18. There is a semiprime ring R such that $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[[x]]) \neq \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[[x]]$. In [6, Example 2.3], there is a commutative (von Neumann) regular ring R (hence right p.q.-Baer), but the ring $R[[x]]$ is not right p.q.-Baer. Thus $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R) = R$ and so $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[[x]] = R[[x]]$. Since $R[[x]]$ is not right p.q.-Baer, $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R[[x]]) \neq \widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)[[x]]$.

Let R be a ring. Then the subring $R\mathbf{B}(Q(R))$ of $Q(R)$ generated by R and $\mathbf{B}(Q(R))$ is called the *idempotent closure* of R (see [2]). From the following lemma, one can see that the idempotent closure of $\mathrm{Mat}_n(R)$ is the matrix ring of n-by-n matrices over the idempotent closure of R and similarly for $T_n(R)$. Let 1_n denote the unity of $\mathrm{Mat}_n(R)$.

Lemma 19. Let $\delta \subseteq \mathbf{B}(Q(R))$ and $\Delta = \{1_n c \mid c \in \delta\}$. Then:

(i) $\mathrm{Mat}_n(\langle R \cup \delta\rangle_{Q(R)}) = \langle \mathrm{Mat}_n(R) \cup \Delta\rangle_{Q(\mathrm{Mat}_n(R))}$.

(ii) $Q(T_n(R)) = Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R))$.

(iii) $T_n(\langle R \cup \delta\rangle_{Q(R)}) = \langle T_n(R) \cup \Delta\rangle_{Q(\mathrm{Mat}_n(R))}$.

Proof. (i) This part follows from straightforward calculation.

(ii) Let $T = T_n(R)$. By routine calculations, T_T is dense in $\mathrm{Mat}_n(R)_T$. So we have that $Q(T_n(R)) = Q(\mathrm{Mat}_n(R))$. From [27, 2.3], $Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R))$. Thus it follows that $Q(T_n(R)) = Q(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(Q(R))$.

(iii) This follows from part (ii) and a routine calculation. □

Theorem 20. Let R be a semiprime ring. Then $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{K}}(R))$, where $\mathfrak{K} = \mathfrak{pq}\mathfrak{B}$, $\mathfrak{p}\mathfrak{F}\mathfrak{I}$, or $\mathfrak{fg}\mathfrak{F}\mathfrak{I}$.

Proof. Assume that $\mathfrak{K} = \mathfrak{pq}\mathfrak{B}$, $\mathfrak{p}\mathfrak{F}\mathfrak{I}$, or $\mathfrak{fg}\mathfrak{F}\mathfrak{I}$. By Theorem 8, it follows that $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \langle \mathrm{Mat}_n(R) \cup \delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(\mathrm{Mat}_n(R))(1_n)\rangle_{Q(\mathrm{Mat}_n(R))}$. Observe that if J is a finitely generated ideal of $\mathrm{Mat}_n(R)$, then there is a finitely generated ideal I of R such that $J = \mathrm{Mat}_n(I)$. Thus $\delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(\mathrm{Mat}_n(R))(1_n) = \{1_n c \mid c \in \delta_{\mathfrak{fg}\mathfrak{F}\mathfrak{I}}(R)(1_n)\}$. So Lemma 19 and Theorem 8 yield that $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{K}}(R))$. □

Theorem 21. Let R be a semiprime ring. Then $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(T_n(R)) = T_n(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$.

Proof. Let $T = T_n(R)$ and S be a right ring of quotients of T. From [9, Proposition 2.6], $T_n(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$ is a right p.q.-Baer ring. Assume that S is a right p.q.-Baer ring. Take $e \in \mathbf{B}_p(Q(R))$. Then there exists $x \in R$ such that $RxR_R \leq^{\text{ess}} eR_R$, hence $RxR_R \leq^{\text{ess}} eQ(R)_R$. Therefore $Q(R)xQ(R)_{Q(R)} \leq^{\text{ess}} eQ(R)_{Q(R)}$. Thus $eQ(R)xQ(R)e_{eQ(R)e} \leq^{\text{ess}} eQ(R)e_{eQ(R)e}$ because $e \in \mathbf{B}_p(Q(R)) \subseteq \mathbf{B}(Q(R))$. Since $eQ(R)e$ is a semiprime ring, $0 = r_{eQ(R)e}(eQ(R)xQ(R)e) = r_{Q(R)}(eQ(R)xq(R)e) \cap eQ(R)e = r_{Q(R)}(Q(R)xQ(R)) \cap eQ(R)$. So we have that $r_{Q(R)}(Q(R)xQ(R))eQ(R) = r_{Q(R)}(Q(R)xQ(R))Q(R)e = 0$. Hence $r_{Q(R)}(Q(R)xQ(R)) \subseteq (1-e)Q(R)$. Obviously, $(1-e)Q(R) \subseteq r_{Q(R)}(Q(R)xQ(R))$. Thus $r_{Q(R)}(Q(R)xQ(R)) = (1-e)Q(R)$.

