MULTIPLICITIES OF THE EIGENVALUES OF PERIODIC DIRAC OPERATORS

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Abstract. Let us consider the Dirac operator
$L = iJ\frac{d}{dx} + U, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & a\cos 2\pi x \\ a\cos 2\pi x & 0 \end{pmatrix},$
where $a \neq 0$ is real, on $I = [0,1]$ with boundary conditions $bc = Per^+$, i.e., $F(1) = F(0)$, and $bc = Per^-$, i.e., $F(1) = -F(0)$, $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^1(I)$. Then $\sigma(L_{bc}) = -\sigma(L_{bc})$, and all $\lambda \in \sigma_{Per^+}(L(U))$ are of multiplicity 2, while $\lambda \in \sigma_{Per^-}(L(U))$ are simple (Thm 15). This is an analogue of E. L. Ince's statement for Mathieu-Hill operator.

Links between spectra of Dirac and Hill operators lead to detailed information about spectra of Hill operators with potentials of the Ricatti form $v = \pm p' + p^2$ (Section 3). It helps to get analogues of Grigis' results [8] on zones of instability of Hill operators with polynomial potentials and their asymptotics for the case of Dirac operators as well (Section 4.2).

keywords: Dirac operator, periodic potential, Hill operator, eigenvalue multiplicity, zones of instability.

1. Introduction

1. Let us consider Dirac operator

$$L = iJ\frac{d}{dx} + V, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \tag{1.1}$$

on $I = [0,1]$ with boundary conditions $bc = Per^+$ or Per^-. If $q(x) = \overline{p(x)}$ then L_{bc} is a self-adjoint operator, and its spectrum consists of the sequence $\{\lambda_n\}_{-\infty}^{\infty}$ of its eigenvalues. Their multiplicities could be 1 or 2.

In the case of Hill-Mathieu operator

$$M = -\frac{d^2}{dx^2} + a\cos 2\pi x, \quad a \in \mathbb{R}, \quad a \neq 0, \tag{1.2}$$

The first author acknowledges the hospitality of The Ohio State University at Newark during the academic year 2003/2004.

on I, with $bc = Per^+$ or Per^-, E. L. Ince [10] showed that *all* eigenvalues in both $\sigma_{Per^+}(M)$ and $\sigma_{Per^-}(M)$ are simple (see [5], [13]).

If M is considered on $\mathbb{R}$ as a selfadjoint (Schrödinger) operator, it follows that *all* spectral gaps are open, i.e.,

$$\sigma(M) = [0, \lambda_1^-] \bigcup \cup_{n=1}^{\infty} [\lambda_n^+, \lambda_{n+1}^-] \tag{1.3}$$

is absolutely continuous and

$$\lambda_1^- < \lambda_1^+ < \lambda_2^- < \lambda_2^+ < \cdots, \qquad \gamma_n = \lambda_n^+ - \lambda_n^- > 0, \tag{1.4}$$

where $\{\lambda_n^+, \lambda_n^-\}$ are eigenvalues of M on I for $bc = Per^+$ if n is even, or for $bc = Per^-$ if n is odd,

$$\lambda_n^\pm \asymp \pi^2 n^2, \qquad n \to \infty.$$

E. Harrel [9] and B. Avron, B. Simon [1] gave the asymptotics of $\gamma_n(M),\ M \in (1.2)$. They showed that

$$\gamma_n = 8\pi^2 \left(\frac{|a|}{4\pi^2} \right)^n \frac{1}{((n-1)!)^2} \left(1 + O(1/n^2)\right).$$

Later, A. Grigis [8] studied the asymptotics of $\gamma_n(M)$ for arbitrary trigonometric polynimial potentials. For information about the asymptotics of $\gamma_n(M)$ in the case of real-valued C^∞ or analytic potentials we refer to [2, 3], and the bibliography there. Recently, we found in [4] the asymptotics of spectral gaps $\gamma_n(L)$ of Dirac operator $L \in (1.1)$ with the cosine potential.

2. However, before we would give any statements on spectra (not semibounded any more) and spectral gaps of Dirac operator, we need to explain carefully some semantic (and mathematical) difficulties related to counting or enumeration of gaps and eigenvalues by index n running over all integers $\mathbb{Z}$.

