A Theory of Hulls for Rings and Modules

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Abstract. In this expository paper, we survey results on the concept of a hull of a ring or a module with respect to a specific class of rings or modules. A hull is a ring or a module which is minimal among essential overrings or essential overmodules from a specific class of rings or modules, respectively. We begin with a brief history highlighting various types of hulls of rings and modules. The general theory of hulls is developed through the investigation of four problems with respect to various classes of rings including the (quasi-) Baer and (FI-) extending classes. In the final section, application to C^*-algebras are provided.

Mathematics Subject Classification (2000). Primary 16N60, 16S90; Secondary 16D50, 16S34, 46L05.

Keywords. FI-extending, ring hull, module hull, quasi-Baer rings, principally quasi-Baer ring, Baer ring, ring of quotients, essential overring, involution, C^*-algebra, local multiplier algebra, boundedly centrally closed.

1. INTRODUCTION

Throughout this paper all rings are associative with unity unless indicated otherwise and R denotes such a ring. Subrings and overrings preserve the unity of the base ring. Ideals without the adjective "right" or "left" mean two-sided ideals. All modules are unital and we use M_R (resp., ${}_RM$) to denote a right (resp., left) R-module.

If N_R is a submodule of M_R, then N_R is called *essential* (resp., *dense* also called *rational*) in M_R if for all $0 \neq x \in M$, there exists $r \in R$ such that $0 \neq xr \in N$ (resp., for $x, y \in M$ with $y \neq 0$, there exists $r \in R$ such that $xr \in N$ and $yr \neq 0$). We use $N_R \leq^{\text{ess}} M_R$ and $N_R \leq^{\text{den}} M_R$ to denote that N_R is an essential submodule of M_R and N_R is a dense submodule of M_R, respectively.

Recall that a *right ring of quotients* T of R is an overring of R such that R_R is dense in T_R. The maximal right (resp., left) ring of quotients of R is denoted by $Q(R)$ (resp., $Q^\ell(R)$). We say that T is a *right essential overring* of R if T is an overring of R such that R_R is essential in T_R. The right injective hull of R is denoted by $E(R_R)$ and we use $\mathcal{E}_R$ to denote $\text{End}(E(R_R))$. Unless noted otherwise, we work with right sided concepts. However most of the results and concepts have left sided analogues.

One of the major efforts in Ring Theory has been, for a given ring R, to find a "well behaved" overring Q in the sense that it has better properties than R and such that a rich information transfer between R and Q can take place. Alternatively, given a "well behaved" ring, to find conditions which describe those subrings for which there is some fruitful transfer of information.

The search for such overrings motivates the notion of a hull (i.e., an overring that is "close to" the base ring, in some sense, so as to facilitate the transfer of information). Since we want the overring to have some "desirable properties" the hull should come from a class of rings possessing these properties.

In 1999, the authors embarked on a research program to develop methods that enable one to select a specific class $\mathfrak{K}$ of rings and then to describe all right essential overrings or all right rings of quotients of a given ring R which lie in $\mathfrak{K}$. Moreover, the transfer of information between the base ring R and the essential overring in the class $\mathfrak{K}$ is also investigated.

We have tried to make our definitions flexible enough to encompass the existing theory, apply to many classes of rings, and shed new light on the relationship between a base ring and its essential overrings.

Much of the current theory of rings of quotients emphasizes investigating when a relatively small number of right rings of quotients of R (e.g., its classical right ring of quotients $Q^r_{c\ell}(R)$, the symmetric ring of quotients $Q^s(R)$, the Martindale right ring of quotients $Q^m(R)$, and $Q(R)$) are in a few standard classes of rings (e.g., semisimple Artinian, right Artinian, right Noetherian, right self-injective, or regular).

Some of the deficiencies of this approach are illustrated in the following examples. First take $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the ring of rational numbers, respectively. The ring R is neither right nor left Noetherian and its prime radical is nonzero. However, $Q(R)$ is simple Artinian. Next take R to be a domain which does not satisfy the right Ore condition. Then $Q(R)$ is a simple right self-injective regular ring which has an infinite set of orthogonal idempotents and an unbounded nilpotent index. The sharp disparity between R and $Q(R)$ in the aforementioned examples limits the transfer of information between R and $Q(R)$. These examples illustrate a need to find overrings of a given ring that have some weaker versions of the properties traditionally associated with right rings of quotients such as mentioned above. Furthermore, this need is reinforced when one studies classes of rings for which $R = Q(R)$ (e.g., right Kasch rings). For these classes the theory of right rings of quotients is virtually useless.

Our theory makes no particular restriction on the classes that we consider for our essential overrings. Further, the properties of the classes determine the existence and characterizations of the hulls which may not coincide with $Q^r_{c\ell}(R)$, $Q^s(R)$,

$Q^m(R)$, or $Q(R)$. However those classes which are generalizations of the class of right self-injective rings, regular rings, or classes which are closed under dense or essential extensions work especially well with our methods.

We recall the definitions of some of the classes that generalize the class of right self-injective rings or the class of regular right self-injective rings. A ring R is: right (*FI-*) *extending* if every (ideal) right ideal of R is essential in a right ideal generated by an idempotent; right (*quasi-*) *continuous* if R is right extending and (if A_R and B_R are direct summands of R_R with $A \cap B = 0$, then $A_R \oplus B_R$ is a direct summand of R_R) R satisfies the (C_2) condition, that is, if X and Y are right ideals of R with $X_R \cong Y_R$ and X_R is a direct summand of R_R, then Y_R is a direct summand of R_R; (*quasi-*) *Baer* if the right annihilator of every (ideal) nonempty subset of R is an idempotent generated right ideal. The classes of Baer rings, quasi-Baer rings, right extending rings, right FI-extending rings, right continuous rings, and right quasi-continuous rings are denoted by $\mathbf{B}, \mathbf{qB}$, $\mathbf{E}$, $\mathbf{FI}$, $\mathbf{Con}$, and $\mathbf{qCon}$, respectively (See [11, 58, 63, 76] for $\mathbf{B}$, [13, 15, 16, 17, 20, 23, 39, 73, 76] for $\mathbf{qB}$, [37, 38, 43] for $\mathbf{E}$, [18, 22, 23, 29] for $\mathbf{FI}$, and [49, 64, 65, 84, 85] for $\mathbf{Con}$ and $\mathbf{qCon}$.)

Recall from [14] that a ring R is called *right principally quasi-Baer* (simply, *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated, as a right ideal, by an idempotent (equivalently, R modulo the right annihilator of each principal right ideal is projective). Left principally quasi-Baer (simply, left p.q.-Baer) rings are defined similarly. Rings which are both right and left principally quasi-Baer are called *principally quasi-Baer* (simply, *p.q.-Baer*) rings. We use $\mathbf{pqB}$ to denote the class of right p.q.-Baer rings (see [14] and [30] for more details on right p.q.-Baer rings). A ring R is called *right PP* (also called *right Rickart*) if the right annihilator of each element is generated, as a right ideal, by an idempotent. Left PP (also called left Rickart) rings are defined in a similar way. Rings which are both right and left PP are called *PP* ring (also called Rickart rings).

The right essential overings, which are in some sense "minimal" with respect to belonging to a specific class of rings, are important tools in our investigations. Hence we define several types of ring hulls to accommodate the various notions of "minimality" among the class of right essential overrings of a given ring. Our search for such minimal overrings for a given ring R includes the seemingly unexplored region that lies between $Q(R)$ and $E(R_R)$ (e.g., when $R = Q(R)$). We consider two basic types (the others are their derivatives). Let S be a right essential overring of R and $\mathfrak{K}$ be a specific class of rings. We say that S is a $\mathfrak{K}$ *right ring hull* of R if S is minimal among the right essential overrings of R belonging to the class $\mathfrak{K}$ (i.e., whenever T is a subring of S where T is a right essential overring of R in the class $\mathfrak{K}$, then $T = S$). For the other basic type, we generate S with R and certain subsets of $E(R_R)$ so that S is in $\mathfrak{K}$ in some "minimal" fashion. This leads to our concepts of a $\mathfrak{C}$ pseudo and $\mathfrak{C}\,\rho$ pseudo right ring hull of R, where ρ is an equivalence relation on a certain set of idempotents from $\mathcal{E}_R$. These ring hull concepts are "tool" concepts in that they appear in the proofs of various results but do not appear in the statements of the results. Let $\mathbf{M}$ be a class of right R-modules and let M_R be a right R-module. The smallest essential extension of M_R (if it exists) in a fixed injective hull of M_R, that belongs to $\mathbf{M}$ is called the *absolute* $\mathbf{M}$ *hull* of M_R (see Definition 8.1 for details).

The following four problems provide the driving force for our program.

Problem I. Assume that a ring R and a class $\mathfrak{K}$ of rings are given.

(i) Determine conditions to ensure the existence of right rings of quotients and that of right essential overrings of R which are, in some sense, "minimal" with respect to belonging to the class $\mathfrak{K}$.

(ii) Characterize the right rings of quotients and the right essential overrings of R which are in the class $\mathfrak{K}$, possibly by using the "minimal" ones obtained in part (i).

Problem II. Given a ring R and a class $\mathfrak{K}$ of rings, determine what information transfers between R and its right essential overrings in $\mathfrak{K}$ (especially the right essential overrings which are, in some sense, "minimal" with respect to belonging to $\mathfrak{K}$).

Problem III. Given classes of rings $\mathfrak{K}$ and $\mathcal{S}$, determine those $T \in \mathfrak{K}$ such that $Q(T) \in \mathcal{S}$.

Problem IV. Given a ring R and a class of rings $\mathfrak{K}$, let $X(R)$ denote some standard type of extension of R (e.g., $X(R) = R[x]$, or $X(R) = \text{Mat}_n(R)$, the n-by-n matrix ring over R, etc.) and let $H(R)$ denote a right essential overring of R which is "minimal" with respect to belonging to the class $\mathfrak{K}$ (i.e., a hull). Determine when $H(X(R))$ is comparable to $X(H(R))$.

We recall from [13] that an idempotent e of a ring R is called *left* (resp., *right*) *semicentral* if $xe = exe$ (resp., $ex = exe$) for all $x \in R$. Observe that $e = e^2 \in R$ is left (resp., right) semicentral if and only if eR (resp., Re) is an ideal of R. We let $\mathbf{S}_\ell(R)$ (resp., $\mathbf{S}_r(R)$) denote the set of all left (resp., right) semicentral idempotents of R. Note that $\mathbf{S}_\ell(R) = \{0, 1\}$ if and only if $\mathbf{S}_r(R) = \{0, 1\}$. A ring R is said to be *semicentral reduced* if $\mathbf{S}_\ell(R) = \{0, 1\}$ or equivalently $\mathbf{S}_r(R) = \{0, 1\}$. We use $\mathcal{B}(R)$ to denote the set of all central idempotents of a ring R. It can be shown that $\mathcal{B}(R) = \mathbf{S}_\ell(R) \cap \mathbf{S}_r(R)$. If R is a semiprime ring, then $\mathcal{B}(R) = \mathbf{S}_\ell(R) = \mathbf{S}_r(R)$.

For a ring R, we use $\mathbf{I}(R), \mathbf{U}(R)$, $Z(R_R)$, $\text{Cen}(R)$, $P(R)$, and $J(R)$ to denote the idempotents, units, right singular ideal, center, prime radical, and Jacobson radical of R, respectively. For ring extensions of R, we use $R\mathcal{B}(Q(R))$ and $T_n(R)$ to denote the idempotent closure (i.e., the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$ [9]) and the n-by-n upper triangular matrix ring over R, respectively. For a nonempty subset X of a ring R, the symbols $r_R(X)$, $\ell_R(X)$, and $\langle X \rangle_R$ denote the right annihilator of X in R, the left annihilator of X in R, and the subring of R generated by X, respectively. Also $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ denote the field of rational numbers, the ring of integers, and the ring of integers modulo n, respectively. We use $I \trianglelefteq R$ to denote that I is an ideal of a ring R. Finally recall that a ring R is called *reduced* if R has no nonzero nilpotent elements and *Abelian* if $\mathbf{I}(R) = \mathcal{B}(R)$. Throughout the paper, a regular ring means a von Neumann regular ring.

We let $\mathcal{Q}_R = \mathrm{End}(_{\mathcal{E}_R}E(R_R))$ (recall that $\mathcal{E}_R = \mathrm{End}(E(R_R))$). Note that $Q(R) = 1 \cdot \mathcal{Q}_R$ (i.e., the canonical image of $\mathcal{Q}_R$ in $E(R_R)$) and that $\mathcal{B}(\mathcal{Q}_R) = \mathcal{B}(\mathcal{E}_R)$ [61, pp.94-96]. Also, $\mathcal{B}(Q(R)) = \{b(1) \mid b \in \mathcal{B}(\mathcal{Q}_R)\}$ [60, p.366]. Thus $R\mathcal{B}(\mathcal{E}_R) = R\mathcal{B}(Q(R))$. Recall that the *extended centroid* of R is $\mathrm{Cen}(Q(R))$. If R is semiprime, then $\mathrm{Cen}(Q(R)) = \mathrm{Cen}(Q^m(R)) = \mathrm{Cen}(Q^s(R))$ [60, pp.389-390], where $Q^m(R)$ and $Q^s(R)$ denote the Martindale right ring of quotients of R and the symmetric ring of quotients of R, respectively. (See [2] for more details on $Q^m(R)$.)

2. BRIEF HISTORY OF HULLS

In this section, we summarize the definitions and results that provide the background for our definitions of various ring hulls. The story begins in 1940 with the famous paper of R. Baer [8]. In that paper, Baer introduced the concept of an injective module by calling a module M_R *complete* (*injective* in current terminology) if to every right ideal I of R and to every R-homomorphism h of I into M_R there is some $m \in M$ with $h(x) = mx$ for all $x \in I$. This definition incorporates the celebrated "Baer Criterion". Moreover he proved the following result.

Theorem 2.1. ([8, Baer]) (i) A module M_R is injective if and only if whenever $M_R \leq N_R$ then M_R is a direct summand of N_R.

(ii) Every module is a submodule of an injective module.

Further, Baer indicated that each module can be embedded in some "essentially smallest" injective module. In 1952, Shoda [79] and independently in 1953 Eckmann and Schopf [44] explicitly established the existence of a minimal (up to isomorphism) injective extension (hull) of a module. Eckmann and Schopf characterized the injective hull of a module as its maximal essential extension.

Johnson and Wong [56], in 1961, defined a module K_R to be *quasi-injective* if for every R-homomorphism $h : S \to K$, of a submodule S of K, there is an $f \in \mathrm{End}(K_R)$ such that $f(s) = h(s)$ for all $s \in S$. They proved that every module M_R has a unique (up to isomorphism) quasi-injective hull in the following result.

Theorem 2.2 ([56, Johnson and Wong]) Let $E(M_R)$ be an injective hull of a module M_R. Take $\mathcal{E}_M = \mathrm{End}(E(M_R))$, and let $\mathcal{E}_M M_R$ denote the R-submodule of $E(M_R)$ generated by all the $h(M)$ where $h \in \mathcal{E}_M$. Then the following hold.

(i) $\mathcal{E}_M M_R$ is quasi-injective.

(ii) $\mathcal{E}_M M_R$ is the intersection of all quasi-injective submodules of $E(M_R)$ containing M_R.

(iii) M_R is quasi-injective if and only if $M_R = \mathcal{E}_M M_R$.

In 1963, J. Kist [59] defined a commutative PP ring $\overline{R}$ to be a *Baer extension* of a commutative PP ring R if the following conditions hold.

(i) R is (isomorphic to) a subring of $\overline{R}$;

(ii) $\mathfrak{C}(R)$ is (isomorphic to) a dense semilattice of $\mathfrak{C}(\overline{R})$, and the Boolean subalgebra of $\mathfrak{C}(\overline{R})$ is generated by this dense subsemilattice is all of $\overline{R}$, where

$\mathfrak{C}(-)$ consists of the open-and-closed sets in the hull-kernel topology on the set of minimal prime ideals of a ring; and

(iii) If $x \in \overline{R}$, then there exist finitely many idempotents $e_1, \dots, e_n$ in $\overline{R}$ which are mutually orthogonal, and whose sum is 1; and elements $x_1 \dots, x_n$ in R such that $x = e_1 x_1 + \cdots + e_n x_n$.

We note that in [59], Kist uses the terminology "Baer ring" for what are more commonly called PP rings. Thus a Baer extension may not be a Baer ring in the sense of Kaplansky [58]. Kist proved the following result.

Theorem 2.3. ([59, Kist]) If R is a commutative semiprime ring, then it has a Baer extension. Moreover, isomorphic rings have isomorphic Baer extensions.

For a commutative semiprime ring R, in 1968 H. Storrer [82], called the intersection of all regular subrings of $Q(R)$ containing R the *epimorphic hull* of R. By showing this intersection was regular, he showed that every commutative semiprime ring has a smallest regular ring of quotients.

The Baer hull, namely the ring $\mathbf{B}(R)$ in the next theorem, for a commutative semiprime ring R, was defined by Mewborn [63] in 1971.

Theorem 2.4. ([63, Mewborn]) Let R be a commutative semiprime ring. Let $\mathbf{B}(R)$ be the intersection of all Baer subrings of $Q(R)$ containing R. Then $\mathbf{B}(R)$ is a Baer ring and it is the subring of $Q(R)$ generated by R and $\mathbf{I}(Q(R))$.

In [66], Oshiro used sheaf theoretic methods to construct the Baer hull of a commutative regular ring.

The absolute π-injective (equivalently, quasi-continuous) hull of a module was defined by Goel and Jain in 1978 [49]. The following theorem is an immediate consequence of their results.

Theorem 2.5. ([49, Goel and Jain]) Let V be the subring of $\mathcal{E}_M = \mathrm{End}(E(M_R))$ generated by $\mathbf{I}(\mathcal{E}_M)$. Then VM_R is the unique (up to isomorphism) absolute quasi-continuous hull of M_R.

For any submodule A of a quasi-continuous module M, there exists a direct summand $P = M \cap E(A)$ of M which contains A as an essential submodule. This P is called *internal quasi-continuous hull* of A in M and was shown to be unique up to isomorphism by Müller and Rizvi [65].

Theorem 2.6. ([65, Müller and Rizvi]) Let M be a quasi-continuous module, A_1, A_2 submodules of M, and P_1, P_2 internal hulls of A_1 and A_2, respectively. If $A_1 \cong A_2$, then $P_1 \cong P_2$.