Next we show that $r_{Q(R)}(RxR) = (1-e)Q(R)$. For this, first note that $(1-e)Q(R) = r_{Q(R)}(Q(R)xQ(R)) \subseteq r_{Q(R)}(RxR)$. Thus by the modular law, $r_{Q(R)}(RxR) = (1-e)Q(R) \oplus [r_{Q(R)}(RxR) \cap eQ(R)]$. Asuume to the contrary that $r_{Q(R)}(RxR) \cap eQ(R) \neq 0$. Take $0 \neq eq \in r_{Q(R)}(RxR) \cap eQ(R)$ with $q \in Q(R)$. Since $RxR_R \leq^{\text{ess}} eQ(R)_R$, there exists $r \in R$ such that $0 \neq eqr \in RxR$. Thus

$eqr \in r_{Q(R)}(RxR) \cap R = r_R(RxR)$. So $eqr \in RxR \cap r_R(RxR) = 0$ because R is semiprime. This is absurd. So $r_{Q(R)}(RxR) \cap eQ(R) = 0$. Therefore $r_{Q(R)}(RxR) = (1-e)Q(R)$.

Let $\theta \in T = T_n(R)$ be the n-by-n matrix with x in the (1,1)-position and 0 elsewhere. Thus $T\theta T$ is the n-by-n matrix with RxR throughout the top row and 0 elsewhere. Moreover, $Q(T)\theta Q(T) = \mathrm{Mat}_n(Q(R)xQ(R))$. Since $T\theta T \subseteq S\theta S \subseteq Q(T)\theta Q(T)$ and $r_{Q(R)}(RxR) = (1-e)Q(R)$, we have that

$$(1-f)Q(T) = r_{Q(T)}(Q(T)\theta Q(T)) \subseteq r_{Q(T)}(S\theta S) \subseteq r_{Q(T)}(T\theta T) = (1-f)Q(T),$$

where f is the diagonal matrix with e on the diagonal. Since S is right p.q.-Baer, there exists $c = c^2 \in S$ such that $cS = r_S(S\theta S) = S \cap r_{Q(R)}(S\theta S) = S \cap (1-f)Q(T)$. Thus $cQ(T) \subseteq (1-f)Q(T)$. Let $0 \neq (1-f)q \in (1-f)Q(T)$ with $q \in Q(T)$. Then $0 \neq (1-f)qQ(T) \cap S \subseteq (1-f)Q(T) \cap S = cS \subseteq cQ(T)$. Hence $0 \neq (1-f)q\alpha \in cQ(T)$ with $\alpha \in Q(T)$. So $cQ(T)_{Q(T)} \leq^{\text{ess}} (1-f)Q(T)_{Q(T)}$ and hence $c = 1-f$. Thus $f = 1-c \in S$. Therefore $T_n(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)) \subseteq S$ by Theorem 8. So $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(T)$ also exists and $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(T) = T_n(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$. □

For a ring R and a nonempty set Γ, $\mathrm{CFM}_\Gamma(R), \mathrm{RFM}_\Gamma(R)$, and $\mathrm{CRFM}_\Gamma(R)$ denote the column finite, the row finite, and the column and row finite matrix rings over R indexed by Γ, respectively.

Theorem 22. ([13, Theorem 19]) (i) $R \in \mathfrak{q}\mathfrak{B}$ if and only if $\mathrm{CFM}_\Gamma(R)$ (resp., $\mathrm{RFM}_\Gamma(R)$ and $\mathrm{CRFM}_\Gamma(R)) \in \mathfrak{q}\mathfrak{B}$.

(ii) If $R \in \mathfrak{F}\mathfrak{I}$, then $\mathrm{CFM}_\Gamma(R)$ (resp., $\mathrm{CRFM}_\Gamma(R)) \in \mathfrak{F}\mathfrak{I}$.

(iii) If R is semiprime, then we have that $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R))$, $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R))$, and $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R))$.

Theorems 15 and 21 motivate the following questions: (1) *Is the right p.q.-Baer property preserved under the various infinite matrix ring extensions?* (2) *Does $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)$ of a ring R have behavior similar to that of $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)$ for the various infinite matrix ring extensions?* Our next example provides negative answers to these questions.