Lemma 1. *(Counting lemma). Let $V \in (1.1)$ be C^∞ function, i.e., $p, q \in C^\infty$, and $q(x) = \overline{p(x)}$. There exists an even integer $m = m(V)$ such that*

$$\sigma^\pm = \sigma_{Per^\pm}(L) \subset I_m \cup \bigcup_{\substack{|k| > m \\ k \in \Gamma^\pm}} D_k, \tag{1.5}$$

where

$$\Gamma^+ = 2\mathbb{Z}, \qquad \Gamma^- = 2\mathbb{Z} + 1, \tag{1.6}$$

and intervals

$$I_m = [-(m+1/2)\pi, (m+1/2)\pi], \qquad D_k = [(k-1/3)\pi, (k+1/3)\pi]. \tag{1.7}$$

Moreover,

(1.8) $\#(\sigma^+ \cap D_k) = 2$ *if* k *is even*; $\#(\sigma^+ \cap D_k) = 0$ *if* k *is odd*

(1.9) $\#\sigma^- \cap D_k = 0$ *if* k *is even*; $\#(\sigma^- \cap D_k) = 2$ *if* k *is odd*

and

$$\#\sigma^+ \cap I_m = \#\left(\{2\mathbb{Z}\} \cap I_m\right) = 2(m+1), \tag{1.10}$$

$$\#\sigma^- \cap I_m = \#\left(\{2\mathbb{Z}+1\} \cap I_m\right) = 2m. \tag{1.11}$$

This statement can be found in [14].

We do not need in this paper a stronger version of a Counting Lemma (for non-C^∞ or non-symmetric potentials) which can be found in [11], [7] and [14].

Now, by Lemma 1, we know that each of the intervals D_k, $|k| > m$, for either even k or odd k, contains two eigenvalues (maybe coinciding, i.e., one eigenvalue of multiplicity 2). We denote and index them as

$$\lambda_k^+,\ \lambda_k^-, \quad \lambda_k^- \le \lambda_k^+, \quad |k| > m. \tag{1.12}$$

Indexes k, $|k| \le m$, are remaining, $2m+1$ of them, but (1.10) and (1.11) tell us that exactly $2(m+1) + 2m = 2(2m+1)$ eigenvalues, or $2m+1$ pairs are remaining without labeling. By (1.10), (1.11) they lie in the interval I_m so moving from the left we index them as

$$\lambda_{-m}^- \le \lambda_{-m}^+ < \lambda_{-(m-1)}^- \le \lambda_{-(m-1)}^+ < \cdots < \lambda_m^- \le \lambda_m^+.$$

This procedure labels each eigenvalue, and (1.10), (1.11) and (1.12) guarantee that nobody (either index or eigenvalue) left behind. Moreover, each eigenvalue with an even index comes from $bc = Per^+$, and each eigenvalue labeled by an odd index comes from $bc = Per^-$.

This procedure is in particular important when we count and index spectral gaps

$$\gamma_n = \lambda_n^+ - \lambda_n^-. \tag{1.13}$$

By this definition, their indexes come from the pair $\{\lambda_n^-, \lambda_n^+\}$. If $\lambda_n^- = \lambda_n^+$ then of course $\gamma_n = 0$, i.e., this gap is closed, but it has not to be forgotten.

Only with this rule of indexation we can write proper asymptotics and count many closed gaps. Proposition 12 and Proposition 24 make this point pronouncedly.

3. In [4] we analyzed spectra $\sigma^\pm$ of Dirac operator

$$L = iJ\frac{d}{dx} + \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}, \quad p = a\cos 2\pi x, \tag{1.14}$$

and showed $\gamma_{-n} = \gamma_n$, and that for $N = N(a)$ sufficiently large

$$\gamma_n = 0 \quad \text{for even } n, \quad |n| > N,$$

(1.15)

$$\gamma_n = 2|a| \left(\frac{a}{4\pi}\right)^{n-1} \left[\left(\frac{n-1}{2}\right)!\right]^{-2} \left[1 + 0\left(\frac{\ln n}{n}\right)\right], \quad \text{for odd} \quad n > N_*.$$

Of course, it implies that *for* $|n| > N$ *odd gaps are open but even gaps are closed.*

One of the main goals (and results) of this paper is to show that the same is true for *all gaps, i.e., for Dirac operator (1.14) with cosine potential*

$$\gamma_n = 0 \quad for\ even\ \ n, \qquad \gamma_n > 0 \quad for\ odd\ \ n, \quad n \in \mathbb{Z}.$$

Links between spectra of Dirac operators (1.14) with any even p, and Hill operators with a potential $v(x) = \pm p'(x) + p^2(x)$ (Sect. 3, Thm 15) help us to reformulate Grigis' results on zones of instability of Hill operators with a polynomial potential for Dirac operators as well.

Acknowledgement. We thank Prof. L. Friedlander of University of Arizona for discussions related to our paper and other topics of spectral analysis of differential operators.