In 1982, Müller and Rizvi [64] defined three types of continuous hulls for modules as follows.

Definitions 2.7. ([64, Müller and Rizvi]) Let M be a module with an injective hull E, and let H be a continuous overmodule of M.

(I) H is called a *type* I *continuous hull* of M, if $M \subseteq X \subseteq H$ for a continuous module X implies $X = H$.

(II) H is called a *type* II *continuous hull* of M, if for every continuous overmodule X of M, there exists a monomorphism $\mu : H \to X$ over M.

(III) H is called a *type* III *continuous hull* of M (in E), if $M \subseteq H \subseteq E$, and if $H \subseteq X$ for every continuous module $M \subseteq X \subseteq E$.

Observe that a type III continuous hull is uniquely determined as a submodule of a fixed injective hull. They gave an example of a module which has neither a type II nor a type III continuous hull. However they proved the following result.

Theorem 2.8. ([64, Müller and Rizvi]) Every cyclic module over a commutative ring whose singular submodule is uniform, has a type III continuous hull.

Also, in 1982, Hirano, Hongan, and Ohori [54] defined the Baer hull and the strongly regular hull for a reduced right Utumi ring. Recall that a right nonsingular ring R is called a *right Utumi* ring if every non-essential right ideal of R has a nonzero left annihilator. They defined the *strongly regular hull* of a reduced right Utumi ring to be the intersection of all regular subrings of $Q(R)$. Note that their definition of a strongly regular hull generalizes the epimorphic hull of Storrer [82].

Threorem 2.9. ([54, Hirano, Hongan, and Ohori]) Let R be a reduced right Utumi ring and let $\mathbf{B}(R)$ be the intersection of all the Baer subrings of $Q(R)$ containing R. Then $\mathbf{B}(R)$ is a Baer ring and coincides with the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$.

This result generalizes Theorem 2.4 to noncommutative rings.

Corollary 2.10. ([54, Hirano, Hongan, and Ohori]) Every reduced PI ring has a Baer hull and a strongly regular hull.

The idempotent closure of a module was introduced by Beidar and Wisbauer [9] in 1993. Recall that $\mathcal{E}_M M_R$ is the quasi-injective hull of M_R by Theorem 2.2. The *idempotent closure* of M_R is the submodule of $\mathcal{E}_M M_R$ generated by $\{e(M) \mid e \in \mathcal{B}(\mathcal{E}_M)\}$. For a ring R, we identify the idempotent closure of R_R with the subring of $Q(R)$ generated by R and $\mathcal{B}(Q(R))$ and denote it by $R\mathcal{B}(Q(R))$. Thus if R is a commutative semiprime ring, then the idempotent closure of R is the Baer hull of R as already shown by Mewborn in 1971 (Theorem 2.4). Beidar and Wisbauer indicated that if $\mathrm{End}(\mathcal{E}_M M_R)$ is Abelian then the idempotent closure of M_R is π-injective (equivalently, quasi-continuous) hull of M_R. In [9] and [10], they showed that information about prime ideals and various types of regularity conditions transfer between R and $R\mathcal{B}(Q(R))$.

Theorem 2.11. ([9, Beidar and Wisbaer]) Let R be a semiprime ring. Then the following hold.

(i) For every prime ideal K of $R\mathcal{B}(Q(R))$, $P = K \cap R$ is a prime ideal of R and $R\mathcal{B}(Q(R))/K = (R + K)/K \cong R/P$.

(ii) For any prime ideal P of R, there exists a prime ideal K of $R\mathcal{B}(Q(R))$ with $K \cap R = P$ (i.e., LO (lying over) holds between R and $R\mathcal{B}(Q(R))$).

Theorem 2.12. ([9, Beidar and Wisbauer]) Let R be a ring. Then R is biregular if and only if R is semiprime and $R\mathcal{B}(Q(R))$ is biregular.

Theorem 2.13. ([10, Beidar and Wisbauer]) Let R be a ring. Then R is regular and biregular if and only if $R\mathcal{B}(Q(R))$ is regular and biregular.

Burgess and Raphael call a regular ring with bounded index an *almost biregular* ring if and only if for each $x \in R$ there is an $e \in \mathcal{B}(R)$ such that $RxR_R \leq^{\text{ess}} eR_R$ [34]. Recall that if R is a right nonsingular ring, then $Q(R)$ is a Baer ring. From [58, p.9] each element of a Baer ring R has a central cover (recall that $e \in R$ is a *central cover* for $r \in R$ if e is the smallest central idempotent in the Boolean algebra of the central idempotents of R such that $er = r$). In the following result, Burgess and Raphael show that every regular ring with bounded index is contained in a smallest almost biregular right ring of quotients.

Theorem 2.14. ([34, Burgess and Raphael]) Let R be a regular ring of bounded index. Define $R^{\#}$ to be the ring generated by R and the central covers from $Q(R)$ of all elements of R. Then $R^{\#}$ is the unique smallest almost biregular ring among the regular rings S such that $R \subseteq S \subseteq Q(R)$. Moreover:

(i) If $R^{\#} \subseteq T \subseteq Q(R)$ and T is generated as a subring of $Q(R)$ by R and $\mathcal{B}(T)$, then T is almost biregular; and

(ii) Let A be the sub-Boolean algebra of $\mathcal{B}(Q(R))$ generated by the central covers of elements of R. Then $\mathcal{B}(R^{\#}) = A$.

In 1980, Picavet defined a commutative ring R to be a *weak Baer* ring if and only if R is a PP ring (in our terminology), that is, for each $a \in R$ there exists $e = e^2 \in R$ such that $r_R(a) = eR$ [71]. He defined the *weak Baer envelope* for a commutative reduced ring to be the subring of $Q(R)$ generated by $R \cap \{aq \mid a \in R \text{ and } q \in Q(R) \text{ such that } aqa = a\}$. He showed that the weak Baer envelope of a commutative reduced ring R is the smallest weak Baer subring of $Q(R)$ that contains R. Various applications of the weak Baer envelope appear in [72] and [42].

3. DEFINITIONS OF A RING HULL

In this section, we provide several definitions of the concept of a ring hull to abstract, unify, and encompass the various definitions of particular ring hulls (e.g., Baer extension, Baer hull, epimorphic hull, strongly regular hull, etc.) given in Section 2. These definitions are established in the context of intermediate rings between a base ring R and its injective hull $E(R_R)$ to insure some flow of information between the base ring R and the overrings under consideration. Moreover, our definitions are in terms of abstract classes of rings so as to guarantee their flexibility and versatility.

Henceforth we assume that all right essential overrings of a ring R are contained as right R-modules in a fixed injective hull $E(R_R)$ of R_R and that all right rings of quotients of R are subrings of a fixed maximal right ring of quotients $Q(R)$ of R.

In our next definition we exploit the notion of a right essential overring which is minimal with respect to belonging to a class $\mathfrak{K}$ of rings.

Definition 3.1. ([24, Definition 2.1]) Let $\mathfrak{K}$ denote a class of rings. For a ring R, let S be a right essential overring of R and T an overring of R. Consider the following conditions.

(i) $S \in \mathfrak{K}$.

(ii) If $T \in \mathfrak{K}$ and T is a subring of S, then $T = S$.

(iii) If S and T are subrings of a ring V and $T \in \mathfrak{K}$, then S is a subring of T.

(iv) If $T \in \mathfrak{K}$ and T is a right essential overring of R, then S is a subring of T.

If S satisfies (i) and (ii), then we say that S is a $\mathfrak{K}$ *right ring hull* of R, denoted by $\widetilde{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iii), then we say that S is the $\mathfrak{K}$ *absolute to* V *right ring hull* of R, denoted by $Q^V_{\mathfrak{K}}(R)$; for the $\mathfrak{K}$ absolute to $Q(R)$ right ring hull, we use the notation $\widehat{Q}_{\mathfrak{K}}(R)$. If S satisfies (i) and (iv), then we say that S is the $\mathfrak{K}$ *absolute right ring hull* of R, denoted by $Q_{\mathfrak{K}}(R)$. Observe that if $Q(R) = E(R_R)$, then $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$. The concept of a $\mathfrak{K}$ absolute right ring hull was already implicit in [64] from their definition of a type III continuous (module) hull (see Definition 2.7).

Moreover, the notions of $\mathfrak{K}$ absolute to $Q(R)$ right ring hull and $\mathfrak{K}$ absolute right ring hull incorporate many of the hull definitions in Section 2 that utilized the intersection of all right rings of quotients from a certain class of rings (e.g., Baer hull, epimorphic hull, etc.) which contain the base ring. This will be illustrated in the next section.

Now we consider generating a right essential overring in a class $\mathfrak{K}$ from a base ring R and some subset of $\mathcal{E}_R$. By using equivalence relations, we can effectively reduce the size of the subsets of $\mathcal{E}_R$ needed to generate a right essential overring of R in $\mathfrak{K}$.

Definition 3.2. ([24, Definition 2.2]) Let $\mathfrak{R}$ denote a class of rings and $\mathfrak{X}$ a class of subsets of rings such that for each $R \in \mathfrak{R}$ all subsets of $\mathcal{E}_R$ are contained in $\mathfrak{X}$. Let $\mathfrak{K}$ be a subclass of $\mathfrak{R}$ such that there exists an assignment $\delta_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{X}$ such that $\delta_{\mathfrak{K}}(R) \subseteq \mathcal{E}_R$ and $\delta_{\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$, where $\delta_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta_{\mathfrak{K}}(R)\}$. Let S be a right essential overring of R and ρ an equivalence relation on $\delta_{\mathfrak{K}}(R)$. Note that there may be distinct assignments for the same $\mathfrak{R}$, $\mathfrak{X}$, and $\mathfrak{K}$ say $\delta_{1\mathfrak{K}}$ and $\delta_{2\mathfrak{K}}$ such that for a given R, $\delta_{1\mathfrak{K}}(R) \neq \delta_{2\mathfrak{K}}(R)$; but $\delta_{1\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$ and $\delta_{2\mathfrak{K}}(R)(1) \subseteq R$ implies $R \in \mathfrak{K}$.

(i) If $\delta_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta_{\mathfrak{K}}(R)(1)\rangle_S$ the $\delta_{\mathfrak{K}}$ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$, then we say that S is a $\delta_{\mathfrak{K}}$ *pseudo right ring hull of* R.

(ii) If $\delta^{\rho}_{\mathfrak{K}}(R)(1) \subseteq S$ and $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S \in \mathfrak{K}$, then we call $\langle R \cup \delta^{\rho}_{\mathfrak{K}}(R)(1)\rangle_S$ a $\delta_{\mathfrak{K}}\,\rho$ *pseudo right ring hull of* R *with respect to* S and denote it by $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, where $\delta^{\rho}_{\mathfrak{K}}(R)$ is a set of representatives of all equivalence classes of ρ and $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \{h(1) \in E(R_R) \mid h \in \delta^{\rho}_{\mathfrak{K}}(R)\}$. If $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$, then we say that S is a $\delta_{\mathfrak{K}}\,\rho$ *pseudo right ring hull of* R.

If a $\delta_{\mathfrak{K}}$ has been fixed for a class $\mathfrak{K}$, then in the above nomenclature we replace $\delta_{\mathfrak{K}}$ (resp., $\delta_{\mathfrak{K}}\,\rho$) with $\mathfrak{K}$ (resp., $\mathfrak{K}\ \rho$) (e.g., $\delta_{\mathfrak{K}}$ pseudo right ring hull becomes $\mathfrak{K}$ pseudo right ring hull) and $\delta_{\mathfrak{K}}$ from the notation (e.g., $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ becomes $R(\mathfrak{K}, S)$). Observe that if $\delta_{\mathfrak{K}}(R)(1) \subseteq Q(R)$ and S is a right essential overring of R such that $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S)$ exists, then $R(\mathfrak{K}, \delta_{\mathfrak{K}}, S) = R(\mathfrak{K}, \delta_{\mathfrak{K}}, Q(R))$.

Throughout the remainder of this paper take $\mathfrak{R}$ to be the class of all rings unless indicated otherwise. Some examples illustrating Definition 3.2 are:

(1) $\mathfrak{K} = \mathbf{SI} = \{\text{right self-injective rings}\}$, $\delta_{\mathfrak{S}\mathfrak{I}}(R) = \mathcal{E}_R$.

(2) $\mathfrak{K} = \mathbf{qCon}$, $\delta_{\mathbf{qCon}}(R) = \mathbf{I}(\mathcal{E}_R)$.

(3) $\mathfrak{K} = \{\text{right P-injective rings}\}$, $\delta_{\mathfrak{K}}(R) = \{h \in \mathcal{E}_R \mid$ there exist $a \in R$ and an R-homomorphism $f : aR \to R$ such that $h|_{aR} = f\}$.

(4) Let $\mathfrak{R} = \{\text{right nonsingular rings}\}$, $\mathfrak{K} = \mathbf{B}$, $\delta_{\mathbf{B}}(R) = \{e \in \mathbf{I}(\mathcal{E}_R) \mid$ there exists $\emptyset \neq X \subseteq R$ such that $r_{Q(R)}(X) = eQ(R)\}$.

Also note that Definition 3.2 allows us the flexibility to consider any right essential overring S of a ring R, such that $S \in \mathfrak{K}$ and $S = \langle R \cup \delta(1)\rangle_S$, to be a $R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$ where $\emptyset \neq \delta \subseteq \delta_{\mathfrak{K}}(R)$ and $\delta(1) = \{e(1) \mid e \in \delta\}$. To see this, choose $f \in \delta$. Let $X = \delta_{\mathfrak{K}}(R) \setminus \{e \mid e \in \delta \text{ and } e \neq f\}$. Then $\{X\} \cup \{\{e\} \mid e \in \delta \text{ and } e \neq f\}$ is a partition of $\delta_{\mathfrak{K}}(R)$. Let ρ be the equivalence relation induced on $\delta_{\mathfrak{K}}(R)$ by this partition and take $\delta^{\rho}_{\mathfrak{K}}(R)(1) = \delta(1)$. Then $S = R(\mathfrak{K}, \delta_{\mathfrak{K}}, \rho, S)$.

Observe that the concept of a $\delta_{\mathfrak{K}}$ pseudo right ring hull incorporates that of Goel and Jain [49] for quasi-continuous ring hull (when it exists) by taking $\delta_{\mathbf{qCon}}(R) = \mathbf{I}(\mathcal{E}_R)$, and that of Mewborn [63] for the Baer hull when $\mathfrak{R}$ is the class of commutative semiprime rings by taking $\delta_{\mathbf{B}}(R) = \mathbf{I}(\mathcal{E}_R)$. Also note that several of the hulls considered in Section 2, are types of hulls indicated in both Definitions 3.1 and 3.2.

Definition 3.3. ([24, Definition 1.6]) Let $\mathfrak{R}$ be a class of rings, $\mathfrak{K}$ a subclass of $\mathfrak{R}$, and $\mathfrak{Y}$ a class containing all sets of subsets of every ring. We say that $\mathfrak{K}$ is a class *determined by a property on right ideals* if there exist an assignment $\mathfrak{D}_{\mathfrak{K}} : \mathfrak{R} \to \mathfrak{Y}$ such that $\mathfrak{D}_{\mathfrak{K}}(R) \subseteq \{\text{right ideals of } R\}$ and a property P such that $\mathfrak{D}_{\mathfrak{K}}(R)$ has P if and only if $R \in \mathfrak{K}$.

If $\mathfrak{K}$ is such a class where P is the property that a right ideal is essential in an idempotent generated right ideal, then we say that $\mathfrak{K}$ is a **D**-**E** *class* and use $\mathfrak{C}$ to designate a **D-E** class.

Some examples illustrating Definition 3.3 are:

(1) $\mathfrak{K}$ is the class of right Noetherian rings, $\mathbf{D}_{\mathfrak{K}}(R) = \{\text{right ideals of } R\}$, and P is the property that a right ideal is finitely generated;

(2) $\mathfrak{K}$ is the class of regular rings, $\mathbf{D}_{\mathfrak{K}}(R) = \{\text{principal right ideals of } R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;

(3) $\mathfrak{K} = \mathbf{B}$, $\mathbf{D}_{\mathbf{B}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$, and P is the property that a right ideal is generated by an idempotent as a right ideal;

(4) $\mathfrak{C} = \mathbf{E}$ (resp., $\mathfrak{C} = \mathbf{FI}$, $\mathfrak{C} = \mathbf{eB}$), $\mathbf{D}_{\mathbf{E}}(R) = \{I \mid I_R \leq R_R\}$ (resp., $\mathbf{D}_{\mathbf{FI}}(R) = \{I \mid I \trianglelefteq R\}$, $\mathbf{D}_{\mathbf{eB}}(R) = \{r_R(X) \mid \emptyset \neq X \subseteq R\}$).

Our primary focus in this paper is on classes of rings which are either **D-E** classes or subclasses of **D-E**. Note that any **D-E** class always contains the class of right extending (and hence all right self-injective) rings. Moreover, many known classes of rings are subclasses of a **D-E** class

Theorem 3.4 illustrates the generality achieved by working in the context of a **D-E** class, while Corollary 3.5 demonstrates its application to concrete **D-E** classes.

Theorem 3.4. ([24, Theorem 1.7]) Assume that $\mathfrak{C}$ is a **D-E** class of rings.

(i) Let T be a right essential overring of R. Suppose that for each $Y \in \mathbf{D}_{\mathfrak{C}}(T)$ there exist $X_R \leq R_R$ and $e \in \mathbf{I}(T)$ such that $X_R \leq^{\text{ess}} eR_R$, $X_R \leq^{\text{ess}} Y_R$, and $eY \subseteq Y$. Then $T \in \mathfrak{C}$.

(ii) Let T be a right ring of quotients of R and $R \in \mathfrak{C}$. If $Y \in \mathbf{D}_{\mathfrak{C}}(T)$ implies $Y \cap R \in \mathbf{D}_{\mathfrak{C}}(R)$, then $T \in \mathfrak{C}$.

Classes of rings which are closed with respect to right rings of quotients (resp., right essential overrings) work especially well with a hull concept in that once one finds a hull from such a class then one has that all right rings of quotients (resp., right essential overrings) of that hull are also in the class. Among our final results of this section, we give several examples of classes of rings that are closed with respect to right rings of quotients or right essential overrings.