Example 23. Let F be a field and $F_n = F$ for $n = 1, 2 \ldots$. Put

$$R = \left\{ (q_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} F_n \mid q_n \text{ is eventually constant} \right\},$$

which is a subring of $\prod_{n=1}^{\infty} F_n$. Then R is a commutative (von Neumann) regular ring. Hence R is a right p.q.-Baer ring. Let $S = \mathrm{CFM}_\Gamma(R)$, where $\Gamma = \{1, 2, \ldots\}$. Take

$$a_1 = (0,1,0,0,\ldots),\ a_2 = (0,1,0,1,0,0,\ldots),\ a_3 = (0,1,0,1,0,1,0,0,\ldots),$$

and so on, in R. Let x be the element in S with a_n in the (n,n)-position for $n = 1, 2, \ldots$ and 0 elsewhere, and let

$$e = (q_n)_{n=1}^{\infty} \in Q(R) = \prod_{n=1}^{\infty} F_n$$

such that $q_{2n} = 1$ and $q_{2n-1} = 0$ for $n = 1, 2, \ldots$. Then $e = e^2 \in \mathbf{B}(Q(R))$, hence

$$eI \in \mathrm{CFM}_{\Gamma}(\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R)) \subseteq Q(S)$$

because $\widehat{Q}_{\mathfrak{q}\mathfrak{B}}(R) = R\mathbf{B}(Q(R))$ from Theorem 12, where I is the unity matrix in S. Therefore $eI \in \mathbf{B}(Q(S))$. Also note that $(\sum a_i R)_R \leq^{\text{ess}} eR_R$. We *claim* that

$$SxS_S \leq^{\text{ess}} (eI)S_S.$$

For convenience, let E_{ij} be the matrix in S with 1 in the (i,j)-position and 0 elsewhere. Take $0 \neq (eI)s \in (eI)S$ with $s = (r_{ij}) \in S$. Then there is a nonzero column, say the m-th column, of $(eI)s$. In this case the m-th column of $(eI)s$ is the same as the first column of $(eI)sE_{m1}$. Thus the first column of $(eI)E_{m1}$ is nonzero and all other columns except the first column of $(eI)E_{m1}$ are zero. So without loss of generality, we may assume that the first column of the matrix $(eI)s$ is nonzero and all the other columns except the first column are zero. In the first column of $(eI)s$, there are only finitely many nonzero entries, say

$$er_{k_1 1}, er_{k_2 1}, \ldots, er_{k_n 1}$$

with

$$k_1 < k_2 < \cdots < k_n.$$

To show that $SxS_S \leq^{\text{ess}} (eI)S_S$, we proceed by induction. Suppose $n = 1$. Since $(\sum a_i R)_R \leq^{\text{ess}} eR_R$, there exist $b_1, \lambda_1, \ldots, \lambda_m \in R$ such that $0 \neq er_{k_1 1}b_1 = a_1\lambda_1 + \cdots + a_m\lambda_m$. Thus $0 \neq (eI)s(b_1E_{11}) = (\lambda_1 E_{k_1 1} + \cdots + \lambda_m E_{k_1 m}) \cdot x \cdot (E_{11} + \cdots + E_{m1}) \in SxS$.

Next consider the case for $n > 1$. Since $(\sum a_i R)_R \leq^{\text{ess}} eR_R$, there is $b_1 \in R$ such that $0 \neq er_{k_1 1}b_1 \in \sum a_i R$. If $er_{k_i 1}b_1 = 0$ for some i with $1 < i \leq n$, then we are done by induction. So $er_{k_i 1}b_1 \neq 0$ for all $i = 1, 2, \ldots, n$. Assume that $er_{k_2 1}b_1 \notin \sum a_i R$. There exists $b_2 \in R$ with $0 \neq er_{k_2 1}b_1 b_2 \in \sum a_i R$. In this case, note that $er_{k_1 1}b_1 b_2 \in \sum a_i R$. Suppose $er_{k_i 1}b_1 b_2 = 0$ for some $i \neq 2$. Again we are done by induction. Next if $er_{k_3 1}b_1 b_2 \notin R$, then there is $b_3 \in R$ such that $0 \neq er_{k_3 1}b_1 b_2 b_3 \in \sum a_i R$ and $er_{k_i 1}b_1 b_2 b_3 \neq 0$ for all i. Also note that $er_{k_1 1}b_1 b_2 b_3$, $er_{k_2 1}b_1 b_2 b_3$, $er_{k_3 1}b_1 b_2 b_3 \in \sum a_i R$. Continue this process, it follows that there are $b_1, b_1, \ldots, b_n \in R$ with $er_{k_i 1}b_1 b_2 \cdots b_n \neq 0$ and $er_{k_i 1}b_1 b_2 \cdots b_n \in \sum a_i R$ for all i. Let $b = b_1 b_2 \cdots b_n$. Then there is a positive integer ℓ and $\lambda_{ij} \in R$ such that