2. Special case of potential with $p(x) = a(1 + e^{-2\pi i x}),\ q(x) = \overline{p(x)}$.

This potential has a series of nice and special features. Its investigation is important for us as a step in finding multiplicities of eigenvalues of Dirac operator with *cosine* potential.

Proposition 2. *In the case*

(2.1)

$$p(x) = a\left(1 + e^{2\pi i x}\right), \quad q(x) = \overline{p(x)} = a\left(1 + e^{-2\pi i x}\right), \quad a \in \mathbb{R} \setminus 0,$$

all eigenvalues $\lambda \in \sigma(L_{bc})$, $bc = Per^+$ *or* Per^-, *are simple, i.e., of multiplicity 1.*

1. This is our main result in Section 1. The conclusive argument is given in Subsection 1.5. Many elements of the proof have claims on potentials that are more general than just (2.1). But we always assume that p and q are periodic, of period 1, i.e.,

$$p(x+1) = p(x), \quad q(x+1) = q(x), \quad \forall x \in \mathbb{R}.$$

Lemma 3. *Suppose* $F = \begin{bmatrix} f \\ g \end{bmatrix}$ *is a* λ*-eigenfunction of* L_{bc}*, i.e.*

$$LF = \lambda F, \qquad F \in D(L_{bc}). \tag{2.2}$$

(a) If

$$q(x) = p(1-x), \tag{2.3}$$

then

$$K = \begin{bmatrix} g(1-x) \\ f(1-x) \end{bmatrix} \tag{2.4}$$

is a λ-eigenfunction as well.

(b) If

$$p(1-x) = p(x), \quad q(1-x) = q(x) \tag{2.5}$$

then

$$\tilde{K} = \begin{bmatrix} f(1-x) \\ -g(1-x) \end{bmatrix}$$

is a $(-\lambda)$-eigenfunction, i.e.

$$L\tilde{K} = -\lambda \tilde{K}. \tag{2.6}$$

Proof. (a) Condition (2.2) means that

$$\begin{aligned} &if'(x) + p(x)g(x) = \lambda f(x), \\ &-ig'(x) + q(x)f(x) = \lambda g(x). \end{aligned} \tag{2.7}$$

Substituting $1-x$ instead of x, and taking into account that

$$f'(1-x) = -[f(1-x)]', \quad g'(1-x) = -[g(1-x)]',$$

we obtain that

$$\begin{aligned} &-i[f(1-x)]' + p(1-x)g(1-x) = \lambda f(1-x), \\ &i[g(1-x)]' + q(1-x)f(1-x) = \lambda g(1-x). \end{aligned} \tag{2.8}$$

Thus (2.3) implies that (2.8) may be written as

$$LK = \lambda K. \tag{2.9}$$

By the definition (2.4), it is clear that

$$F \in Per^+ \Leftrightarrow K \in Per^+; \tag{2.10}$$

and

$$F \in Per^- \Leftrightarrow K \in Per^-. \tag{2.11}$$

Therefore

$$K \in D(L_{bc}) \quad \text{iff} \quad F \in D(L_{bc}), \tag{2.12}$$

so (2.9) and (2.12) mean that K is a λ-eigenfunction of L_{bc}.

(b) To prove part (b), with (2.5), we can rewrite (2.7) as

$$\begin{aligned} &-i[f(1-x)]' + p(x)[-g(1-x)] = (-\lambda) f(1-x), \\ &-i[-g(1-x)]' + q(x) f(1-x) = (-\lambda)[-g(1-x)]. \end{aligned} \tag{2.13}$$

This is an equivalent of (2.6). (2.10) and (2.11) hold as well. Lemma 3 is proven. □

2. Lemma 3(a) leads to a decomposition of λ-eigenfunctions into "even" and "odd" components D and H :

$$2 \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} d(x) \\ d(1-x) \end{bmatrix} + \begin{bmatrix} h(x) \\ -h(1-x) \end{bmatrix} = D + H, \tag{2.14}$$

where

$$d(x) = f(x) + g(1-x), \qquad h(x) = f(x) - g(1-h).$$

If we know this special structure of vector functions D or H, then the system (2.2), or (2.7), will be equivalent to *one* differential equation for a function $d(x)$ or a function $h(x)$. For D we write (2.7) as

$$\begin{aligned} &id'(x) + p(x) d(1-x) = \lambda d(x), \\ &-i[d(1-x)]' + q(1-x) d(1-x) = \lambda d(1-x). \end{aligned} \tag{2.15}$$

These lines are identical if (see (2.3)

$$q(1-x) = p(x).$$

The same type formulas show that $LH = \lambda H$ is equivalent to *one* differential equation

$$ih'(x) - p(x) h(1-x) = \lambda h(x). \tag{2.16}$$

We explained that the following is true.