Recall the following definitions:

1. A ring R is called right *finitely* Σ-*extending* if any finitely generated right free R-module is extending [43].

2. A ring R is said to be right *uniform extending* if each uniform right ideal of R is essential as a right R-module in a direct summand of R_R [43].

3. A ring R is said to be right C_{11} if every right ideal of R has a complement which is a direct summand [80].

4. A ring R is called right *G-extending* if for each right ideal Y of R there is a direct summand D of R_R with $(Y \cap D)_R \leq^{\text{ess}} Y_R$ and $(Y \cap D)_R \leq^{\text{ess}} D_R$ [1].

5. A ring R is called *ideal intrinsic over its center*, IIC, if every nonzero ideal of R has nonzero intersection with the center of R [6].

As a consequence of Theorem 3.4, the next corollary exhibits the transfer of the right (FI-) extending property from R to its (right essential overrings) right rings of quotients. Also note that whenever a property is carried from R to its (right essential overrings) right rings of quotients, then a Zorn's lemma argument can be used to show that R has a (right essential overring) right ring of quotients which is maximal with respect to having that property.

Corollary 3.5. ([24, Corollary 1.8]) (i) Any right essential overring of a right FI-extending ring is right FI-extending.

(ii) Any right ring of quotients of a right extending ring is right extending.

(iii) Any right ring of quotients of a right finitely Σ-extending ring is right finitely Σ-extending.

(iv) Any right ring of quotients of a right uniform extending ring is right uniform extending.

Theorem 3.6. (i) ([31, Theorem 3.5]) If R is a right C_{11}-ring and T is a right essential overring of R, then T is a right C_{11}-ring.

(ii) ([1]) If R is a right G-extending ring and T is a right essential overring of R, then T is a right G-extending ring.

(iii) If R is an IIC-ring and T is a right essential overring with $\mathrm{Cen}(R) \subseteq \mathrm{Cen}(T)$ (e.g., $T = Q(R)$), then T is an IIC-ring.

We say that a ring R is *right essentially Baer* (resp., *right essentially quasi-Baer*) if the right annihilator of any nonempty subset (resp., ideal) of R is essential in a right ideal generated by an idempotent ([24, Definition 1.1]). We use **eB** (resp., **eqB**) to denote the class of right essentially Baer (resp., right essentially quasi-Baer) rings.

Note that the classes **B** and **qB** are not $\mathfrak{C}$ classes, but they are contained in the $\mathfrak{C}$ classes **eB** and **eqB**, respectively. It can be seen that **eB** (resp., **eqB**) properly contains **E** (resp., **FI**) and **B** (resp., **qB**): If $S = A \oplus B$, where A is a domain which is not right Ore and B is a prime ring with $Z(B_B) \neq 0$ [33, Example 4.4], then S is neither right extending nor Baer. But $S \in$ **eB**. Next take

$$R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}.$$

Then the ring R is neither right FI-extending nor quasi-Baer. However $R \in$ **eqB**.

The following two results provide connections between the classes **FI**, **B**, **qB**, **eB**, and **eqB**.

Proposition 3.7. ([24, Proposition 1.2]) Assume that R is a right nonsingular ring.

(i) If $R \in$ **eB** (resp., $R \in$ **eqB**), then $R \in$ **B** (resp., $R \in$ **qB**).

(ii) If $R \in$ **FI**, then $R \in$ **qB**.

Proposition 3.8. ([12, Lemma 2.2] and [18, Theorem 4.7]) Assume that R is a semiprime ring. Then the following are equivalent.

(i) $R \in$ **FI**.

(ii) For any $I \trianglelefteq R$, there is $e \in \mathcal{B}(R)$ such that $I_R \leq^{\text{ess}} eR_R$.

(iii) $R \in$ **qB**.

(iv) $R \in$ **eqB**.

Theorem 3.9. ([24, Theorem 1.9]) (i) Let T be a right and left essential overring of R. If $R \in$ **qB**, then $T \in$ **qB**.

(ii) Let T be a right essential overring of R which is also a left ring of quotients of R. If $R \in$ **B** (resp., $R \in$ **eqB**), then $T \in$ **B** (resp., $T \in$ **eqB**).

(iii) Let T be a right and left ring of quotients of R. If $R \in$ **eB**, then $T \in$ **eB**.

The following corollary generalizes the well known result that a right ring of quotients of a Prüfer domain is a Prüfer domain [48, pp.321-323].

Corollary 3.10. ([24, Corollary 1.10]) Let T be a right and left ring of quotients of R. If R is right semihereditary and every finitely generated free right R-module satisfies the ACC on direct summands, then T is right and left semihereditary.

4. EXISTENCE AND UNIQUENESS OF RING HULLS

In this section, we not only explicitly show how our theory encompasses the particular hulls indicated in Section 2 but how it can be used in a much wider context by applying the theory to many classes of rings not considered in Section 2. Also our results will often show an interplay between the ring hull concept (Definition 3.1) and the pseudo ring hull concept (Definition 3.2). These results also provide answers to Problem I of Section 1.

Our first result illustrates Definitions 3.1 and 3.2 by taking advantage of several well known facts to provide ring hulls for the classes of semisimple Artinian rings, right self-injective rings, and right duo rings.

Proposition 4.1. ([24, Proposition 2.3]) (i) Let $\mathbf{A}$ be the class of semisimple Artinian rings and R a right nonsingular ring with finite right uniform dimension. Then $Q_{\mathbf{A}}(R) = Q(R)$.

(ii) If $Q(R) = E(R_R)$, then $Q_{\mathbf{SI}}(R) = Q(R) = R(\mathbf{SI}, \delta_{\mathbf{SI}}, Q(R))$, where $\mathbf{SI}$ is the class of right self-injective rings.

(iii) If $Q(R) = E(R_R)$, then $Q_{\mathbf{qCon}}(R) = \langle R \cup \mathbf{I}(Q(R))\rangle_{Q(R)} = R(\mathbf{qCon}, \delta_{\mathbf{qCon}}, Q(R))$.

(iv) If R is a commutative semiprime ring, then $Q_{\mathbf{B}}(R) = \langle R \cup \mathbf{I}(Q(R))\rangle_{Q(R)} = Q_{\mathbf{qCon}}(R)$.

(v) Assume that R has finite right uniform dimension and S is a right ring of quotients of R. Then $\mathrm{Mat}_n(S) = \widetilde{Q}_{\mathbf{B}}(\mathrm{Mat}_n(R))$ for all positive integers n if and only if S is a right and left semihereditary right ring hull of R.

(vi) If R is a right Ore domain, then R has a right duo absolute right ring hull.

For Proposition 4.1(vi), the next example is that of a right Ore domain R which is *not* right duo, but it has a right duo absolute right ring hull properly between R and $Q(R)$.

Example 4.2. ([24, Example 2.4]) Take $A = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k$, the integer quaternions. Let $P = 5\mathbb{Z}$ and $\widehat{\mathbb{Z}}_P$ the P-adic completion of $\mathbb{Z}$. Also let

$$R = \widehat{\mathbb{Z}}_P + \widehat{\mathbb{Z}}_P i + \widehat{\mathbb{Z}}_P j + \widehat{\mathbb{Z}}_P k.$$

Then R is a right Ore domain. Note that R is not right duo because $(3+i)R$ is not a left ideal. Take

$$\lambda = (1/2)(1 + i + j + k) \in Q(A) = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k.$$

Let $S = A + \lambda A$. Then by [74, p.131, Exercise 2] S is a maximal $\mathbb{Z}$-order in $Q(A)$. Thus the P-adic completion $\widehat{S}_P = \widehat{\mathbb{Z}}_P \otimes_{\mathbb{Z}} S$ of S is a maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R) = Q(\widehat{\mathbb{Z}}_P) \otimes_{\mathbb{Q}} Q(A)$ by [74, p.134, Corollary 11.6]. Since $\widehat{\mathbb{Z}}_P$ is a complete discrete valuation ring and $Q(R)$ is a division ring, $\widehat{S}_P$ is the unique maximal $\widehat{\mathbb{Z}}_P$-order in $Q(R)$, thus $\widehat{S}_P$ is right duo by [74, p.139, Theorem 13.2]. So $\widehat{S}_P$ is a proper intermediate right duo ring between R and $Q(R)$. Thus, by Proposition 4.1(vi), there exists a right duo absolute right ring hull properly between R and $Q(R)$.

Let $\mathfrak{U}$ denote the class $\{R \mid R \cap \mathbf{U}(Q(R)) = \mathbf{U}(R)\}$ of rings, where $\mathbf{U}(-)$ is the set of units of a ring. Recall from [84] and [85] that R is called *directly finite* if every one-sided inverse of an element of R is two-sided. Note that if R has finite right uniform dimension, or if R satisfies the condition that $r_R(x) = 0$ implies $\ell_R(x) = 0$, or if R is Abelian, then R is directly finite.

For our next result, let $i < j$ be ordinal numbers. We define $R_1 = \langle R \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R\}\rangle_{Q(R)}$, $R_j = \langle R_i \cup \{q \in \mathbf{U}(Q(R)) \mid q^{-1} \in R_i\}\rangle_{Q(R)}$ for $j = i+1$, and $R_j = \bigcup_{i<j} R_i$ for j a limit ordinal. The following theorem characterizes $Q^r_{c\ell}(R)$ as a $\mathfrak{U}$ absolute to $Q(R)$ right ring hull.

Theorem 4.3. ([24, Theorem 2.7]) (i) $\widehat{Q}_{\mathfrak{U}}(R)$ exists and $\widehat{Q}_{\mathfrak{U}}(R) = R_j$ for any j with $|j| > |Q(R)|$.

(ii) Assume that T is a directly finite right essential overring of R and T_T satisfies (C_2). Then $\widehat{Q}_{\mathfrak{U}}(R)$ is a subring of T.

(iii) If R is a right Ore ring, then $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$.

Note that from Theorem 4.3, $\widehat{Q}_{\mathfrak{U}}(R)$ may be thought of as a generalization of $Q^r_{c\ell}(R)$ since $\widehat{Q}_{\mathfrak{U}}(R) = Q^r_{c\ell}(R)$ whenever $Q^r_{c\ell}(R)$ exists. But $\widehat{Q}_{\mathfrak{U}}(R)$ has the advantage in that it always exists which is not the case, in general, for $Q^r_{c\ell}(R)$.

The next results are inspired by the work on continuous module hulls in [64] or [75].

Proposition 4.4. ([24, Proposition 2.9]) Assume that R is a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$. If $Q^r_{c\ell}(R)$ is Abelian and right extending, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.

Corollary 4.5. ([24, Corollary 2.10]) Let R be a right Ore ring. If any one of the following conditions is satisfied, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.

(i) R is Abelian, right extending, and $r_R(x) = 0$ implies $\ell_R(x) = 0$.

(ii) R is right uniform and $r_R(x) = 0$ implies $\ell_R(x) = 0$.

(iii) R is Abelian, right extending, and $Z(R_R) = 0$.

The following theorem is an adaptation of [75, Theorem 4.25].

Theorem 4.6. ([24, Theorem 2.11]) Let R be a right nonsingular ring and S the intersection of all right continuous right rings of quotients of R. Then $Q_{\mathbf{Con}}(R) = S$.

Theorem 4.7. ([24, Theorem 2.12]) Let R be a ring such that $Q(R)$ is Abelian.

(i) $Q(R)$ is a right extending ring if and only if $\widehat{Q}_{\mathbf{E}}(R) = \widehat{Q}_{\mathbf{qCon}}(R) = R\mathcal{B}(Q(R))$, where **qCon** is the class of right quasi-continuous rings.

(ii) Assume that R is a right Ore ring such that $r_R(x) = 0$ implies $\ell_R(x) = 0$ for $x \in R$ and $Z(R_R)$ has finite right uniform dimension. Then $Q(R) \in \mathbf{E}$ if and only if $\widehat{Q}_{\mathbf{Con}}(R)$ exists and $\widehat{Q}_{\mathbf{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a right continuous strongly regular ring and H_2 is a direct sum of right continuous local rings.

For commutative rings, the preceding results yield the following corollary which is related to [64, Corollaries 3 and 7], in particular Corollary 4.8 is related to Theorem 2.6.

Corollary 4.8. ([24, Corollary 2.13]) Let R be a commutative ring.

(i) If R or $Q^r_{c\ell}(R)$ is extending, then $\widehat{Q}_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$.

(ii) If R is uniform, then $Q_{\mathbf{Con}}(R) = Q^r_{c\ell}(R)$ and is also a local ring.

(iii) If $Z(R_R) = 0$, then $Q_{\mathbf{Con}}(R) = \bigcap\{T \mid \mathcal{B}(Q(R)) \subseteq T$ and T is a regular right ring of quotients of $R\}$.

(iv) Assume that $Z(R_R)$ has finite uniform dimension. Then $Q(R)$ is right extending if and only if $\widehat{Q}_{\mathbf{Con}}(R)$ exists and $\widehat{Q}_{\mathbf{Con}}(R) = H_1 \oplus H_2$ (ring direct sum), where H_1 is a continuous regular ring and H_2 is a direct sum of continuous local rings.

We note that in Corollary 4.8(i), the hypothesis "R or $Q^r_{c\ell}(R)$ is extending" is not superfluous. Let T be a countably infinite direct product of copies of a field F. Take $R = \langle \bigoplus_{i=1}^{\infty} F_i \cup \{1\}\rangle_T$. Then $Q^r_{c\ell}(R)$ is the subring of T whose elements are eventually constant. It can be seen that neither R nor $Q^r_{c\ell}(R)$ is extending. Hence $Q^r_{c\ell}(R)$ is not continuous. Also, in general, R may not satisfy the (C_2) property (e.g., take $F = \mathbb{Q}$); but $Q^r_{c\ell}(R)$ does satisfy the (C_2) property since it is regular.

To develop the theory of pseudo hulls for **D-E** classes $\mathfrak{C}$, we define (and fix)

$$\delta_{\mathfrak{C}}(R) = \{e \in \mathbf{I}(\mathrm{End}\,(E(R_R)) \mid X_R \leq^{\mathrm{ess}} eE(R_R) \text{ for some } X \in \mathbf{D}_{\mathfrak{C}}(R)\}.$$

To find a right essential overring S of R such that $S \in \mathfrak{C}$, one might naturally look for a right essential overring T of R with $\delta_{\mathfrak{C}}(R)(1) \subseteq T$. Then take $S = \langle R \cup \delta_{\mathfrak{C}}(R)(1)\rangle_T$. In order to obtain a right essential overring with some hull-like behavior, we need to determine subsets Ω of $\delta_{\mathfrak{C}}(R)(1)$ for which $\langle R \cup \Omega\rangle_T \in \mathfrak{C}$ in some minimal sense. Moreover, to facilitate the transfer of information between R and $\langle R \cup \Omega\rangle_T$, one would want to include in Ω enough of $\delta_{\mathfrak{C}}(R)(1)$ so that for all (or almost all) $X \in \mathbf{D}_{\mathfrak{C}}(R)$ there is $e \in \delta_{\mathfrak{C}}(R)$ with $X_R \leq^{\mathrm{ess}} e(1) \cdot \langle R \cup \Omega\rangle_T$ and $e(1) \in \Omega$. To accomplish this, we use equivalence relations on $\delta_{\mathfrak{C}}(R)$.

Since we have fixed the $\delta_{\mathfrak{C}}$ assignment for all **D-E** classes $\mathfrak{C}$, we will use the terminology $\mathfrak{C}$ (resp., $\mathfrak{C}\ \rho$) pseudo right ring hull for $\delta_{\mathfrak{C}}$ pseudo right ring hull and use $R(\mathfrak{C}, S)$ for $R(\mathfrak{C}, \delta_{\mathfrak{C}}, S)$ and $R(\mathfrak{C}, \rho, S)$ for $R(\mathfrak{C}, \delta_{\mathfrak{C}}, \rho, S)$.

In the next few results, we show that for the concept of idempotent closure [9], we can find a **D-E** class of rings **IC** such that $R\mathcal{B}(Q(R))$ becomes an **IC** absolute to $Q(R)$ right ring hull and a $\delta_{\mathbf{IC}}$ pseudo right ring hull, where $\delta_{\mathbf{IC}}(R) = \mathcal{B}(\mathcal{E}_R)$.

Definition 4.9. ([26, Definition 2.1]) (i) For a ring R, let $\mathbf{D_{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0 \text{ and } \ell_R(I) \cap \ell_R(\ell_R(I)) = 0\}$.

(ii) Let **IC** denote the class of rings R such that for each $I \in \mathbf{D_{IC}}(R)$ there exists some $e \in \mathbf{I}(R)$ such that $I_R \leq^{\text{ess}} eR_R$. We call the class **IC** the *idempotent closure class.*

The set $\mathbf{D_{IC}}(R)$ of ideals of R was studied by Johnson and denoted by $\mathfrak{F}'(R)$, who showed that if $Z(R_R) = 0$, then $\mathbf{D_{IC}}(R) = \{I \trianglelefteq R \mid I \cap \ell_R(I) = 0\}$ [55, p.538].

Remark 4.10. ([26, Remark 2.2]) (i) R is semiprime if and only if $\mathbf{D_{IC}}(R)$ is the set of all ideals of R.

(ii) Let $e \in \mathbf{I}(R)$ with $eR \trianglelefteq R$. Then $eR \in \mathbf{D_{IC}}(R)$ if and only if $e \in \mathbf{B}(R)$.

(iii) For a prime ideal P of R, $P \in \mathbf{D_{IC}}(R)$ if and only if $P \cap \ell_R(P) = 0$.

(iv) Let P be a prime ideal of R and $P \in \mathbf{D_{IC}}(R)$. If $I \trianglelefteq R$ such that $P \subseteq I$, then $I \in \mathbf{D_{IC}}(R)$.

(v) If $I \trianglelefteq R$ such that $\ell_R(I) \cap P(R) = 0$, then $I \in \mathbf{D_{IC}}(R)$.

(vi) If $Z(R_R) = 0$ and $I \trianglelefteq R$ such that $I \cap P(R) = 0$, then $I \in \mathbf{D_{IC}}(R)$.

Proposition 4.11. ([26, Proposition 2.4]) Let R be a ring. Then $\mathbf{D_{IC}}(R) = \{I \trianglelefteq R \mid \text{there exists } J \trianglelefteq R \text{ with } I \cap J = 0 \text{ and } (I \oplus J)_R \leq^{\text{den}} R_R\}$.

Theorem 4.12. ([26, Theorem 2.11]) (i) $\mathbf{D_{IC}}(R)$ is a sublattice of the lattice of ideals of R.