$$er_{k_1 1}b = a_1\lambda_{11} + a_2\lambda_{12} + \cdots + a_\ell\lambda_{1\ell},\ er_{k_2 1}b = a_1\lambda_{21} + a_2\lambda_{22} + \cdots + a_\ell\lambda_{2\ell}, \ldots,$$

and

$$er_{k_n 1}b = a_1\lambda_{n1} + a_2\lambda_{n2} + \cdots + a_\ell\lambda_{n\ell}.$$

Thus

$$0 \neq (eI)s(bE_{11}) = (\lambda_{11}E_{k_1 1} + \cdots + \lambda_{1\ell}E_{k_1\ell} + \lambda_{21}E_{k_2 1} + \cdots + \lambda_{2\ell}E_{k_2\ell} + \cdots + \lambda_{n1}E_{k_n 1} + \cdots + \lambda_{n\ell}E_{k_n\ell}) \cdot x \cdot (E_{11} + \cdots E_{\ell 1}) \in SxS.$$

Therefore $SxS_S \leq^{\text{ess}} (eI)S_S$, hence $eI \in \mathbf{B}_p(Q(S))$. But note that $eI \notin S$. Observe that S is a semiprime ring because R is semiprime. Thus the ring S is not right p.q.-Baer by Theorem 8(ii). Furthermore, since R is right p.q.-Baer, $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R) = R$. Thus we have that $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(\text{CFM}_\Gamma(R)) \not\subseteq \text{CFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$. Also $\text{CFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$ is not right p.q.-Baer.

For $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(\text{CRFM}_\Gamma(R)) \not\subseteq \text{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$, let x and e be as in the case of the column finite matrix ring. Then, by the same method, we can show that $eI \in \mathbf{B}_p(Q(\text{CRFM}_\Gamma(R)))$; but $eI \notin \text{CRFM}_\Gamma(R)$. So $\text{CRFM}_\Gamma(R)\,(= \text{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$ is not right p.q.-Baer by Theorem 8(ii). Also we have that

$$\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(\text{CRFM}_\Gamma(R)) \not\subseteq \text{CRFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R)).$$

Finally for $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(\text{RFM}_\Gamma(R)) \not\subseteq \text{RFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$, let $U = \text{RFM}_\Gamma(R)$ and x, e be as before. Then by modifying the method used for the case of column finite matrix rings, it can be shown that

$$_U UxU \leq^{\text{ess}} {}_U(eI)U = {}_U(eI)U,$$

where I is the unity matrix in U. Note eI is a central idempotent. So we have that $_{(eI)U(eI)}UxU \leq^{\text{ess}} {}_{(eI)U(eI)}(eI)U(eI)$. Since UxU is an ideal of the semiprime ring $(eI)U(eI)$, $r_{(eI)U(eI)}(UxU) = \ell_{(eI)U(eI)}(UxU) = 0$. So $UxU_{(eI)U(eI)} \leq^{\text{ess}} (eI)U(eI)_{(eI)U(eI)}$. Thus $UxU_U \leq^{\text{ess}} (eI)U_U$. Moreover, since $e \in \mathbf{B}(Q(R)) = \mathbf{B}(Q^m(R))$, there exists $J \trianglelefteq R$ such that $\ell_R(J) = 0$ and $eJ \subseteq R$. Thus

$$\text{RFM}_\Gamma(J) \trianglelefteq \text{RFM}_\Gamma(R),\ \ell_{\text{RFM}_\Gamma(R)}(\text{RFM}_\Gamma(J)) = 0,$$

and

$$(eI)\text{RFM}_\Gamma(J) \subseteq \text{RFM}_\Gamma(R),$$

where I is the unity matrix in $\text{RFM}_\Gamma(R)$. So $eI \in Q^m(\text{RFM}_\Gamma(R))$. Hence we have that $eI \in \mathbf{B}(Q^m(\text{RFM}_\Gamma(R)))$. So $eI \in \mathbf{B}(Q(U))$, hence $eI \in \mathbf{B}_p(Q(U))$. But $eI \notin U$. Therefore $U = \text{RFM}_\Gamma(R)\,(= \text{RFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$ is not right p.q.-Baer by Theorem 8. Thus $\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(\text{RFM}_\Gamma(R)) \not\subseteq \text{RFM}_\Gamma(\widehat{Q}_{\mathfrak{pq}\mathfrak{B}}(R))$.

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