Lemma 4. *Under assumptions of Lemma 3, if λ has a multiplicity 2, then both equations (2.15) and (2.16) have non-zero solutions $d(x), h(x) \in Per^+(Per^-)$ if $bc = Per^+(Per^-)$.*

Proof. Indeed (let us assume $bc = Per^+$),

$$D(x) = \begin{bmatrix} d(x) \\ d(1-x) \end{bmatrix}, \quad \text{so} \quad D(0) = \begin{bmatrix} d(0) \\ d(1) \end{bmatrix} = D(1) = \begin{bmatrix} d(1) \\ d(0) \end{bmatrix} \tag{2.17}$$

and $D \in Per^+$ is equivalent to $d(0) = d(1)$. In an analogous way we have
(2.18)

$$H(x) = \begin{bmatrix} h(x) \\ -h(1-x) \end{bmatrix}, \quad \text{so} \quad H(0) = \begin{bmatrix} h(0) \\ -h(1) \end{bmatrix} = H(1) = \begin{bmatrix} h(1) \\ -h(0) \end{bmatrix}$$

and $H \in Per^+$ implies (and is equivalent to) $h(0) = h(1)$.

The same type formulas do the case Per^-.

Both D and H are nonzero functions. Indeed, as (2.14) shows

$$E(\lambda) = \{LF = \lambda F, \quad F \in bc\}$$

is

$$Lin\ Span\{D \in (2.17)\ \ and\ \ H \in (2.15), \quad D, H \in bc\},$$

and by Lemma's assumption

$$\dim E(\lambda) = 2. \tag{2.19}$$

If, say, all $H \in (2.18) + (2.16)$ are zero functions then $F(0) = \tau \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and if $C(x)$ is the (unique) solution of an initial value problem

$$LC = \lambda C, \quad C(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

we have $F = \tau C$ and $\dim E(\lambda) \le 1$, in contradiction to (2.19).

If all $D \in (2.17) + (2.15)$ are zero functions then $F(0) = \sigma \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and if $C(x)$ is the unique (!) solution of an initial value problem

$$LC = \lambda C, \quad C(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

then we have $F = \sigma C$ and again $\dim E(\lambda) \le 1 < 2$, in contradiction to (2.19). Lemma 4 is proven. □

3. Now we'll deal with Equations (2.15) and (2.16) in terms of Fourier coefficients of functions d and h. But at the start it is important to make clear that we have two cases Per^+ and Per^-, and

$$d(x) = \sum_{k \in \Gamma} d_k e^{i\pi k x}, \qquad x \in [0,1], \tag{2.20}$$

$$h(x) = \sum_{k \in \Gamma} h_k e^{i\pi k x}, \qquad x \in [0,1], \tag{2.21}$$

where

$$\Gamma = 2\mathbb{Z} \quad \text{if} \quad bc = Per^+, \qquad \Gamma = 2\mathbb{Z}+1 \quad \text{if} \quad bc = Per^-. \tag{2.22}$$

Lemma 5. *If $p \in$ (2.1), and $\Gamma \in$ (2.22), the equations (2.15) and (2.16), with conditions $d, h \in bc$, are equivalent to equations*

$$-(\pi k+\lambda)d_k + a(d_{-k}+d_{-k+2}) = 0, \quad k \in \Gamma \tag{2.23}$$

$$(\pi k+\lambda)h_k + a(h_{-k}+h_{-k+2}) = 0, \quad k \in \Gamma \tag{2.24}$$

correspondingly.

Proof. Equations (2.23) and (2.24) come if we compare Fourier coefficients, $k \in \Gamma$, of the functions on the left and on the right in (2.15) and (2.16). □

With $a \neq 0$, put

$$B = \pi/a, \qquad \lambda = \pi\mu. \tag{2.25}$$

Then $a^{-1}(\pi k+\lambda) = B(k+\mu)$, and we rewrite (2.23) and (2.24) as

$$-B(k+\mu)d_k + d_{-k} + d_{-k+2} = 0, \quad k \in \Gamma \tag{2.26}$$

$$B(k+\mu)h_k + h_{-k} + h_{-k+2} = 0, \quad k \in \Gamma \tag{2.27}$$

Lemma 6. *For any $S(k)$, $k \in \Gamma$, $\Gamma = 2\mathbb{Z}$ or $2\mathbb{Z}+1$, the recurrences*

$$S(k)x_k + x_{-k} + x_{-k+2} = 0, \quad k \in \Gamma \tag{2.28}$$

determine the sequence $(x_k)_{k\in\Gamma}$ by the value of x_0, if $\Gamma = 2\mathbb{Z}$, or x_1, if $\Gamma = 2\mathbb{Z}+1$. In particular, $x_k = 0 \ \forall k$ if x_0, or respectively x_1, is zero.