(ii) If $\mathbf{D_{IC}}(R)$ is a complete sublattice of the lattice of ideals of R, then $\mathbf{B}(Q(R))$ is a complete Boolean algebra.

(iii) If R is a ring with unity which is right and left FI-extending, then $\mathbf{D_{IC}}(R)$ is a complete sublattice of the lattice of ideals of R.

The following result answers the question: *Which ideals of a ring R are dense in ring direct summands of $Q(R)$?*

Theorem 4.13. ([26, Theorem 2.10]) Let $I \trianglelefteq R$. Then $I_R \leq^{\text{den}} eQ(R)_R$ for some unique $e \in \mathcal{B}(Q(R))$ if and only if $I \in \mathbf{D_{IC}}(R)$.

The next result indicates that $R\mathcal{B}(Q(R))$ is a ring hull according to Definitions 3.1 and 3.2 for the **IC** class of rings. Thus these hulls exist for every ring R. We observe that $\delta_{\mathbf{IC}}(R) = \mathcal{B}(\mathcal{E}_R)$ and $\delta_{\mathbf{IC}}(R)(1) = \mathcal{B}(Q(R))$.

Theorem 4.14. ([26, Theorem 2.7]) (i) Let T be a right ring of quotients of R. Then $T \in \mathbf{IC}$ if and only if $\mathcal{B}(Q(R)) \subseteq T$.

(ii) $R \in \mathbf{IC}$ if and only if $\mathcal{B}(Q(R)) \subseteq R$.

(iii) $R\mathcal{B}(Q(R)) = \widehat{Q}_{\mathbf{IC}}(R) = R(\mathbf{IC}, \delta_{\mathbf{IC}}, Q(R))$.

Our next result is a structure theorem for the idempotent closure $R\mathcal{B}(Q(R))$ when R is a semiprime ring with only finitely many minimal prime ideals. It is used for a characterization of C^*-algebras with only finitely many minimal prime ideals in Section 9. Many well known finiteness conditions on a ring imply that it has only finitely many minimal prime ideals (see [60, p.336, Theorem 11.43]).

Theorem 4.15. ([26, Theorem 3.15]) The following are equivalent for a ring R.

(i) R is semiprime and has exactly n minimal prime ideals.

(ii) $\widehat{Q}_{\mathbf{IC}}(R) = R\mathcal{B}(Q(R))$ is a direct sum of n prime rings.

(iii) $\widehat{Q}_{\mathbf{IC}}(R) = R\mathcal{B}(Q(R)) \cong R/P_1 \oplus \cdots \oplus R/P_n$, where each P_i is a minimal prime ideal of R.

The following example illustrates Definitions 3.1 and 3.2. In [24] we develop, in detail, the general consequences of Definitions 3.1 and 3.2. The independence of these definitions is beneficial in the sense that they provide distinct tools for analyzing interconnections between a ring and its right essential overrings relative to a class $\mathfrak{K}$. Also the following example shows that there is a quasi-Baer ring R (hence R itself is a quasi-Baer right ring hull of R), but R does not have a unique right FI-extending right ring hull.

Example 4.16. ([28, Example 1.7]) Let F be a field. Consider the following subrings of $\mathrm{Mat}_3(F)$:

$$R = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & b \end{pmatrix} \mid a, b, x, y \in F \right\}, \ H_1 = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix},$$

$$H_2 = \left\{ \begin{pmatrix} a+b & a & x \\ 0 & b & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in F \right\},$$

and

$$H_3 = \left\{ \begin{pmatrix} a+b & a & x \\ a & b & y \\ 0 & 0 & c \end{pmatrix} \mid a, b, c, x, y \in F \right\}.$$

Then the following facts are illustrated in [24, Example 3.19].

(i) $Z(R_R) = 0$ and R is quasi-Baer, but R is not right FI-extending.

(ii) H_1, H_2, and H_3 are right FI-extending right ring hulls of R with $H_1 \cong H_2$, but $H_1 \ncong H_3$ for appropriate choices of F.

(iii) H_1 is not a right FI-extending pseudo right ring hull of R.

(v) $R(\mathbf{FI},\ Q(R)) = \begin{pmatrix} F & F & F \\ F & F & F \\ 0 & 0 & F \end{pmatrix}$.

The following example also illustrates Definition 3.1. In fact, there is a ring R which has mutually isomorphic right FI-extending right ring hulls, but R has no quasi-Baer right essential overring.

Recall from [25, p.30] that a ring R is right *Osofsky compatible* if $E(R_R)$ has a ring multiplication that extends its R-module scalar multiplication (i.e., $E(R_R)$ has a ring structure that is compatible with its R-module scalar multiplication).

Example 4.17. ([28, Example 1.8]) Assume that $n = p^m$, where p is a prime integer and $m \geq 2$. Let $A = \mathbb{Z}_n$, the ring of integers modulo n and let

$$R = \begin{pmatrix} A & A/J(A) \\ 0 & A/J(A) \end{pmatrix}.$$

Then $Q(R) = R$ by [19]. Further, from [19, Theorem 1]

$$E = \begin{pmatrix} A \oplus A/J(A) & A/J(A) \\ A/J(A) & A/J(A) \end{pmatrix}$$

is an injective hull of R_R, where the addition is componentwise and the R-module scalar multiplication is given by

$$\begin{pmatrix} s+\overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} t & \overline{x} \\ 0 & \overline{y} \end{pmatrix} = \begin{pmatrix} st+\overline{at} & \overline{sx}+\overline{ax}+\overline{by} \\ \overline{ct} & \overline{cx}+\overline{dy} \end{pmatrix},$$

where $\overline{a}, \overline{x} \in A/J(A)$, etc. denote canonical images of a, $x \in A$.

It is shown in [19, Theorem 1] that the ring R is right Osofsky compatible. Let $\mathrm{Soc}(A)$ denote the socle of A. By a direct computation using the associativity of multiplication and the distributivity of multiplication over addition, we get that $\{(E, +, \circ_{(\alpha,\beta)}) \mid \alpha, \beta \in \mathrm{Soc}(A)\}$ is the set of all compatible ring structures on $E(R_R)$, where the addition is componentwise and the multiplication $\circ_{(\alpha,\beta)}$ is defined by

$$\begin{pmatrix} s_1+\overline{a_1} & \overline{b_1} \\ \overline{c_1} & \overline{d_1} \end{pmatrix} \circ_{(\alpha,\beta)} \begin{pmatrix} s_2+\overline{a_2} & \overline{b_2} \\ \overline{c_2} & \overline{d_2} \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix},$$

where

$$x = s_1s_2 + \alpha a_1a_2 + \beta c_1a_2 + (-\beta)s_1c_2 + \alpha b_1c_2 + \beta d_1c_2 + \overline{a_1a_2} + \overline{a_1s_2} + \overline{s_1a_2} + \overline{b_1c_2},$$

$$y = \overline{a_1b_2} + \overline{s_1b_2} + \overline{b_1d_2},\ z = \overline{c_1a_2} + \overline{c_1s_2} + \overline{d_1c_2},\ \text{and } w = \overline{c_1b_2} + \overline{d_1d_2}.$$

Thus E has exactly $|\mathrm{Soc}(A)|^2 = p^2$ ring structures extending the R-module scalar multiplication (i.e., compatible ring structures). Define $\theta_{(\alpha,\beta)} : (E, +, \circ_{(\alpha,\beta)}) \to (E, +, \circ_{(0,0)})$ by

$$\theta_{(\alpha,\beta)} \left[\begin{pmatrix} s+\overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix} \right] = \begin{pmatrix} s+\overline{a}+(-\alpha)a+(-\beta)c & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}.$$

Then $\theta_{(\alpha,\beta)}$ is a ring isomorphism. Hence $(E, +, \circ_{(\alpha,\beta)})$ are all isomorphic. Let $e = \begin{pmatrix} 1-\overline{1} & 0 \\ 0 & 0 \end{pmatrix} \in (E, +, \circ_{(0,0)})$ and $f = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix} \in (E, +, \circ_{(0,0)})$. Then e and f are central idempotents in $(E, +, \circ_{(0,0)})$ and $e + f = 1$. Thus $(E, +, \circ_{(0,0)}) \cong e(E, +, \circ_{(0,0)}) \oplus f(E, +, \circ_{(0,0)}) \cong A \oplus \mathrm{Mat}_2(A/J(A))$. Hence $(E, +, \circ_{(0,0)})$ is a QF-ring, and so all $(E, +, \circ_{(\alpha,\beta)})$ are QF-rings for $\alpha, \beta \in \mathrm{Soc}(A)$. Let

$$T = \begin{pmatrix} A \oplus A/J(A) & A/J(A) \\ 0 & A/J(A) \end{pmatrix}.$$

Then T is the only proper R-submodule of E with $R \subseteq T \subseteq E$ (and $R \neq T \neq E$) which can have a ring structure that is compatible with its R-module scalar multiplication. Also, $\{(T, +, \circ_{(\alpha,0)}) \mid \alpha \in \mathrm{Soc}(A)\}$ is the set of all compatible ring structures on T, where the multiplication $\circ_{(\alpha,0)}$ is the restriction of $\circ_{(\alpha,\beta)}$ on E

to T for $\beta \in \mathrm{Soc}(A)$. Hence $(T, +, \circ_{(\alpha,0)})$ is a subring of $(E, +, \circ_{(\alpha,\beta)})$ for each $\beta \in \mathrm{Soc}(A)$. Define $\lambda_{(\alpha,0)} : (T, +, \circ_{(\alpha,0)}) \to (T, +, \circ_{(0,0)})$ by

$$\lambda_{(\alpha,0)}\left[\begin{pmatrix} s+\overline{a} & \overline{b} \\ 0 & \overline{d} \end{pmatrix}\right] = \begin{pmatrix} s+(-\alpha)a+\overline{a} & \overline{b} \\ 0 & \overline{d} \end{pmatrix}.$$

Then we see that $\lambda_{(\alpha,0)}$ is a ring isomorphism.

We note that all right essential overrings of R are $\{(E, +, \circ_{(\alpha,\beta)}) \mid \alpha, \beta \in \mathrm{Soc}(A)\}$, $\{(T, +, \circ_{(\alpha,0)}) \mid \alpha \in \mathrm{Soc}(A)\}$, and R itself.

Take $g = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R$. Then $g = g^2 \in R$ and $gRg \cong A$. Note that A is not quasi-Baer. Thus R is not quasi-Baer by [39, Lemma 2] or [23, Theorem 3.2]. Next observe that $e = \begin{pmatrix} 1-\overline{1} & 0 \\ 0 & 0 \end{pmatrix} \in T$. Then $e(T, +, \circ_{(0,0)})e \cong A$, which is not quasi-Baer. Thus $(T, +, \circ_{(0,0)})$ is not quasi-Baer by [39, Lemma 2] or [23, Theorem 3.2]. So all $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ cannot be quasi-Baer since $(T, +, \circ_{(\alpha,0)}) \cong (T, +, \circ_{(0,0)})$. Further, $e(E, +, \circ_{(0,0)})e \cong A$ is not quasi-Baer, so $(E, +, \circ_{(0,0)})$ is not quasi-Baer again from [39, Lemma 2] or [23, Theorem 3.2]. Thus $(E, +, \circ_{(\alpha,\beta)})$ cannot be quasi-Baer for $\alpha, \beta \in \mathrm{Soc}(A)$ since $(E, +, \circ_{(\alpha,\beta)}) \cong (E, +, \circ_{(0,0)})$. Hence R has no quasi-Baer right essential overring.

Finally, let $I = \begin{pmatrix} J(A) & 0 \\ 0 & 0 \end{pmatrix} \trianglelefteq R$. Then there is no $h = h^2 \in R$ with $I_R \leq^{\mathrm{ess}} hR_R$. Hence R is not right FI-extending. Note that $f = \begin{pmatrix} \overline{1} & 0 \\ 0 & \overline{1} \end{pmatrix} \in T$. Thus $(T, +, \circ_{(0,0)}) = e(T, +, \circ_{(0,0)}) \oplus f(T, +, \circ_{(0,0)}) \cong A \oplus T_2(A/J(A))$, where $T_2(-)$ is the 2-by-2 upper triangular matrix ring over a ring. From [18, Theorem 1.3 and Corollary 2.5], $(T, +, \circ_{(0,0)})$ is right FI-extending. Thus all $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ are right FI-extending. Therefore the $(T, +, \circ_{(\alpha,0)})$ with $\alpha \in \mathrm{Soc}(A)$ are right FI-extending right ring hulls of R.

In Example 4.16, we have seen that, in general, $\mathfrak{C}$ right ring hulls and $\mathfrak{C}$ pseudo right ring hulls are distinct and may not be unique (when they exist) even if the ring is right nonsingular. Also in Example 4.17, there is a ring where all right FI-extending ring hulls are mutually isomorphic, but it does not have a quasi-Baer right ring hull. However, the semiprime condition on the ring rescues us from this somewhat chaotic situation, for the classes $\mathfrak{C} = \mathbf{FI}$ or $\mathfrak{C} = \mathbf{eqB}$. In the following theorem, we establish the existence and uniqueness of quasi-Baer and right FI-extending right ring hulls of a semiprime ring. This result indicates the ubiquity of the right FI-extending and quasi-Baer ring hulls by showing that every nonzero ring R has a nontrivial homomorphic image, $R/P(R)$, which has each of these hulls. Mewborn [63] (see Theorem 2.4) showed the existence of a Baer (absolute) hull for a commutative semiprime ring. Our next theorem also generalizes Mewborn's result since a commutative quasi-Baer ring is a Baer ring.

Theorem 4.18. ([28, Theorem 3.3]) Let R be a semiprime ring. Then:

(i) $\widehat{Q}_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R)) = R(\mathbf{FI}, Q(R))$.

(ii) $\widehat{Q}_{\mathbf{qB}}(R) = \widehat{Q}_{\mathbf{eqB}}(R) = R\mathcal{B}(Q(R)) = R(\mathbf{eqB}, Q(R))$.

(iii) If R is right Osofsky compatible, then $R\mathcal{B}(Q(R)) = Q_{\mathbf{FI}}(R) = Q_{\mathbf{qB}}(R) = Q_{\mathbf{eqB}}(R)$.

Corollary 4.19. ([28, Corollary 3.16]) Let R be a semiprime ring and T a right ring of quotients of R. Then T is quasi-Baer (hence right FI-extending) if and only if $\mathcal{B}(Q(R)) \subseteq T$.

Our first corollary to Theorem 4.18 generalizes both the result of Mewborn, Theorem 2.4, and the result of Hirano, Hongan, and Ohori, Theorem 2.8.

Corollary 4.20. (see [28, Theorem 3.8]) If R is a reduced ring, then $Q_{\mathbf{B}}(R) = R\mathcal{B}(Q(R))$ (i.e., R has a Baer hull).

Corollary 4.21. ([28, Corollary 3.17]) (i) If R is a semiprime ring, then the central closure of R, the normal closure of R, $Q^m(R)$, $Q^s(R)$, and $Q(R)$ are all quasi-Baer and right FI-extending.

(ii) Assume that $Q(R)$ is semiprime. Then $Q(R)$ is quasi-Baer and right FI-extending. Also there exists a right essential overring of R containing $Q(R)$ which is maximal with respect to being quasi-Baer (or right FI-extending).

In [47], Ferrero has shown that $Q^s(R) \in \mathbf{qB}$ for a semiprime ring R. There is a semiprime ring R for which neither $Q^m(R)$ nor $Q^s(R)$ is Baer. In fact, there is a simple ring R given by Zalesski and Neroslavskii [50] which is not a domain and 0, 1 are its only idempotents. Then $Q^m(R) = R$ (and hence $Q^s(R) = R$). In this case, $Q^m(R)$ is not a Baer ring.

In [67] Osofsky poses the question: *If $E(R_R)$ has a ring multiplication which extends its right R-module scalar multiplication, must $E(R_R)$ be a right self-injective ring?* Example 4.23 below shows that this is not true in general. We can, however, show that the ring $E(R_R)$ does satisfy the right FI-extending property - a generalization of right self-injectivity, for the case when the ring R is right FI-extending or when $Q(R)$ is semiprime.

Corollary 4.22. ([28, Corollary 3.18]) Let R be a right Osofsky compatible ring. If R has a right FI-extending right essential overring which is a subring of $E(R_R)$, then $E(R_R)$ is a right FI-extending ring. In particular, if $Q(R)$ is semiprime, then $E(R_R)$ is a right FI-extending ring.

The following example, due to Camillo, Herzog, and Nielsen [36] illustrates Corollary 4.22. In fact, in the following example, there exists a right Osofsky compatible ring R which is right extending, but the compatible ring structure on $E(R_R)$ is not right self-injective. However, by Corollary 4.22, the compatible ring structure on $E(R_R)$ is right FI-extending.

Example 4.23. ([28, Example 3.19]) Let $\mathbb{R}\{X_1, X_2, \dots\}$ be the free algebra over the field $\mathbb{R}$ of real numbers with indeterminates $X_1, X_2, \dots$. Put

$$R = \mathbb{R}\{X_1, X_2, \dots\}/\langle X_i X_j - \delta_{ij}\, X_1^2 \rangle,$$

where $\langle X_iX_j - \delta_{ij}\, X_1^2\rangle$ is the ideal of $\mathbb{R}\{X_1, X_2, \dots\}$ generated by $X_iX_j - \delta_{ij}\, X_1^2$ with $i, j = 1, 2, \dots$ and δ_{ij} the Kronecker delta. We denote the canonical image of X_i by x_i in R. Set $V = \mathbb{R}x_1 \oplus \mathbb{R}x_2 \oplus \cdots$, $P = \mathbb{R}x_1^2$ and let the bilinear form on V be given by $B(x_i, x_j) = \delta_{ij}$. Then B is non-degenerate and symmetric. Hence we see that

$$R = \left\{ \begin{pmatrix} k & v & p \\ 0 & k & v \\ 0 & 0 & k \end{pmatrix} \;\middle|\; k \in \mathbb{R}, v \in V, \text{ and } p \in P \right\},$$

where the addition is componentwise and the multiplication is defined by

$$\begin{pmatrix} k_1 & v_1 & p_1 \\ 0 & k_1 & v_1 \\ 0 & 0 & k_1 \end{pmatrix} \begin{pmatrix} k_2 & v_2 & p_2 \\ 0 & k_2 & v_2 \\ 0 & 0 & k_2 \end{pmatrix} =$$

$$\begin{pmatrix} k_1k_2 & k_1v_2 + k_2v_1 & k_1p_2 + k_2p_1 + B(v_1, v_2)x_1^2 \\ 0 & k_1k_2 & k_1v_2 + k_2v_1 \\ 0 & 0 & k_1k_2 \end{pmatrix}.$$

Let $E_R = [\mathrm{Hom}_{\mathbb{R}}(R_{\mathbb{R}}, \mathbb{R}_{\mathbb{R}})]_R$. Then it is shown in [23] that E_R is an injective hull of R_R. Further, E_R has a compatible ring structure with its R-module scalar multiplication, but it is not right self-injective. Note that R is a commutative local ring. Also

$$\begin{pmatrix} 0 & 0 & P \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the smallest nonzero ideal of R and it is essential in R. Hence R is uniform, so it is extending. Thus by Corollary 4.22, the compatible ring structure on the injective hull E_R is right FI-extending.

The following example provides a ring R which is neither semiprime, right (nor left) nonsingular, right (nor left) FI-extending, nor quasi-Baer. However, we have that $Q_{\mathbf{FI}}(R) = R\mathcal{B}(Q(R))$. Thus, even without the semiprime condition, a ring can have a natural unique FI-extending absolute right ring hull. Recall from [22] that a ring R is right *strongly FI-extending* if for each $I \trianglelefteq R$ there is $e = e^2 \in R$ such that $I_R \leq^{\text{ess}} eR_R$ and $eR \trianglelefteq R$.

Example 4.24. Let A be a QF-ring with $J(A) \neq 0$. Assume that A is right strongly FI-extending, and A has nontrivial central idempotents while the subring of A generated by 1_A contains no nontrivial idempotents (e.g., $A = \mathbb{Q} \oplus \mathrm{Mat}_2(\mathbb{Z}_4)$). Let $1_{\prod_{i=1}^{\infty} A_i}$ denote the unity of $\prod_{i=1}^{\infty} A_i$, where $A_i = A$. Take R to be the subring of $\prod_{i=1}^{\infty} A_i$ generated by $1_{\prod_{i=1}^{\infty} A_i}$ and $\bigoplus_{i=1}^{\infty} A_i$. Observe that $Q(R) = \prod_{i=1}^{\infty} A_i = E(R_R)$ by [85, 2.1]. Now R has the following properties:

(i) R is neither semiprime nor right FI-extending.

(ii) $R\mathcal{B}(Q(R)) = R(\mathbf{FI}, Q(R)) = Q_{\mathbf{FI}}(R)$.

(iii) $R\mathcal{B}(Q(R))$ is neither right extending nor quasi-Baer.

Let c be a nontrivial idempotent of A. Let π_i and κ_i denote the i-th projection and injection , respectively, of the direct product. Let K be the ideal of R generated by $\{\kappa_i(c) \mid 1 \leq i < \infty\}$. Then there exists no $b = b^2 \in R$ such that $K_R \leq^{\text{ess}} bR_R$. Thus R is not right FI-extending.

Now let $I \trianglelefteq R$. Then $\pi_i(I) \trianglelefteq A_i$. By [60, p.421, Exercise 16], there exists $e_i \in \mathcal{B}(A_i)$ such that $\pi_i(I)_{A_i} \leq^{\text{ess}} e_i A_{i A_i}$, since A_i is right strongly FI-extending by assumption. Let $e \in Q(R)$ such that $\pi_i(e) = e_i$. Then

$$I_R \leq^{\text{ess}} eQ(R)_R \text{ and } e \in \mathcal{B}(Q(R)).$$

Hence $\mathcal{B}(\mathcal{E}_R) = \delta_{\mathbf{FI}}(R)$. Let $S = \langle R \cup \delta_{\mathbf{FI}}(R)(1)\rangle_{Q(R)} = R\mathcal{B}(Q(R))$. Then $\mathbf{D}_{\mathbf{FI}}(S \to R)$ holds (see [24, p.638]). By [24, Lemma 2.19 and Corollary 2.18], $S = R(\mathbf{FI}, Q(R))$.

Next we show that $S = Q_{\mathbf{FI}}(R)$. Let T be a right FI-extending right ring of quotients of R. Take $e \in \mathcal{B}(Q(R)) = \delta_{\mathbf{FI}}(R)(1)$. Then $eQ(R) \cap T \trianglelefteq T$. Since T is right FI-extending, there is $f = f^2 \in T$ such that $(eQ(R) \cap T)_T \leq^{\text{ess}} fT_T$. So $(eQ(R) \cap T)_R \leq^{\text{ess}} fT_R$ from [24, Lemma 1.4]. Since $fT_R \leq^{\text{ess}} fQ(R)_R$, $(eQ(R) \cap T)_R \leq^{\text{ess}} fQ(R)_R$. Hence $(eQ(R) \cap R)_R \leq^{\text{ess}} fQ(R)_R$. Also $(eQ(R) \cap R)_R \leq^{\text{ess}} eQ(R)_R$. Since $e \in \mathcal{B}(Q(R))$, $fQ(R) \cap eQ(R) = efQ(R)$ and $ef = (ef)^2$. Thus $fQ(R) = eQ(R)$, so $e = f \in T$. Therefore $\mathcal{B}(Q(R)) \subseteq T$. Hence S is a subring of T. Consequently, S is the right FI-extending absolute right ring hull of R.

To see that S, in general, is not right extending, take $A = \mathbb{Q} \oplus \text{Mat}_2(\mathbb{Z}_4)$ and let V be a right ideal of S generated by

$$\left\{\kappa_i \left[\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\right] \mid 1 \leq i < \infty\right\}.$$

Then V is not right essential in a right direct summand of S_S.

Since $Q(R)$ is a QF-ring, $Q(R) = Q^{\ell}(R) = E(R_R)$. By [60, p.421, Exercise 16], $\mathbf{S}_{\ell}(Q(R)) = \mathcal{B}(Q(R))$. Note that $Q(R)$ is not semiprime, so $Q(R)$ cannot be right p.q.-Baer from [14, Proposition 1.7]. By Theorem 3.9(i), $R\mathcal{B}(Q(R))$ is not quasi-Baer.

After giving some preliminary results on the class **pqB** of right p.q.-Baer rings, we describe ring hulls for this and related classes over semiprime rings.

Proposition 4.25. (i) ([15, Proposition 1.8] and [14, Proposition 1.12]) The center of a quasi-Baer (resp., right p.q.-Baer) ring is Baer (resp., PP).

(ii) ([14, Proposition 3.11]) Assume that a ring R is semiprime. Then R is quasi-Baer if and only if R is p.q.-Baer and the center of R is Baer.

(iii) ([81, pp.78-79] and [15, Theorem 3.5]) Let a ring R be regular (resp., biregular). Then R is Baer (resp., quasi-Baer) if and only if the lattice of principal right ideals (resp., principal ideals) is complete.

(iv) A ring R is biregular if and only if R is right (or left) p.q.-Baer ring and $r_R(\ell_R(RaR)) = RaR$, for all $a \in R$.

Recall from [30], we say that a ring R is *principally right FI-extending* (resp., *finitely generated right FI-extending*) if every principal ideal (resp., finitely generated ideal) of R is essential as a right R-module in a right ideal of R generated by an idempotent. We use **pFI** (resp., **fgFI**) to denote the class of principally (resp., finitely generated) right FI-extending rings.

Lemma 4.26. ([14, Corollary 1.11]) Let R be a semiprime ring. Then the following conditions are equivalent.

(i) R is right p.q.-Baer.

(ii) R is principally right FI-extending.

(iii) R is finitely generated right FI-extending.

Our next result, when applied to a commutative reduced ring yields Picavet's weak Baer envelope [71]; and when it is applied to a regular ring of bounded index, it yields the unique smallest almost biregular ring of Burgess and Raphael [34, Theorem 1.7] (see Section 2).

Theorem 4.27. ([30, Theorem 8]) Let R be a semiprime ring. Then:

(i) $\langle R \cup \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{pFI}}(R) = R(\mathbf{pFI}, Q(R))$.

(ii) $\langle R \cup \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{pqB}}(R)$.

(iii) $\langle R \cup \delta_{\mathbf{pFI}}(R)(1)\rangle_{Q(R)} = \widehat{Q}_{\mathbf{fgFI}}(R) = R(\mathbf{fgFI}, Q(R))$.

Note that $\delta_{\mathbf{pFI}}(R)(1) = \{c \in \mathcal{B}(Q(R)) \mid \text{there is } x \in R \text{ with } RxR_R \leq^{\text{ess}} cR_R\}$.

Corollary 4.28. ([30, Theorem 15]) Let R be a reduced ring. Then $Q_{\mathbf{pqB}}(R)$ exists and is the PP absolute right ring hull.

The next two equivalence relations are particularly important to our study.

Definition 4.29. ([24, Definition 2.4]) (i) Let A be a ring and let $\delta \subseteq \mathbf{I}(A)$. We define an equivalence relation α on δ by $e\ \alpha\ c$ if and only if $ce = e$ and $ec = c$.

(ii) We define an equivalence relation β on $\delta_{\mathfrak{C}}(R)$ by $e\ \beta\ c$ if and only if there exists $X_R \leq R_R$ such that $X_R \leq^{\text{ess}} eE(R_R)$ and $X_R \leq^{\text{ess}} cE(R_R)$.

Note that for $e, c \in \delta_{\mathfrak{C}}(R), e\ \alpha\ c$ implies $e\ \beta\ c$. Also note that $\alpha = \beta$ if and only if every element of $\mathbf{D}_{\mathfrak{C}}(R)$ has a unique essential closure in $E(R_R)$. So if $Z(R_R) = 0$, then $\alpha = \beta$.

The following example again indicates the independence of Definition 3.1 and 3.2 for **D**-**E** classes. Moreover, it shows that a nonsemiprime commutative ring R can have an absolute self-injective right ring hull even when R is not right Osofsky compatible. Recall from [60, Corollary 8.28] that a ring R is right *Kasch* if the left annihilator of every maximal right ideal of R is nonzero.

Example 4.30. ([24, Example 2.15]) For a field F, let $T = F[x]/x^4F[x]$ and $\overline{x}$ be the canonical image of x in T. Then $T = F + F\overline{x} + F\overline{x}^2 + F\overline{x}^3$. Let $R = F + F\overline{x}^2 + F\overline{x}^3$ which is a subring of T. Now R and T have the following properties.

(i) R is right Kasch, so $R = Q(R)$ [60, Corollary 13.24].

(ii) T is a QF right essential overring of R. There is no proper intermediate ring between R and T. Hence $T = Q_{\mathbf{FI}}(R) = Q_{\mathbf{E}}(R) = Q_{\mathbf{SI}}(R)$.

(iii) T is not a $\mathfrak{C}$ ρ pseudo right ring hull of R for any choice of $\mathfrak{C}$ and any equivalence relation ρ on $\delta_{\mathfrak{C}}(R)$. Indeed, there is no $c \in \delta_{\mathfrak{C}}(R)$ such that $c(1) \in T \backslash R$ and $I_R \leq^{\text{ess}} cE(R_R)$ for any nonzero ideal I of R.

(iv) T_R is not FI-extending (hence not extending). In fact, $\overline{x}R_R \trianglelefteq R_R$. But there does not exist $e \in \mathbf{I}(\text{End}\,(T_R))$ such that $\overline{x}R_R \leq^{\text{ess}} eT_R$.

(v) Since T_T is injective, T is maximal among right extending right essential overrings of R.

(vi) By [62, Theorem 4] $E(R_R)$ has no ring multiplication which extends its R-module scalar multiplication.

Our next result shows that when $Q(R) = E(R_R)$ the α pseudo right ring hulls and β pseudo right ring hulls also exist, respectively for the right FI-extending and right essentially quasi-Baer properties.

Corollary 4.31. ([24, Corollary 2.21]) Assume that $Q(R) = E(R_R)$.

(i) For each $\delta^{\alpha}_{\mathbf{E}}(R)$ (resp., $\delta^{\beta}_{\mathbf{FI}}(R)$), $R(\mathbf{E}, \alpha, Q(R))$ (resp., $R(\mathbf{FI}, \beta, Q(R))$ exists. Moreover, every right ring of quotients of R containing $R(\mathbf{E}, \alpha, Q(R))$ (resp., $R(\mathbf{FI}, \beta, Q(R))$ is right extending (resp., right FI-extending).

(ii) Let $S = \langle R \cup \delta(1)\rangle_{Q(R)}$. If $\delta(1) = \delta^{\alpha}_{\mathbf{eB}}(R)(1)$ (resp., $\delta(1) = \delta^{\beta}_{\mathbf{eqB}}(R)(1)$) and S is a left ring of quotients of R, then $R(\mathbf{eB}, \alpha, Q(R))$ (resp., $R(\mathbf{eqB}, \beta, Q(R))$) exists. Moreover, any right and left ring of quotients of R which also lies between $R(\mathbf{eB}, \alpha, Q(R))$ (resp., $R(\mathbf{eqB}, \beta, Q(R))$) and $Q(R)$ is right essentially Baer (resp., right essentially quasi-Baer). If $Z(R_R) = 0$, then these intermediate rings are Baer (resp., quasi-Baer).

We remark that the $\mathfrak{K}$ absolute (absolute to $Q(R)$) right ring hull of R is the intersection of all right essential overrings (of all right rings of quotients) of R which are in $\mathfrak{K}$ (see for example, Theorem 2.18). Our next result shows that under suitable conditions, these intersections coincide with the intersections of the α pseudo or the β pseudo right ring hulls for various **D-E** classes (e.g., $\mathbf{E}, \mathbf{FI}$, **eB**, and **eqB**). Also under these conditions a $\mathfrak{C}$ right ring hull will be a $\mathfrak{C}$ α or a $\mathfrak{C}$ β pseudo right ring hull. We note that the condition $X \trianglelefteq R$ implies $XT \trianglelefteq T$ holds for example when T is a centralizing extension of R or when R is a right Noetherian ring and T is a right ring of quotients of R contained in $Q^r_{c\ell}(R)$ [60, pp.314-315]. This condition is useful in the following result.

Corollary 4.32. ([24, Corollary 2.23]) Let T be a right ring of quotients of R.

(i) Suppose that either $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{E}}(R)(1) \subseteq \text{Cen}(T)$. Then $T \in \mathbf{E}$ if and only if there exists an $R(\mathbf{E}, \alpha, Q(R)$ which is a subring of T.

(ii) If $X \trianglelefteq R$ implies $XT \trianglelefteq T$, then $T \in \mathbf{FI}$ if and only if there exists a $R(\mathbf{FI}, \beta, Q(R))$ which is a subring of T.

(iii) Suppose that either $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{eB}}(R)(1) \subseteq \text{Cen}(T)$. If T is also a left ring of quotients of R, then $T \in \mathbf{eB}$ if and only if there exists a $R(\mathbf{eB}, \alpha, Q(R))$ which is a subring of T.

(iv) If T is also a left ring of quotients of R and $X \trianglelefteq R$ implies $TX \trianglelefteq T$, then $T \in \mathbf{eqB}$ if and only if there is a $R(\mathbf{eqB}, \beta, Q(R))$ which is a subring of T.

Proposition 4.33. ([24, Corollary 2.24]) Assume that $E(R_R) = Q(R)$, $Q(R)$ is a left ring of quotients of R, and T is a right ring of quotients of R. Then:

(i) $\delta_{\mathbf{E}}(R) = \delta_{\mathbf{eB}}(R)$.

(ii) Assume that $\alpha = \beta$ or some $\delta^{\beta}_{\mathbf{E}}(R)(1) \subseteq \mathrm{Cen}(T)$. Then $T \in \mathbf{E}$ if and only if $T \in \mathbf{eB}$. Also every right extending α pseudo right ring hull of R is a right essentially Baer α pseudo right ring hull of R and conversely.

(iii) Assume that $Z(R_R) = 0$. Then $T \in \mathbf{E}$ if and only if $T \in \mathbf{B}$. Moreover every right extending α pseudo right ring hull of R is a right essentially Baer α pseudo right ring hull of R which is Baer and conversely.

The following result provides an answer to Problem I of Section 1 for the case when $\mathfrak{K} = \mathbf{E}$, the class of right extending rings, and $R = T_2(W)$ by characterizing the right extending right rings of quotients which are intermediate between $T_2(W)$ and $\mathrm{Mat}_2(W)$, where W is from a large class of local right finitely Σ-extending rings (see [43] for finitely Σ-extending modules).

Theorem 4.34. ([24, Theorem 3.11]) Let W be a local ring, V a subring of W with $J(W) \subseteq V$, $R = \begin{pmatrix} V & W \\ 0 & W \end{pmatrix}$, $S = \begin{pmatrix} V & W \\ J(W) & W \end{pmatrix}$, and $T = \mathrm{Mat}_2(W)$. Then:

(i) For each $e \in \mathbf{I}(T)$, there exists $f \in \mathbf{I}(S)$ such that $e\ \alpha\ f$.

(ii) $S \in \mathbf{E}$ if and only if $T \in \mathbf{E}$ if and only if $S = R(\mathbf{E}, \rho, T)$ for some ρ.

(iii) If W is right self-injective, then $S = R(\mathbf{E}, \alpha, T)$, and $Q_{\mathbf{qCon}}(R) = R(\mathbf{E}, T) = T$.

(iv) If $T \in \mathbf{E}$ (resp., W is right self-injective) and at least one of the following conditions is satisfied, then $S = Q^T_{\mathbf{E}}(R)$ (resp., $S = Q_{\mathbf{E}}(R)$):

(a) $J(W) \subseteq \mathrm{Cen}(W)$; (b) $\mathbf{U}(W) \subseteq \mathrm{Cen}(W)$; (c) $J(W)$ is nil; (d) W is right nonsingular.

(v) Assume that $S = Q^T_{\mathbf{E}}(R)$ and M is an intermediate ring between R and T. Then $M \in \mathbf{E}$ if and only if $M = \begin{pmatrix} A & W \\ J(W) & W \end{pmatrix}$ or $M = T$, where A is an intermediate ring between V and W.

(vi) $R \in \mathbf{FI}$ if and only if $W \in \mathbf{FI}$.