Proof. (i) Case $\Gamma = 2\mathbb{Z}$. Put $k = 0$ in (2.28); then

$$x_2 = -[1+S(0)]x_0. \tag{2.29}$$

If $k = 2$ we have

$$S(2)x_2 + x_{-2} + x_0 = 0, \tag{2.30}$$

so

$$x_{-2} = -x_0 - S(2)x_2 = x_0[-1+S(2)(1+S(0))]. \tag{2.31}$$

If we know all x_i, $-2m \le i \le 2m$, $i \in \Gamma$, then (2.28) with $k = -2m$ gives

$$S(-2m)x_{-2m} + x_{2m} + x_{2m+2} = 0 \tag{2.32}$$

which determines x_{2m+2}. In an analogous way from (2.28) with $k = 2m$ it follows

$$S(2m+2)x_{2m+2} + x_{-2m-2} + x_{-2m} = 0, \tag{2.33}$$

thus

$$x_{-(2m+2)} = -x_{-2m} - S(2m+2)x_{2m+2} \tag{2.34}$$

is defined as well.

This induction process determines the sequence $(x_k)_{k\in\Gamma}$. Of course, if $x_0 = 0$ we obtain by induction that all $x_k = 0,\ k \in \mathbb{Z}$.

(ii) Case $\Gamma = 2\mathbb{Z} + 1$.

First we choose $k = 1$ in (2.28), so

$$S(1)x_1 + x_{-1} + x_1 = 0 \tag{2.35}$$

and

$$x_{-1} = -[1 + S(1)]x_1.$$

If we know all $x_i,\ |i| \le 2m + 1,\ m \ge 0$, then (2.28) with $k = -(2m + 1)$ gives

$$S(-2m-1)x_{-(2m+1)} + x_{2m+1} + x_{2m+3} = 0 \tag{2.36}$$

which determines x_{2m+3}. Next, from (2.28) with $k = 2m + 3$ we obtain

$$S(2m+3)x_{2m+3} + x_{-(2m+3)} + x_{-(2m+1)} = 0, \tag{2.37}$$

so $x_{-(2m+3)}$ is defined as well.

This induction process determines the entire sequence $(x_k)_{k\in\Gamma}$. Of course, if $x_1 = 0$, we obtain by induction that all $x_k = 0,\ k \in \Gamma$. Lemma 6 is proven. $\square$

4. The specific form of $S(k)$ was not important in Lemma 6. Of course, it covers the cases

$$S(k) = \varepsilon B(k + \mu), \quad \varepsilon = 1 \text{ or } -1, \tag{2.38}$$

so (2.26) and (2.27) are particular examples of (2.28). Therefore Lemma 6 implies that d_0 for $\Gamma = 2\mathbb{Z}$ or d_1 for $\Gamma = 2\mathbb{Z} + 1$ in (2.23) and (2.26), (and h_0 or h_1 in (2.24) and (2.27)), uniquely determine the entire sequence $(d_k)_{k\in\Gamma}$ (and $(h_k)_{k\in\Gamma}$). But now these coefficients depend on a parameter μ.

Lemma 7. *With $S \in$ (2.38) if*

(a) $x_0 = 1$ *for* $\Gamma = 2\mathbb{Z}$

or

(b) $x_1 = 1$ *for* $\Gamma = 2\mathbb{Z} + 1$

then the elements of the sequence (x_k) defined by (2.28) in Lemma 6 are polynomials of μ.

Proof. (a) First, we consider the case $\Gamma = 2\mathbb{Z}$.

By (2.29) we have

$$x_2 = -[1 + \varepsilon B\mu] := P_2(\mu), \tag{2.39}$$

and by (2.30) and (2.31)

$$x_{-2} = -1 - \varepsilon B(\mu + 1)P_1(\mu) := P_{-2}(\mu) \tag{2.40}$$

is a polynomial of degree 2. By induction, (2.32), (2.33) and (2.34) define polynomials $P_k(\mu),\ k \in \Gamma$. Indeed, if these polynomials are known for $|k| \le 2m$ then we have

$$x_{2m+2} = -x_{2m} - \varepsilon x_{-2m} B(-2m+\mu) = P_{2m+2}(\mu) \tag{2.41}$$

where

$$P_{2m+2}(\mu) := -P_{-2m}(\mu) + \varepsilon B(2m-\mu) P_{-2m},$$

and

$$x_{-2(m+1)} = -x_{-2m} - \varepsilon x_{2m+2} B(2m+2+\mu) = P_{-2(m+1)}(\mu) \tag{2.42}$$

where

$$P_{-2(m+1)}(\mu) := -P_{-2m}(\mu) + \varepsilon P_{2m+2}(\mu) B(2m+2+\mu).$$

These formulae prove Lemma 7 if $\Gamma = 2\mathbb{Z}$.