5. TRANSFERENCE BETWEEN R AND OVERRINGS

In this section, we consider Problem II from the introduction. Since $R\mathcal{B}(Q(R))$ is used in the construction of several hulls, we show how various types of information transfer between R and $R\mathcal{B}(Q(R))$. Indeed, we prove that the properties of lying over, going up, and incomparability of prime ideals hold between R and $R\mathcal{B}(Q(R))$ and so do the π-regularity and classical Krull dimension properties. Moreover, we show that $\varrho(R) = \varrho(R\mathcal{B}(Q(R))) \cap R$, where ϱ is a special radical. We use LO, GU, and INC for "lying over", "going up", and "incomparability" [77, p.292], respectively.

Lemma 5.1. ([28, Lemma 2.1]) Assume that R is a subring of a ring T and $\mathbb{E}$ is a subset of $\mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. Let S be the subring of T generated by R and $\mathbb{E}$.

(i) If K is a prime ideal of S, then $R/(K \cap R) \cong S/K$.

(ii) LO, GU, and INC hold between R and S. In particular, LO, GU, and INC hold between R and $R\mathcal{B}(Q(R))$.

We note that Lemma 5.1 generalizes results of Beidar and Wisbauer [9] for $R\mathcal{B}(Q(R))$ (see Theorem 2.9). Recall that a ring R is *left π-regular* if for each $a \in R$ there exist $b \in R$ and a positive integer n such that $a^n = ba^{n+1}$. Observe from [41] that the class of special radicals includes most well known radicals (e.g., the prime radical, the Jacobson radical, the Brown-McCoy radical, the nil radical, the generalized nil radical, etc.). For a ring R, the *classical Krull dimension* $\mathrm{kdim}(R)$ is the supremum of all lengths of chains of prime ideals of R.

Theorem 5.2. ([28, Theorem 2.2]) Assume that R is a subring of a ring T and $\mathbb{E} \subseteq \mathbf{S}_\ell(T) \cup \mathbf{S}_r(T)$. Let S be the subring of T generated by R and $\mathbb{E}$. Then we have the following.

(i) $\varrho(R) = \varrho(S) \cap R$, where ϱ is a special radical. In particular, $\varrho(R) = \varrho(R\mathcal{B}(Q(R))) \cap R$.

(ii) R is left π-regular if and only if S is left π-regular. Hence, R is left π-regular if and only if $R\mathcal{B}(Q(R))$ is left π-regular.

(iii) $\mathrm{kdim}\,(R) = \mathrm{kdim}\,(S)$. Thus, $\mathrm{kdim}\,(R) = \mathrm{kdim}\,(R\mathcal{B}(Q(R)))$.

(iv) If S is regular, then so is R.

The following corollary complements Theorems 2.11 and 2.12.

Corollary 5.3. ([28, Corollary 3.6]) For a ring R, the following are equivalent.

(i) R is regular.

(ii) $R\mathcal{B}(Q(R))$ is regular.

(iii) R is semiprime and $\widehat{Q}_{\mathbf{qB}}(R)$ is regular.

Lemma 5.1 and Corollary 5.3 show a transference of properties between R and $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$. Our next example indicates that this transference, in general, fails between R and its right rings of quotients which properly contain $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$.

Example 5.4. ([28, Example 3.7]) Let $\mathbb{Z}[G]$ be the group ring of the group $G = \{1, g\}$ over the ring $\mathbb{Z}$. Then $\mathbb{Z}[G]$ is semiprime and $Q(\mathbb{Z}[G]) = \mathbb{Q}[G]$. Note that $\mathbf{B}(\mathbb{Q}[G]) = \{0,\, 1,\, (1/2)(1+g),\, (1/2)(1-g)\}$. Thus, using Theorem 4.17(ii),

$$\mathbb{Z}[G] \neq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G])$$

and $\mathbb{Z}[G] \subseteq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) = \{(a + c/2 + d/2) + (b + c/2 - d/2)g \mid a, b, c, d \in \mathbb{Z}\} \subseteq \mathbb{Z}[1/2][G] \subseteq \mathbb{Q}[G]$, and $\widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) \neq \{(a+c/2+d/2)+(b+c/2-d/2)g \mid a, b, c, d \in \mathbb{Z}\} \subseteq \mathbb{Z}[1/2][G] \subseteq \mathbb{Q}[G]$, where $\mathbb{Z}[1/2] = \langle \mathbb{Z} \cup \{1/2\}\rangle_{\mathbb{Q}}$.

In this case, for example, LO does not hold between $\mathbb{Z}[G]$ and $\mathbb{Z}[1/2][G]$. Assume to the contrary that LO holds. From [77, Theorem 4.1], LO holds between $\mathbb{Z}$ and $\mathbb{Z}[G]$. Hence there exists a prime ideal P of $\mathbb{Z}[G]$ such that $P \cap \mathbb{Z} = 2\mathbb{Z}$. By LO, there is a prime ideal K of $\mathbb{Z}[1/2][G]$ such that $K \cap \mathbb{Z}[G] = P$. Now $K \cap \mathbb{Z}[1/2] = K_0$ is a prime ideal of $\mathbb{Z}[1/2]$. So $K_0 \cap \mathbb{Z} = K \cap \mathbb{Z}[1/2] \cap \mathbb{Z} = K \cap \mathbb{Z} = 2\mathbb{Z}$. Thus $2 \in K_0$. But since K_0 is an ideal of $\mathbb{Z}[1/2]$, $1 = 2\cdot(1/2) \in K_0$, a contradiction.

Next, $\mathbb{Q}[G]$ is regular but $\mathbb{Z}[G]$ is not, so Corollary 3.6 does not hold for right rings of quotients properly containing $R\mathcal{B}(Q(R))$ or $\widehat{Q}_{\mathbf{qB}}(R)$.

By [53, Proposition 4] a semiprime ring R with bounded index is right and left nonsingular. Thus in this case $\widehat{Q}_{\mathbf{qB}}(R) = Q_{\mathbf{qB}}(R)$.

Theorem 5.5. ([28, Theorem 3.8]) Let R be a semiprime ring. Then R has bounded index at most n if and only if $Q_{\mathbf{qB}}(R)$ ($Q_{\mathbf{pqB}}(R)$) has bounded index at most n. In particular, if R is reduced, then $Q_{\mathbf{qB}}(R) = Q_{\mathbf{B}}(R)$ and it is reduced.

We note that if R is a domain which is not right Ore, then $R = Q_{\mathbf{qB}}(R)$ has bounded index 1, but $Q(R)$ does not have bounded index. So we cannot replace "$Q_{\mathbf{qB}}(R)$" with "$Q(R)$" in Theorem 5.5. An immediate consequence of Corollary 5.3 and Theorem 5.5 is the next result.

Corollary 5.6. ([28, Corollary 3.9]) A ring R is strongly regular if and only if $R\mathcal{B}(Q(R))$ ($Q_{\mathbf{pqB}}(R)$) is strongly regular.

In Theorem 4.18, for every semiprime ring R, we show that $\widehat{Q}_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{FI}}(R)$ exist. Also as we see in Theorem 5.5, a semiprime ring with bounded index 1 (i.e., a reduced ring) always has a Baer absolute right ring hull. However a Baer absolute right ring hull does not always exist even for prime PI-rings with bounded index 2, as shown in our next example.

Example 5.7. ([28, Example 3.10]) For a field F and a positive integer $k > 1$, let $R = \mathrm{Mat}_k(F[x, y])$, where $F[x, y]$ is the ordinary polynomial ring over F. Then R is a prime PI-ring with bounded index k. (In particular, if $k = 2$, then R has bounded index 2.) Now R has the following properties (observe that $Q(R) = E(R_R)$, hence $\widehat{Q}_{\mathfrak{K}}(R) = Q_{\mathfrak{K}}(R)$ for any class $\mathfrak{K}$ of rings).

(i) $Q_{\mathbf{B}}(R)$ does not exist.

(ii) $Q_{\mathbf{E}}(R)$ does not exist.

Since R is prime, $R = Q_{\mathbf{qB}}(R) = Q_{\mathbf{FI}}(R)$. We *claim* that $Q_{\mathbf{B}}(R)$ does not exist (the same argument shows that $Q_{\mathbf{E}}(R)$ does not exist). Assume to the contrary that $Q_{\mathbf{B}}(R)$ exists. Note that $F(x)[y]$ and $F(y)[x]$ are Prüfer domains. So $\mathrm{Mat}_k(F(x)[y])$ and $\mathrm{Mat}_k(F(y)[x])$ are Baer rings [58, p.17, Exercise 3] (and right extending rings [43, pp.108-109]). Note that $Q(R) = \mathrm{Mat}_k(F(x, y))$. Hence $Q_{\mathbf{B}}(R) \subseteq \mathrm{Mat}_k(F(x)[y]) \cap \mathrm{Mat}_k(F(y)[x]) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x])$. To see that $F(x)[y] \cap F(y)[x] = F[x, y]$, let

$$\gamma(x, y) = f_0(x)/g_0(x) + (f_1(x)/g_1(x))y + \cdots + (f_m(x)/g_m(x))y^m =$$
$$h_0(y)/k_0(y) + (h_1(y)/k_1(y))x + \cdots + (h_n(y)/k_n(y))x^n \in F(x)[y] \cap F(y)[x]$$

with $f_i(x), g_i(x) \in F[x]$, $h_j(y), k_j(y) \in F[y]$, and $g_i(x) \neq 0$, $k_j(y) \neq 0$ for $i = 0, 1, \ldots, m, j = 0, 1, \ldots, n$. Let $\overline{F}$ be the algebraic closure of F. If $\deg g_0(x) \geq 1$, then there is $\alpha \in \overline{F}$ with $g_0(\alpha) = 0$. So $\gamma(\alpha, y)$ cannot be defined. But $\gamma(\alpha, y) = h_0(y)/k_0(y) + (h_1(y)/k_1(y))\alpha + \cdots + (h_n(y)/k_n(y))\alpha^n$, a contradiction. Thus $g_0(x) \in F$. Similarly, $g_1(x), \ldots, g_m(x) \in F$. Hence $\gamma(x, y) \in F[x, y]$. Therefore $F(x)[y] \cap F(y)[x] = F[x, y]$. Hence $Q_{\mathbf{B}}(R) = \mathrm{Mat}_k(F(x)[y] \cap F(y)[x]) = \mathrm{Mat}_k(F[x, y])$. Thus

$\mathrm{Mat}_k(F[x,y]) \in \mathbf{B}$, a contradiction because $F[x,y]$ is a non-Prüfer domain [58, p.17, Exercise 3].

A ring is called right *Utumi* [81, p.252] if it is right nonsingular and right cononsingular (Recall that a ring R is called *right cononsingular* if any right ideal I of R with $\ell_R(I) = 0$ is right essential in R).

Corollary 5.8. ([28, Corollary 3.11]) A reduced ring R is right Utumi if and only if $R\mathcal{B}(Q(R)) = Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R)$.

There is a non-reduced right Utumi ring R for which the equalities $R\mathcal{B}(Q(R)) = Q_{\mathbf{qCon}}(R)$ and $Q_{\mathbf{E}}(R) = Q_{\mathbf{qCon}}(R)$ in Corollary 5.8 do not hold, as the following example shows.

Example 5.9. ([28, Example 3.12]) Let $R = \mathrm{Mat}_k(F[x])$, where $F[x]$ is the polynomial ring over a field F and $k > 1$. Then R is right Utumi by [81, p.252, Proposition 4.9]. We show that R is *not* right quasi-continuous. For this, let E_{ij} denote the matrix in R with 1 in the (i,j)-position and 0 elsewhere. Take

$$f_1 = xE_{11} + (1-x)E_{12} + xE_{21} + (1-x)E_{22}$$

and

$$f_2 = xE_{12} + E_{22}$$

in R. Then $f_1 = f_1^2$, $f_2 = f_2^2$ and $f_1R \cap f_2R = 0$. Also $(f_1R \oplus f_2R)_R \leq^{\mathrm{ess}} fR_R$ since the uniform dimension of fR_R is 2, where $f = E_{11} + E_{22} \in R$. If there is an idempotent $g \in R$ such that $f_1R \oplus f_2R = gR$, then $gR_R \leq^{\mathrm{ess}} fR_R$. So $gR = fR$ by the modular law. But this is impossible because $(x^2+1)E_{11} + E_{12} \in fR \setminus gR$. Therefore R is *not* right quasi-continuous. Now $R\mathcal{B}(Q(R)) = R \neq Q_{\mathbf{qCon}}(R)$. Also by [43, Lemma 12.8 and Corollary 12.10], $R \in \mathbf{E}$, so $R = Q_{\mathbf{E}}(R)$. Thus $Q_{\mathbf{E}}(R) \neq Q_{\mathbf{qCon}}(R)$.

6. HOW DOES $Q(R)$ DETERMINE R?

In this section, we investigate Problem III listed in the Introduction (i.e., Given classes $\mathfrak{K}$ and $\mathcal{S}$ of rings, determine those $T \in \mathfrak{K}$ such that $Q(T) \in \mathcal{S}$). We take the class $\mathcal{S}$ to be

$$\mathcal{S} := \{\mathrm{Mat}_2(D) \mid D \text{ is a division ring}\}$$

and $\mathfrak{K}$ to be $\mathbf{E}$, $\mathbf{B}$, or related classes.

Our first result of the section characterizes any right extending ring whose maximal right ring of quotients is the 2×2 matrix ring over a division ring.

Theorem 6.1. ([24, Theorem 3.1]) Let D be a division ring and assume that T is a ring such that $Q(T) = \mathrm{Mat}_2(D)$ (resp., $Q(T) = Q^{\ell}(T) = \mathrm{Mat}_2(D)$). Then T is right extending (resp., T is Baer) if and only if the following conditions are satisfied:

(i) there exist v, $w \in D$ such that $\begin{pmatrix} 1 & v \\ 0 & 0 \end{pmatrix} \in T$ and $\begin{pmatrix} 0 & 0 \\ w & 1 \end{pmatrix} \in T$; and

(ii) for each $0 \neq d \in D$ at least one of the following conditions is true:

(1) $\begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} \in T$,

(2) $\begin{pmatrix} 1 & 0 \\ d^{-1} & 0 \end{pmatrix} \in T$, or

(3) there exists $a \in D$ such that $a - a^2 \neq 0$ and $\begin{pmatrix} a & (1-a)d \\ d^{-1}a & d^{-1}(1-a)d \end{pmatrix} \in T$.

Corollary 6.2. ([24, Corollary 3.3]) (i) Let T be a ring such that $Q(T) = \mathrm{Mat}_2(D)$, where D is a division ring and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T$. If $\begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \subseteq T$ or $\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix} \subseteq T$, then T is right extending and Baer.

(ii) Let A be a right Ore domain with $D = Q^r_{c\ell}(A)$. Then $\begin{pmatrix} A & D \\ 0 & A \end{pmatrix}$ is a right extending right ring hull of $T_2(A)$ and it is Baer.

As a consequence of Corollary 6.2, our next example provides a right extending generalized 2-by-2 triangular matrix ring T such that $Q(T) = \mathrm{Mat}_2(D)$, where $D = Q^r_{c\ell}(A)$ and A is a right Ore domain, but T is not necessarily an overring of $T_2(A)$.

Example 6.3. ([24, Example 3.4]) Let A be a right Ore domain with $D = Q^r_{c\ell}(A)$ and B any subring of D. Then $T = \begin{pmatrix} B & D \\ 0 & A \end{pmatrix}$ is right extending and $Q(T) = \mathrm{Mat}_2(D)$. For an explicit example, take $A = \mathbb{Z}[x]$ or $\mathbb{Q}[x]$, and $B = \mathbb{Z}$.

From [58, p.16, Exercise 2] it is well known that if A is a commutative domain with F as its field of fractions and $A \neq F$, then $T_n(A)$ $(n > 1)$ is not Baer, but by Theorem 3.9 any right ring of quotients of $T_n(A)$ which contains $T_n(F)$ is Baer. This result motivates the question: *If A is a commutative domain, can we find $\mathfrak{C}$ right ring hulls or $\mathfrak{C}\,\rho$ pseudo right ring hulls for $T_n(A)$ and use these to describe all $\mathfrak{C}$ right rings of quotients of $T_n(A)$ when $\mathfrak{C}$ is a class related to the Baer class?* (See Problem I and Problem II in Section 1). Using Theorem 6.1, we answer this question when A is either a PID or a Bezout domain (i.e., every finitely generated ideal is principal [48]) and $n = 2$.

Theorem 6.4. ([24, Theorem 3.7]) Let A be a commutative Bezout domain with F as its field of fractions, $A \neq F$, and T be a right ring of quotients of $T_2(A)$. If any one of the following conditions holds, then T is right extending and Baer.

(i) $\begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ is a subring of T.

(ii) $\begin{pmatrix} A & a^{-1}A \\ aA & A \end{pmatrix}$ is a subring of T for some $0 \neq a \in A$.

(iii) $\begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1}A \\ aA & A \end{pmatrix}$ is a subring of T for some $0 \neq a \in A$, where $a = p_1^{k_1} \cdots p_m^{k_m}$, each p_i is a distinct prime, and each k_i is a positive integer.

The following corollary illustrates how both Definitions 3.1 and 3.2 can be used to characterize all right rings of quotients from a **D-E** class $\mathfrak{C}$ (see Problem I in Section 1).

Corollary 6.5. ([24, Corollary 3.9]) Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let $R = T_2(A)$.

(i) Let T be a right ring of quotients of R. Then T is right extending if and only if either the ring

$$U = \begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$$

is a subring of T, or the ring

$$V = \begin{pmatrix} A & (p_1^{k_1-1} \cdots p_m^{k_m-1})^{-1}A \\ aA & A \end{pmatrix}$$

is a subring of T for some nonzero $a = p_1^{k_1} \cdots p_m^{k_m}$, where each p_i is a distinct prime of A.

(ii) $\begin{pmatrix} A & F \\ 0 & A \end{pmatrix}$ is the unique right extending right ring hull of R.

(iii) R has no right extending absolute right ring hull.

(iv) In (i)-(iii) we can replace "right extending" with "Baer", "right PP", or "right semihereditary".

We remark that U and V, in Corollary 6.5, are right extending α pseudo right ring hulls of R; whereas $Q(R) = R(\mathbf{E}, Q(R))$. Moreover, if $\{p_1, p_2, \dots\}$ is an infinite set of distinct primes of A and

$$V_i = \begin{pmatrix} A & A \\ p_1 \cdots p_i A & A \end{pmatrix},$$

then $V_1 \supseteq V_2 \cdots$ forms an infinite descending chain of right extending α pseudo right ring hulls none of which contains U. Thus no V_i is a right extending right ring hull.

Corollary 6.6. ([24, Corollary 3.10]) Let A be a commutative PID with F as its field of fractions, $A \neq F$, and let T be a right ring of quotients of $R = T_2(A)$. Take

$$S = \begin{pmatrix} A & F \\ 0 & F \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} A & ({p_1}^{k_1-1} \cdots {p_m}^{k_m-1})^{-1}A \\ {p_1}^{k_1} \cdots {p_m}^{k_m} A & A \end{pmatrix},$$

where each p_i is a distinct prime of A.

(i) If T is right hereditary, then either S or V is a subring of T. The converse holds when T is right Noetherian.

(ii) The ring S is the unique right hereditary right ring hull of R; but R has no right hereditary absolute right ring hull.

7. HULLS OF RING EXTENSIONS

In this section, we seek solutions to Problem IV of Section 1 (i.e., Given a ring R and a class of rings $\mathfrak{K}$, let $\mathbf{X}(R)$ denote some standard type of extension of R (e.g., $\mathbf{X}(R) = R[x]$, or $\mathbf{X}(R) = \mathrm{Mat}_n(R)$, etc.) and let $\mathbf{H}(R)$ denote a right essential overring of R which is "minimal" with respect to belonging to the class $\mathfrak{K}$. Determine when $\mathbf{H}(\mathbf{X}(R))$ is comparable to $\mathbf{X}(\mathbf{H}(R)))$, where $\mathfrak{K}$ is **qB** or **FI** and the types of ring extensions include monoid rings, full and triangular matrix rings, infinite matrix rings, etc.

Theorem 7.1. ([27, Theorem 4]) Let $R[G]$ be a semiprime monoid ring of a monoid G over a ring R. Then:

(i) $\widehat{Q}_{\mathbf{qB}}(R)[G] \subseteq \widehat{Q}_{\mathbf{qB}}(R[G])$.

(ii) If G is a u.p.-monoid, then $\widehat{Q}_{\mathbf{qB}}(R[G]) = \widehat{Q}_{\mathbf{qB}}(R)[G]$.

In [49] Goel and Jain posed the open question: *If G is an infinite cyclic group and A is a prime right quasi-continuous ring, is it true that $A[G] \in$ **qCon**?* Since a semiprime right quasi-continuous ring is quasi-Baer (see [24, Proposition 1.3]) and $A[G]$ is semiprime, Theorem 7.1 and [24, Proposition 1.3] show that $A[G] \in$ **FI**. Thus, from Theorem 7.1, when A is a commutative semiprime quasi-continuous ring and G is torsion-free Abelian, then $A[G] \in \mathbf{E}$, hence $A[G] \in$ **qCon**. This provides an affirmative answer to this question when A is a commutative semiprime quasi-continuous ring.

Corollary 7.2. ([27, Corollary 5]) Let R be a semiprime ring. Then:

(i) $\widehat{Q}_{\mathbf{qB}}(R[x, x^{-1}]) = \widehat{Q}_{\mathbf{qB}}(R)[x, x^{-1}]$.

(ii) $\widehat{Q}_{\mathbf{qB}}(R[X]) = \widehat{Q}_{\mathbf{qB}}(R)[X]$ and $\widehat{Q}_{\mathbf{qB}}(R[[X]]) = \widehat{Q}_{\mathbf{qB}}(R)[[X]]$ for a nonempty set X of not necessarily commuting indeterminates.

Example 7.3. (i) ([28, Example 3.7]) Let $\mathbb{Z}[G]$ be the group ring of the group $G = \{1, g\}$ over $\mathbb{Z}$. Then $\mathbb{Z}[G]$ is semiprime, $\widehat{Q}_{\mathbf{qB}}(\mathbb{Z})[G] = \mathbb{Z}[G] \subseteq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G]) = \mathbb{Z}[G]\mathbf{B}(\mathbb{Q}[G])$, and $\mathbb{Z}[G] \neq \widehat{Q}_{\mathbf{qB}}(\mathbb{Z}[G])$. Thus the "u.p.-monoid" condition is not superfluous in Theorem 7.1(ii).

(ii) Let F be a field. Then $F[x]$ is a semiprime u.p.-monoid ring and $F[x] = Q(F)[x] \neq Q(F[x]) = F(x)$, where $F(x)$ is the field of fractions of $F[x]$. Thus "Q" cannot replace "$\widehat{Q}_{\mathbf{qB}}$" in Theorem 7.1(ii).

Theorem 7.4. ([27, Theorem 7]) Let $\mathfrak{K}$ be a class of rings such that $\Lambda \in \mathfrak{K}$ if and only if $\mathrm{Mat}_n(\Lambda) \in \mathfrak{K}$ for any positive integer n, and let $H_{\mathfrak{K}}(-)$ denote any of the right ring hulls indicated in Definition 1 for the class $\mathfrak{K}$. Then for a ring R, $H_{\mathfrak{K}}(R)$ exists if and only if $H_{\mathfrak{K}}(\mathrm{Mat}_n(R))$ exists for any n. In this case, $H_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(H_{\mathfrak{K}}(R))$.

Corollary 7.5. ([27, Corollary 9]) Let R be a ring and n a positive integer. Then:

(i) $\widehat{Q}_{\mathbf{IC}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathbf{IC}}(R)) = \mathrm{Mat}_n(R\mathcal{B}(Q(R))$.

(ii) $\widehat{Q}_{\mathbf{IC}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{IC}}(R)) = T_n(R\mathcal{B}(Q(R)))$.

(iii) If R is semiprime, then $\widehat{Q}_{\mathfrak{K}}(\mathrm{Mat}_n(R)) = \mathrm{Mat}_n(\widehat{Q}_{\mathfrak{K}}(R))$, where $\mathfrak{K} =$ **qB** or **FI**.

Theorem 7.6. ([27, Theorem 11]) Let R be a semiprime ring. If R and a ring S are Morita equivalent, then $\widehat{Q}_{\mathbf{qB}}(R)$ and $\widehat{Q}_{\mathbf{qB}}(S)$ are Morita equivalent.

In contrast to Theorem 4.17, the following result provides a large class of nonsemiprime rings T for which $Q_{\mathbf{qB}}(T) = \widehat{Q}_{\mathbf{FI}}(T) = T\mathcal{B}(Q(T))$.

Theorem 7.7. ([27, Theorem 18]) Let R be a semiprime ring and n a positive integer. Then:

(i) $\widehat{Q}_{\mathbf{qB}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{qB}}(R)) = T_n(R)\mathbf{B}(Q(T_n(R)))$.

(ii) $\widehat{Q}_{\mathbf{FI}}(T_n(R)) = T_n(\widehat{Q}_{\mathbf{FI}}(R)) = T_n(R)\mathcal{B}(Q(T_n(R)))$.

For a ring R and a nonempty set Γ, $\mathrm{CFM}_\Gamma(R), \mathrm{RFM}_\Gamma(R)$, and $\mathrm{CRFM}_\Gamma(R)$ denote the column finite, the row finite, and the column and row finite matrix rings over R indexed by Γ, respectively.

In [35, Theorem 1], it was shown that $\mathrm{CRFM}_\Gamma(R)$ is a Baer ring for all infinite index sets Γ if and only if R is semisimple Artinian. Our next result shows that the quasi-Baer property is always preserved by infinite matrix rings.

Theorem 7.8. ([27, Theorem 19]) (i) $R \in \mathbf{qB}$ if and only if $\mathrm{CFM}_\Gamma(R)$ (resp., $\mathrm{RFM}_\Gamma(R)$ and $\mathrm{CRFM}_\Gamma(R)) \in \mathbf{qB}$.

(ii) If $R \in \mathbf{FI}$, then $\mathrm{CFM}_\Gamma(R)$ (resp., $\mathrm{CRFM}_\Gamma(R)) \in \mathbf{FI}$.

(iii) If R is semiprime, then we have that $\widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R))$, $\widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R))$, and $\widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R))$.

Example 7.9. There exist a commutative regular ring R and a set Γ such that

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \subseteq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),\ \widehat{Q}_{\mathbf{qB}}(\mathrm{CFM}_\Gamma(R)) \neq \mathrm{CFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \subseteq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),\ \widehat{Q}_{\mathbf{qB}}(\mathrm{RFM}_\Gamma(R)) \neq \mathrm{RFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),$$

and

$$\widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \subseteq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R)),\ \widehat{Q}_{\mathbf{qB}}(\mathrm{CRFM}_\Gamma(R)) \neq \mathrm{CRFM}_\Gamma(\widehat{Q}_{\mathbf{qB}}(R))$$

(see [27, Example 20] for details).

8. MODULES WITH FI-EXTENDING HULLS

In module theory the class of injective modules and, its generalization, the class of extending modules have the property that every submodule of a member is essential in a direct summand of the member. This property, originated by Chatters and Hajarnavis in [37], ensures a rich structure theory for these classes. Although every module has an injective hull, it is usually hard to compute. For many modules a minimal essential extension which belongs to the class of extending modules may not exist (e.g., $\oplus_{n=1}^{\infty}\mathbb{Z}_{\mathbb{Z}}$, see comment above Proposition 8.4). Moreover the class of extending modules lacks some important closure properties (e.g., it is not closed under direct sums).

Recall from [18] that a right R-module M_R is *FI-extending* if every fully invariant submodule of M_R is essential in a direct summand of M_R. A ring R is *right FI-extending* if R_R is FI-extending. Note that the set of fully invariant submodules of a module M_R includes the socle, Jacobson radical, torsion submodule for a torsion theory (e.g., $Z(M_R)$ the singular submodule), and MI for all right ideals I of R, etc. Hence, the FI-extending condition provides an "economical use" of the extending condition by targeting only the fully invariant submodules, and thus some of the most significant submodules of M_R for an essential splitting of M_R. Natural examples of FI-extending modules abound: direct sums of uniform modules, more specifically all finitely generated Abelian groups, and semisimple modules.

We show that over a semiprime ring R, every finitely generated projective module P_R has a smallest FI-extending essential extension $H_{\mathbf{FI}}(P_R)$ (called the absolute FI-extending hull of P_R) in a fixed injective hull of P_R. Moreover, $H_{\mathbf{FI}}(P_R)$ is easily computable (see Theorem 8.2 and Proposition 8.4), it is from a class for which direct sums and direct summands are FI-extending, and since $H_{\mathbf{FI}}(P_R)$ is finitely generated and projective over $\widehat{Q}_{\mathbf{FI}}(R)$, we are assured of a reasonable transfer of information between P_R and $H_{\mathbf{FI}}(P_R)$ (e.g., see Theorem 8.5 and Corollary 8.6).

Since many well known types of Banach algebras are semiprime (e.g., C^*-algebras), all our results for semiprime rings are applicable. Finitely generated modules over a Banach algebra are considered in [52]. Kaplansky [57] defined AW^*-modules over a C^*-algebra and used them to answer several questions concerning automorphisms and derivations on certain types of C^*-algebras. Furthermore work using this module appeared in [7]. Moreover, from [32, p.352], every algebraically finitely generated C^*-module M is projective, hence $H_{\mathbf{FI}}(M)$ exists. Since every C^*-algebra A is both semiprime and nonsingular, $\widehat{Q}_{\mathbf{FI}}(A)$ always exists by Theorem 4.18. Also in [28], we characterized all C^*-algebras with only finitely many minimal prime ideals and showed that for such A, $\widehat{Q}_{\mathbf{FI}}(A)$ is also a C^*-algebra. Thus our results should yield fruitful applications to projective modules over C^*-algebras, as well as many other algebras of Functional Analysis. We shall discuss some of these applications to C^*-algebras in the next section in more detail.

Definition 8.1. ([29, Definition 1]) We fix an injective hull $E(M_R)$ of M_R and a maximal right ring of quotients $Q(R)$ of R. Let $\mathbf{M}$ be a class of right R-modules and M_R a right R-module. We call, when it exists, a module $H_{\mathbf{M}}(M_R)$ the *absolute* $\mathbf{M}$ *hull* of M_R if $H_{\mathbf{M}}(M_R)$ is the smallest essential extension of M_R in $E(M_R)$ that belongs to $\mathbf{M}$.

We first obtain the existence of the absolute FI-extending hull for every finitely generated projective module over a semiprime ring. Also this module hull is explicitly described.

Theorem 8.2. ([29, Theorem 6]) Every finitely generated projective module P_R over a semiprime ring R has the absolute FI-extending hull $H_{\mathbf{FI}}(P_R)$. Explicitly, $H_{\mathbf{FI}}(P_R) \cong e(\oplus^n \widehat{Q}_{\mathbf{FI}}(R)_R)$ where $P \cong e(\oplus^n R_R)$, for some n and $e = e^2 \in \mathrm{End}(\oplus^n R_R)$.

Corollary 8.3. ([29, Corollary 7]) Assume that R is a semiprime right Goldie ring. Then every projective right R-module P_R has the absolute FI-extending hull. Moreover, if $P \cong e(\oplus_\Lambda R_R)$ with $e = e^2 \in \mathrm{End}_R(\oplus_\Lambda R_R)$, then $H_{\mathbf{FI}}(P_R) \cong e(\oplus_\Lambda \widehat{Q}_{\mathbf{FI}}(R)_R)$.

The FI-extending hull of a module, in general, is distinct from the injective hull of the module or its extending hull (if it exists). From Corollary 8.3, $H_{\mathbf{FI}}(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Z}_\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. However in $E(\oplus_\Lambda \mathbb{Z}_\mathbb{Z}) = \oplus_\Lambda \mathbb{Q}_\mathbb{Z}$, where Λ is infinite and $\mathbb{Q}$ is the field of rational numbers, there is not even a minimal extending essential extension of $\oplus_\Lambda \mathbb{Z}_\mathbb{Z}$. Our next result gives an alternative description of $H_{\mathbf{FI}}(P_R)$ different from Theorem 8.2.

Proposition 8.4. ([29, Proposition 8]) Assume that P_R is a finitely generated projective module over a semiprime ring R. Then $H_{\mathbf{FI}}(P_R) \cong P \otimes_R \widehat{Q}_{\mathbf{FI}}(R)$ as $\widehat{Q}_{\mathbf{FI}}(R)$-modules. Hence $H_{\mathbf{FI}}(P_R)$ is also a finitely generated projective $\widehat{Q}_{\mathbf{FI}}(R)$-module.

From Osofsky [68], there is a prime ring R with $J(R) = 0$ such that $E(R_R)$ is a non-rational extension of R_R. So $Q(R)_R$ is not injective, thus $\mathrm{End}(E(R_R)) \ncong Q(R)$ as rings by [61, p.95, Proposition 3]. Hence $Q(\mathrm{End}(R_R)) \ncong \mathrm{End}(E(R_R))$ (see also [25, Proposition 2.6]). However, a special case of our next result shows that $\widehat{Q}_{\mathbf{FI}}(R) \cong \mathrm{End}(H_{\mathbf{FI}}(R_R))$ for a semiprime ring R.

Theorem 8.5. ([29, Theorem 12]) Assume that R is a semiprime ring and P_R is a finitely generated projective module. Then:

(i) $\widehat{Q}_{\mathbf{FI}}(\mathrm{End}(P_R)) \cong \mathrm{End}(H_{\mathbf{FI}}(P_R))$ as rings.

(ii) $\mathrm{Rad}(H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}) \cap P = \mathrm{Rad}(P_R)$, where $\mathrm{Rad}(-)$ is the Jacobson radical of a module.

When P_R is a progenerator, we have the following.

Corollary 8.6. ([29, Corollary 13]) Let R be a semiprime ring.

(i) If P_R is a progenerator of the category Mod-R, then $H_{\mathbf{FI}}(P_R)_{\widehat{Q}_{\mathbf{FI}}(R)}$ is a progenerator of the category Mod-$\widehat{Q}_{\mathbf{FI}}(R)$.

(ii) If R and S are Morita equivalent, then $\widehat{Q}_{\mathbf{FI}}(R)$ and $\widehat{Q}_{\mathbf{FI}}(S)$ are Morita equivalent.

Recall from [76] that a module M_R is a *quasi-Baer module* if for any $N_R \trianglelefteq M_R$, there exists $h = h^2 \in \Lambda = \mathrm{End}(M_R)$ such that $\ell_\Lambda(N) = \Lambda h$, where $\ell_\Lambda(N) = \{\lambda \in \Lambda \mid \lambda N = 0\}$. It is clear that R_R is a quasi-Baer module if and only if R is a quasi-Baer ring. Also it is shown in [76] that M_R is quasi-Baer if and only if for any $I \trianglelefteq \Lambda$ there exists $g = g^2 \in \Lambda$ such that $r_M(I) = gM$, where $r_M(I) = \{m \in M \mid Im = 0\}$. Moreover, if M_R is quasi-Baer, then $\mathrm{End}(M_R)$ is a quasi-Baer ring [76, Theorem 4.1]. Close connections between quasi-Baer modules and FI-extending modules are investigated in [37].

In the next result, we obtain another close connection between FI-extending modules and quasi-Baer modules which also generalizes some of the equivalences in [18, Theorem 4.7].

Theorem 8.7. ([29, Theorem 14]) Assume that P_R is a finitely generated projective module over a semiprime ring R. Then the following are equivalent.

(i) P_R is FI-extending.

(ii) P_R is quasi-Baer.

(iii) $\mathrm{End}(P_R)$ is a quasi-Baer ring.

(iv) $\mathrm{End}(P_R)$ is a right FI-extending ring.

9. APPLICATIONS TO RINGS WITH INVOLUTION

In this section, C^*-algebras are assumed to be nonunital unless indicated otherwise. Recall from [11] and [58] that a ring with an involution $*$ is called a *Baer $*$-ring* if the right annihilator of every nonempty subset is generated by a projection (i.e., an idempotent which is invariant under $*$) as a right ideal. (Recall that an ideal I of a ring R with an involution $*$ is called *self-adjoint* if $I^* = I$.) This condition is naturally motivated in the study of Functional Analysis. For example, every von Neumann algebra is a Baer $*$-algebra. With an eye toward returning to the roots of the theory of Baer and Baer $*$-rings (i.e., Functional Analysis), in this section we apply some of our previous results to rings with an involution.