(b) Now, let $\qquad \Gamma = 2\mathbb{Z}+1.$
Put

$$Q_0(\mu) := 1 \tag{2.43}$$

and by (2.35)

$$x_1 = -(1 + \varepsilon B(\mu+1)) := Q_1(\mu). \tag{2.44}$$

We omit details. As in (2.36), (2.37) gives a sequence of polynomials $Q_k(\mu),\ k \in \Gamma$, such that

$$x_k = Q_k(\mu). \tag{2.45}$$

Lemma 7 is proven. □

Lemma 8. *If* $(x_k,\ k \in \Gamma)$ *is a solution of (2.26) or (2.27) then for any* k *such that* $k \ne \mu, \mu+2$

$$H(k) x_k + \frac{x_{k+2}}{k-2} + \frac{x_{k-2}}{k-2-\mu} = 0, \tag{2.46}$$

where

$$H(k) = B^2(k+\mu) + \frac{1}{k-\mu} + \frac{1}{k-2-\mu}. \tag{2.47}$$

Proof. With parameter $\varepsilon = 1$ or -1 we can rewrite (2.26) and (2.27) in a unified form as

$$\varepsilon B(k+\mu) x_k + x_{-k} + x_{-k+2} = 0 \tag{2.48}$$

or

$$B(k+\mu) x_k + \varepsilon x_{-k} + \varepsilon x_{-k+2} = 0 \tag{2.49}$$

Therefore

$$\varepsilon x_k = -\frac{1}{B(k+\mu)}(x_{-k} + x_{-k+2}) \tag{2.50}$$

and (2.48) or (2.50) implies for $-k$ and $-(k-2)$ that

$$\varepsilon x_{-k} = \frac{1}{B(k-\mu)}(x_k + x_{k+2}) \tag{2.51}$$

and

$$\varepsilon x_{-k+2} = \frac{1}{B(k-2-\mu)}(x_{k-2} + x_k) \tag{2.52}$$

Now if we put these εx_{-k} and x_{-k+2} into (2.49) we'll come exactly to (2.46)-(2.47). □

It is important that Equation (2.46) does not depend on ε but both $(d_k) \in$ (2.23) and $(h_k) \in$ (2.24) which come from D and H of Lemma 2.1 satisfy the same equations (2.46).

For any two sequences $(x_k), (y_k),\ k \in \Gamma$ let us define a Wronskian

$$W(x,y;i) = x_{i+2}y_i - x_i y_{i+2}, \quad i \in \Gamma. \tag{2.53}$$

Lemma 9. *If (2.53) holds, and x, y are two solutions of (2.46), $\mu \notin \Gamma$, then*

$$w(k)/(k-\mu) = w(k-2)/(k-2-\mu), \tag{2.54}$$

where $w(i) = W(x,y;i),\ i \in \Gamma$.

Proof. Write Equation (2.46) for y so

$$H(k)y_k + \frac{y_{k+2}}{k-\mu} + \frac{y_{k-\mu}}{k-2-\mu} = 0. \tag{2.55}$$

If we multiply both sides of (2.46) by y_k and both sides of (2.55) by x_k and subtract these equations we come to the identity (2.54). □

5. By Lemma 6 and Lemma 7 we have two uniquely defined sequences $(d) \in$ (2.26) and $(h) \in$ (2.27) with (if $\Gamma = 2\mathbb{Z}$)

$$d_0 = 1, \quad h_0 = 1, \tag{2.56}$$

and

$$d_k = P_k^+(\mu), \qquad h_k = P_k^-(\mu), \tag{2.57}$$

where $+, -$ means that in (2.41), (2.42) we put $\varepsilon = +1$ for (d) and $\varepsilon = -1$ for (h).

If $\Gamma = 2\mathbb{Z}+1$ we have

$$d_1 = 1, \qquad h_1 = 1, \tag{2.58}$$

and

$$d_k = Q_k^+(\mu), \qquad h_k = Q_k^-(\mu), \tag{2.59}$$

where $+, -$ means that in (2.44), (2.45) we put $\varepsilon = +1$ for (d) and $\varepsilon = -1$ for (h).

Lemma 9 helps us to evaluate explicitly Wronskian

$$w(k) = W(d, h; k), \qquad k \in \Gamma. \tag{2.60}$$

Of course, everything depends on μ, so we should write $w(k; \mu)$ for $w(k)$ in (2.60). By Lemma 7

$$w(k; \mu) = d_{k+2}(\mu) h_k(\mu) - d_k(\mu) h_{k+2}(\mu) \tag{2.61}$$

is a polynomial of μ of degree $\leq |k| + 2$. For any $\mu \notin \mathbb{Z}$ we can use (2.54), $k \in \mathbb{Z}$, to realize that

$$z(k; \mu) = \frac{w(k; \mu)}{k - \mu} \tag{2.62}$$

does not depend on k, i.e.

$$\frac{w(k; \mu)}{k - \mu} = \frac{w(j; \mu)}{j - \mu}, \qquad \forall k, j \in \Gamma, \quad \mu \notin \mathbb{Z}. \tag{2.63}$$

But if $j \neq k$ the right-hand side is analytic at $\mu_* = k$; therefore the left-hand side is regular at $\mu_* = k$ as well, and the polynomial $w(k : \mu)$ should vanish for $\mu = k$, i.e.

$$w(k; \mu) = R_k(\mu) \cdot (k - \mu), \tag{2.64}$$

where R_k is a polynomial, and (2.63) can be rewritten as

$$R_k(\mu) = R_j(\mu), \quad \forall k, j \in \Gamma, \ \forall \mu \in \mathbb{C}. \tag{2.65}$$

If $\Gamma = 2\mathbb{Z}$ then $R_0(\mu) = w(0; \mu)/(-\mu)$. By (2.39)

$$w(0; \mu) = P_1^+(\mu) - P_1^-(\mu) = -(1 + B\mu) + (1 - B\mu) = -2B\mu, \tag{2.66}$$

so

$$R_0(\mu) = 2B \quad \text{and} \quad R_k(\mu) = 2B \quad \forall k. \tag{2.67}$$

Finally, (2.64) becomes

$$w(k; \mu) = 2B(k - \mu), \quad \forall k \in \Gamma, \ \mu \in \mathbb{C}. \tag{2.68}$$

If $\Gamma = 2\mathbb{Z} + 1, \ \ \mu + 1 \neq 0$, by (2.44)

$$R_{-1}(\mu) = \frac{w(-1; \mu)}{-1 - \mu} \tag{2.69}$$

where

$$w(-1; \mu) = Q_1^+ - Q_1^-(\mu) = -(1 + B(\mu + 1)) + (1 - B(\mu + 1)) = -2B(\mu + 1), \tag{2.70}$$

so

$$R_{-1}(\mu) = 2B \tag{2.71}$$

and by (2.63) and (2.65)

$$R_k(\mu) = 2B, \quad \forall k \in \Gamma. \tag{2.72}$$

Finally, as in (2.68) we conclude

$$w(k;\mu) = 2B(k-\mu), \qquad \forall k \in \Gamma, \;\; \mu \in \mathbb{C}. \tag{2.73}$$

We have proven the following

Lemma 10. *Let* (d) *and* (h) *be defined by (2.26) and (2.27). Then*

$$w(k;\mu) = W(d,h;k) = 2B(k-\mu), \quad \forall k \in \Gamma, \;\; \forall \mu \in \mathbb{C}. \tag{2.74}$$

6. It immediately leads to the main claim of this section.

Proposition 11. *For each scalar* λ *in (2.23) and (2.24) the two nonzero sequences* (d) *and* (h) *do not belong to* $\ell^2(\Gamma)$ *simultaneously.*

Proof. Without loss of generality (by Lemma 6) we can assume that (2.56) if Γ is evens or (2.58) if Γ is odds hold. If both d and h belong to $\ell^2(\Gamma)$ then their Wronskian sequence (2.60) goes to zero as $k \to \pm\infty$. It contradicts to (2.74) because $B \neq 0$ [see (2.25)] and the right-hand side of (2.74) is unbounded. Proposition 11 is proven. □

Proof of Proposition 2. If λ is of multiplicity 2 then by Lemma 4 there are two L^2-(even analytic) functions $d(x), h(x)$ which are linearly independent eigenfunctions such that D of (2.17) and H of (2.18) are eigenvector functions of L_{bc}. Then by Lemma 5 their Fourier coefficient sequences (d) and (h) are nonzero ℓ^2-solutions of (2.23) and (2.24), correspondingly. By Prop. 11, this is impossible. Proposition 2 is proven.