In the first part of this section, we indicate that a ring R with a certain (i.e., semiproper) involution has a quasi-Baer $*$-ring absolute to $Q(R)$ right ring hull. For a reduced ring this hull coincides with a Baer $*$-ring absolute right ring hull. The section culminates with applications to C^*-algebras. We show that a unital C^*-algebra is boundedly centrally closed if and only if it is quasi-Baer. The existence of the boundedly centrally closed hull of a C^*-algebra A (i.e., the smallest boundedly centrally closed intermediate C^*-algebra between A and its local multiplier algebra $M_{\mathrm{loc}}(A)$) is established. Moreover, it is shown that for an intermediate C^*-algebra B between A and $M_{\mathrm{loc}}(A)$, B is boundedly centrally closed if and only if $B\mathcal{B}(Q(A)) = B$. All of the definitions, examples, and results of this section appear in [28].

Definition 9.1. Let R be a ring with an involution $*$.

(i) R is a *quasi-Baer $*$-ring* if the right annihilator of every ideal is generated by a projection as a right ideal ([16 or 20]).

(ii) We say that $*$ is *semiproper* if $xRx^* = 0$ implies $x = 0$.

As in the case for a Baer $*$-ring, the involution can be used to show that the definition of a quasi-Baer $*$-ring is left-right symmetric. If $*$ is a *proper* involution (i.e., $xx^* = 0$ implies $x = 0$ [20, p.10]), then it is semiproper. Thus all C^*-algebras have a semiproper involution since they have a proper one [11, p.11]. There is a semiproper involution on a prime ring which is not proper [20, p.4266]. If R is a (quasi-) Baer $*$-ring, then $*$ is a (semi-) proper involution [11, p.13] and [16, Proposition 3.4]. Part (ii) of the next lemma is known, but we include it for the readers' convenience.

Lemma 9.2. (i) Let $*$ be a semiproper involution on a ring R. Then R is semiprime and every central idempotent is a projection. If R is reduced, then $*$ is a proper involution.

(ii) If $*$ is a proper involution on a ring R, then R is right and left nonsingular.

Since many rings from Functional Analysis have a (semi-) proper involution (e.g., C^*-algebras), Lemma 9.2 and Theorem 4.17 guarantee that such rings have quasi-Baer right ring hulls.

Proposition 9.3. Let R be a $*$-ring (resp., reduced $*$-ring). Then the following are equivalent.

(i) R is a quasi-Baer $*$-ring (resp., Baer $*$-ring).

(ii) R is a quasi-Baer ring (resp., Baer ring) in which $*$ is a semiproper (resp., proper) involution.

(iii) R is a semiprime quasi-Baer ring and every central idempotent is a projection.

Thereby the center of a quasi-Baer $*$-ring is a Baer $*$-ring.

Note that Baer $*$-rings are quasi-Baer $*$-rings. But the converse does not hold as follows.

Example 9.4. (i) ([16, Example 2.2]) Let $R = \mathrm{Mat}_2(\mathbb{C}[x])$. Then R is a Baer ring. We can extend the conjugation on $\mathbb{C}$ to that on $\mathbb{C}[x]$. Let $*$ denote the conjugate transpose involution on R. Then $*$ is a proper involution. The right annihilator $r_R\left[\begin{pmatrix} x & 2 \\ 0 & 0 \end{pmatrix}\right]$ cannot be generated by a projection as a right ideal. So R is not a Baer $*$-ring; but, by Proposition 9.3, R is a quasi-Baer $*$-ring.

(ii) Let $^-$ be the conjugation on $\mathbb{C}$. If G is a polycyclic-by-finite group and $*$ is the involution on the group algebra $\mathbb{C}[G]$ defined by $(\sum a_g g)^* = \sum \overline{a}_g g^{-1}$, then the involution $*$ is proper. From [16, Corollary 1.9], $\mathbb{C}[G] \in \mathbf{qB}$. So $\mathbb{C}[G]$ is a quasi-Baer $*$-ring by Proposition 9.3. But in general $\mathbb{C}[G]$ is not a Baer $*$-ring. In fact, let $G = D_\infty \times C_\infty$, where D_∞ is the infinite dihedral group and C_∞ is the infinite cyclic group. Then the group G is polycyclic-by-finite. By [16, Example 1.10] $\mathbb{C}[G]$ is not a Baer $*$-ring.

There is a quasi-Baer ring R with an involution such that R has only finitely many minimal prime ideals, but not all minimal prime ideals are self-adjoint. For example, let F be a field and $R = F \oplus F$, where $*$ is the exchange involution. Then R is a Baer ring with only finitely many minimal prime ideals which are not self-adjoint.

Proposition 9.5. Let R be a semiprime $*$-ring with only finitely many minimal prime ideals. Then $\widehat{Q}_{\mathbf{qB}}(R)$ is a quasi-Baer $*$-ring if and only if every minimal prime ideal of R is self-adjoint.

Proposition 9.6. Let R be a $*$-ring and T a right essential overring of R.

(i) If $*$ extends to T and $*$ is semiproper on R, then $*$ is semiproper on T.

(ii) If $*$ extends to T, then $*$ is proper on R if and only if $*$ is proper on T.

Theorem 9.7. Let R be a ring (resp., reduced ring) with a semiproper involution $*$ and T be a right ring of quotients of R. If $*$ extends to T, then the following are equivalent.

(i) T is a quasi-Baer $*$-ring (resp., Baer $*$-ring).

(ii) $\widehat{Q}_{\mathbf{qB}}(R)$ is a subring of T.

(iii) $\mathcal{B}(Q(R)) \subseteq T$.

Thus $Q^s(R)$ is a quasi-Baer $*$-ring. Also $\widehat{Q}_{\mathbf{qB}}(R)$ is the quasi-Baer $*$-ring absolute to $Q(R)$ right ring hull of R. If R is reduced, then $\widehat{Q}_{\mathbf{qB}}(R)$ is the Baer $*$-ring absolute right ring hull of R.

In the remainder of this section, we focus on C^*-algebras. Recall that for a C^*-algebra A, the algebra of all double centralizers on A is called its multiplier algebra, $M(A)$, which coincides with the maximal unitization of A in the category of C^*-algebras. It is an important tool in the classification of C^*-algebras and in the study of K-theory and Hilbert C^*-modules.

For a C^*-algebra A, recall that $A^1 = \{a + \lambda 1_{Q(A)} \mid a \in A \text{ and } \lambda \in \mathbb{C}\}$. Then $A^1 = \{a + \lambda 1_{M(A)} \mid a \in A \text{ and } \lambda \in \mathbb{C}\}$ because $1_{Q(A)} = 1_{M(A)}$. Note that $M(A)$ and A^1 are C^*-algebras. For $X \subseteq A$, $\overline{X}$ denotes the norm closure of X in A.

Let A be a C^*-algebra. Then the set $\mathcal{I}_{ce}$ of all norm closed essential ideals of A forms a filter directed downwards by inclusion. The ring $Q_b(A)$ denotes the algebraic direct limit of $\{M(I)\}_{I\in\mathcal{I}_{ce}}$, where $M(I)$ denotes the C^*-algebra multipliers of I; and $Q_b(A)$ is called the *bounded symmetric algebra of quotients* of A in [5, p.57, Definition 2.23]. The norm closure, $M_{\text{loc}}(A)$, of $Q_b(A)$ (i.e., the C^*-algebra direct limit $M_{\text{loc}}(A)$ of $\{M(I)\}_{I\in\mathcal{I}_{ce}}$) is called the *local multiplier algebra* of A [5, p.65, Definition 2.3.1]. The local multiplier algebra $M_{\text{loc}}(A)$ was first used by Elliott in [45] and Pedersen in [69] to show the innerness of certain $*$-automorphisms and derivations. Its structure has been extensively studied in [5]. Since A is a norm closed essential ideal of A^1, $M_{\text{loc}}(A) = M_{\text{loc}}(A^1)$ by [5, p.66, Proposition 2.3.6]. Also note that $Q_b(A) = Q_b(A^1)$. See [5], [45], and [70] for more details on $M_{\text{loc}}(A)$ and $Q_b(A)$.

Lemma 9.8. Let A be a C^*-algebra. Then we have the following.

(i) $\mathcal{B}(M_{\text{loc}}(A)) = \mathcal{B}(Q(A)) = \mathbf{B}(Q^s(A)) = \mathcal{B}(Q_b(A))$.

(ii) $\text{Cen}(M_{\text{loc}}(A))$ is the norm closure of the linear span of $\mathcal{B}(Q(A))$.

When A is a unital C^*-algebra, Theorem 4.17, Lemma 9.2, and Theorem 9.7 yield that $A\mathcal{B}(Q(A)) = \widehat{Q}_{\mathbf{qB}}(A) = Q_{\mathbf{qB}}(A)$ exists and is the quasi-Baer $*$-ring absolute right ring hull of A. Thus it is of interest to consider unital C^*-algebras which are quasi-Baer $*$-rings.

Recall from [11] that a C^*-algebra is called an *AW^*-algebra* if it is a Baer $*$-ring. In analogy, we say that a unital C^*-algebra A is a *quasi-AW^*-algebra* if it is a quasi-Baer $*$-ring. Thus by Proposition 9.3, a unital C^*-algebra A is a quasi-AW^*-algebra if $A \in \mathbf{qB}$.

The next lemma shows that $Q_b(A)$ is a quasi-Baer $*$-algebra for any C^*-algebra A.

Lemma 9.9. Let A be a C^*-algebra. Then we have the following.

(i) $Q_{\mathbf{qB}}(A^1)$ is a $*$-subalgebra of $Q_b(A)$.

(ii) $Q_b(A)$ is a quasi-Baer $*$-algebra.

By Lemma 9.9, if A is a unital C^*-algebra, then $Q_{\mathbf{qB}}(A)$ is a $*$-subalgebra of $M_{\text{loc}}(A)$.

Definition 9.10. ([5, p.73, Definition 3.2.1]) For a C^*-algebra A, the C^*-subalgebra $\overline{A\mathrm{Cen}(Q_b(A))}$ (the norm closure of $A\mathrm{Cen}(Q_b(A))$ in $M_{\mathrm{loc}}(A)$) of $M_{\mathrm{loc}}(A)$ is called the *bounded central closure* of A. If $A = \overline{A\mathrm{Cen}(Q_b(A))}$, then A is said to be *boundedly centrally closed*.

A boundedly centrally closed C^*-algebra and the bounded central closure of a C^*-algebra are the C^*-algebra analogues of a centrally closed subring and the central closure of a semiprime ring, respectively. These have been used to obtain a complete description of all centralizing additive mappings on C^*-algebras [3] and for investigating the central Haagerup tensor product of multiplier algebras [4]. Boundedly centrally closed algebras are important for studying local multiplier algebras and have been treated extensively in [5].

It is shown in [5, pp.75-76, Theorem 3.2.8 and Corollary 3.2.9] that $M_{\mathrm{loc}}(A)$ and $\overline{A\mathrm{Cen}(Q_b(A))}$ are boundedly centrally closed. Every AW^*-algebra and every prime C^*-algebra are boundedly centrally closed [5, pp.76-77, Example 3.3.1]. Moreover, A is boundedly centrally closed if and only if $M(A)$ is so [5, p.74, Proposition 3.2.3]. However, there exists A which is boundedly centrally closed, but A^1 is not so [5, p.80, Remarks 3.3.10]. Hence it is of interest to investigate the boundedly centrally closed intermediate C^*-algebras between A and $M_{\mathrm{loc}}(A)$.

Definition 9.11. Let A be a C^*-algebra. The smallest boundedly centrally closed C^*-subalgebra of $M_{\mathrm{loc}}(A)$ containing A is called the *boundedly centrally closed hull* of A.

The following lemma shows that a unital C^*-algebra A is boundedly centrally closed if and only if $A \in \mathbf{qB}$. It is shown that the boundedly centrally closed hull of A is $\overline{Q_{\mathbf{qB}}(A)}$. Moreover, this lemma is a unital C^*-algebra analogue of Theorem 4.17. It generalizes [5, pp.72-73, Lemma 3.1.3 and Remark 3.1.4].

Lemma 9.12. Let A be a unital C^*-algebra. Then:

(i) A is boundedly centrally closed if and only if $A \in \mathbf{qB}$ (i.e., a quasi-AW^*-algebra).

(ii) $\overline{Q_{\mathbf{qB}}(A)} = \overline{A\mathrm{Cen}(Q_b(A))}$.

(iii) $\overline{Q_{\mathbf{qB}}(A)}$ is the boundedly centrally closed hull of A.

(iv) Let B be an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then B is boundedly centrally closed if and only if $\mathcal{B}(Q(A)) \subseteq B$.

From Proposition 9.3 and Lemma 9.12(i), the center of a quasi-AW^*-algebra (i.e., a unital boundedly centrally closed C^*-algebra by Lemma 9.12(i)) is an AW^*-algebra. The next example shows that the class of quasi-AW^*-algebras encompasses more variety than its subclass of AW^*-algebras.

Example 9.13. (i) ([11, p.15, Example 1]) There is a quasi-AW^*-algebra which is not an AW^*-algebra. Let A be the set of all compact operators on an infinite dimensional Hilbert space over $\mathbb{C}$. Then the heart of A^1 is the set of bounded linear operators with finite dimensional range space. So A^1 is subdirectly irreducible.

Since A^1 is semiprime, A^1 is prime and so $A^1 \in \mathbf{qB}$. Hence A^1 is a quasi-AW^*-algebra. But as shown in [11, p.15, Example 1], A^1 is not a Baer $*$-ring, thus A^1 is not an AW^*-algebra.

(ii) Every unital prime C^*-algebra is a quasi-AW^*-algebra. There are prime finite Rickart unital C^*-algebras (hence quasi-AW^*-algebras) which are not AW^*-algebras [51].

(iii) From [11, p.43, Corollary], $\mathbb{C}$ is the only prime projectionless unital AW^*-algebra. Various unital prime projectionless C^*-algebras (hence quasi-AW^*-algebras) are provided in [40, pp.124-129 and 205-214].

Our next example provides a nonunital C^*-algebra A such that both A and $M_{\mathrm{loc}}(A)$ are boundedly centrally closed, but A^1 is not so.

Example 9.14. Let A be the C^*-direct sum of $\aleph_0$ copies of $\mathbb{C}$. Then $M_{\mathrm{loc}}(A)$ is the C^*-direct product of $\aleph_0$ copies of $\mathbb{C}$. So both A and $M_{\mathrm{loc}}(A)$ are boundedly centrally closed, but A^1 is not so.

Thus Example 9.14 motivates one to seek a characterization of the boundedly centrally closed (not necessarily unital) intermediate C^*-algebras between A and $M_{\mathrm{loc}}(A)$. Our next result provides such a characterization in terms of $\mathcal{B}(Q(A))$ and shows the existence of the boundedly centrally closed hull of a C^*-algebra A.

Theorem 9.15. Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then:

(i) $\mathcal{B}(Q(A)) \subseteq \mathrm{Cen}(Q_b(B^1\mathcal{B}(Q(A)))) = \mathrm{Cen}(Q_b(B)) \subseteq \mathrm{Cen}(M_{\mathrm{loc}}(A))$.

(ii) $\overline{B\mathcal{B}(Q(A))} = \overline{B\mathrm{Cen}(Q_b(B))}$.

(iii) B is boundedly centrally closed if and only if $B = \overline{B\mathcal{B}(Q(A))}$.

(iv) $\overline{A\mathcal{B}(Q(A))}$ is the boundedly centrally closed hull of A.

Assume that A is a C^*-algebra and B is an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then $M(B)$ may not be contained in $M_{\mathrm{loc}}(A)$. However, the next corollary characterizes $M(B)$ to be boundedly centrally closed via $\mathcal{B}(Q(A))$. Moreover, parts (i) and (ii) are of interest in their own rights.

Corollary 9.16. Let A be a C^*-algebra and B an intermediate C^*-algebra between A and $M_{\mathrm{loc}}(A)$. Then:

(i) $\mathcal{B}(Q(B)) = \mathcal{B}(Q(A))$.

(ii) $\mathrm{Cen}(M_{\mathrm{loc}}(B)) = \mathrm{Cen}(M_{\mathrm{loc}}(A))$.

(iii) $\overline{M(B)\mathrm{Cen}(Q_b(M(B)))} = \overline{M(B)\mathcal{B}(Q(A))}$.

(iv) $M(B)$ is boundedly centrally closed if and only if $\mathcal{B}(Q(A)) \subseteq M(B)$.

Surprisingly, the next result shows that under a mild finiteness condition, $Q_{\mathbf{qB}}(A^1)$ is norm closed.

Corollary 9.17. Let A be a C^*-algebra and n a positive integer. Then the following are equivalent.

(i) A has exactly n minimal prime ideals.

(ii) $Q_{\mathbf{qB}}(A^1)$ is a direct sum of n prime C^*-algebras.

(iii) The extended centroid of A is $\mathbb{C}^n$.

(iv) $M_{\rm loc}(A)$ is a direct sum of n prime C^*-algebras.

(v) $\mathrm{Cen}(M_{\rm loc}(A)) = \mathbb{C}^n$.

(vi) Some boundedly centrally closed intermediate C^*-algebra between A and $M_{\rm loc}(A)$ is a direct sum of n prime C^*-algebras.

(vii) Every boundedly centrally closed intermediate C^*-algebra between A and $M_{\rm loc}(A)$ is a direct sum of n prime C^*-algebras.

Open Questions and Problems. (i) Determine which classes of rings are closed with respect to right essential overrings. In particular, is the class of right extending rings closed with respect to right essential overrings?

(ii) If a ring R is semiprime, then is $R\mathcal{B}(Q(R)) = Q_{\mathbf{FI}}(R)$?

Note that in [76, Example 4.2], there is an example of a module M_R such that $\mathrm{End}(M_R)$ is a quasi-Baer ring, but M_R is not quasi-Baer. In [29], we have shown that for $\mathfrak{M} = \mathbf{FI}$, if R is a semiprime ring then $H_{\mathbf{FI}}(R_R) = \widehat{Q}_{\mathbf{FI}}(R)$. This motivates:

(iii) For a given class $\mathfrak{M}$ of modules, determine necessary and/or sufficient conditions on R such that $H_{\mathfrak{M}}(R_R) = \widehat{Q}_{\mathfrak{M}}(R)$.

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Acknowledgment

The authors express their thanks to the referee for his/her helpful comments for improving the paper. The authors are grateful for the support they received from the Mathematics Research Institute, Columbus, Ohio and for the kind hospitality and support of Busan National University, South Korea, the Ohio State University at Lima, and the University of Louisiana at Lafayette.

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