3. Transformation of potentials and change of the spectra

0. In section 1 we showed that a potential $V = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$ with

$$p(x) = a\left((1+e^{2\pi i x}\right), \quad a \in \mathbb{R}, \quad q(x) = \overline{p(x)} \tag{3.1}$$

leads to Dirac operator

$$L = iJ\frac{d}{dx} + V, \qquad bc = Per^{\pm}, \tag{3.2}$$

such that *all* eigenvalues in both Per^+ and Per^- cases are simple. It implies that all gaps (zones of instability) are open, i.e.

$$\lambda_n^+ - \lambda_n^- = \gamma_n(V) \neq 0, \quad \forall n \in \mathbb{Z} \tag{3.3}$$

Now we transform the potential (3.1) into the cosine-potential. It is done in a few steps by using some special transformations that are quite general. We explain them in a more general setting than we would just need to analyze the cosine potential. Sometimes, we present well-known facts (compare [12], Ch. 1), at least as a folklore, in the framework that fits better to these manipulations with changing potentials.

1. *Increasing frequency.*

A system (3.2) could be rewritten as an evolution equation

$$(a) \quad F'(t) = A(t)F(t), \qquad (b) \quad A(t+1) = A(t), \tag{3.4}$$

where

$$A(t) = i \begin{pmatrix} -\lambda & p(t) \\ -q(t) & \lambda \end{pmatrix}. \tag{3.5}$$

For any initial data

$$F(0) = h \in \mathbb{C}^2 \tag{3.6}$$

its solution is given by

$$F(t) = U(t)h, \quad t \in \mathbb{R}, \tag{3.7}$$

where $U(t)$ is a fundamental matrix-solution, i.e.

$$U'(t) = A(t)U(t), \quad U(0) = 1_{\mathbb{C}^2}. \tag{3.8}$$

A monodromy matrix

$$S = U(1), \tag{3.9}$$

and periodicity (3.4)(b) implies that

$$U(m) = S^m, \qquad \forall m \in \mathbb{Z}. \tag{3.10}$$

For $A \in$ (3.5) Lyapunov function is defined as

$$\delta(\lambda) = Trace\, S, \tag{3.11}$$

and

$$\sigma(L_{Per^+}) = \{\lambda : \ \delta(\lambda) = +2\}, \tag{3.12}$$

$$\sigma(L_{Per^-}) = \{\lambda : \ \delta(\lambda) = -2\}. \tag{3.13}$$

Then

$$\det S = 1, \tag{3.14}$$

so the eigenvalues of S are c and $1/c$, with

$$c = +1 \quad \text{iff} \quad \delta(\lambda) = +2, \tag{3.15}$$

$$c = -1 \quad \text{iff} \quad \delta(\lambda) = -2. \tag{3.16}$$

Moreover, $\lambda \in (3.12)$ [or $\in (3.13)$] has a multiplicity 1 if

$$\delta'(\lambda) \neq 0, \tag{3.17}$$

and a multiplicity 2 if

$$\delta'(\lambda) = 0. \tag{3.18}$$

After this basic information on the monodromy matrix, let us follow carefully to its changes if we increase frequency of a potential.

Fix $m \geq 2, \ m \in \mathbb{Z}$. If

$$w(t) = F(mt), \tag{3.19}$$

with F being defined by (3.4) and (3.6), then

$$w'(t) = mF'(mt) = mA(mt)F(mt), \quad w(0) = F(0) = h \tag{3.20}$$

i.e. $w(t)$ is a solution of an evolution equation

$$w'(t) = B(t)w(t), \qquad w(0) = h, \tag{3.21}$$

where

$$B(t) = mA(mt) = i \begin{pmatrix} -m\lambda & mp(mt) \\ -mq(mt) & m\lambda \end{pmatrix}. \tag{3.22}$$

But by (3.19)

$$w(t) = F(mt) = U(mt)h, \tag{3.23}$$

so a fundamental matrix-solution $W(t)$ for (3.21) is determined by U :

$$W(t) = U(mt) \tag{3.24}$$

and the corresponding monodromy matrix by (3.10) is equal to

$$T = W(1) = U(m) = S^m. \tag{3.25}$$

A matrix-function $B \in (3.22)$ would come from Dirac potential Q

$$Q = \begin{pmatrix} 0 & mp(mt) \\ mq(mt) & 0 \end{pmatrix}, \tag{3.26}$$

and if $\Delta(\mu)$ denotes Lyapunov function of this Dirac operator then by (3.22), (3.26) and (3.14)

$$\Delta(m\lambda) = Trace\, T = c^m + 1/c^m. \tag{3.27}$$

This rational function of c

$$\Delta = c^m + 1/c^m \tag{3.28}$$

is a polynomial of $\delta = c + 1/c$, i.e.

$$c^m + 1/c^m = P_m(c + 1/c). \tag{3.29}$$

Remark. P_m is essentially the Chebyshev polynomial $T_m(x) = \cos(m \arccos x)$. To be precise, $P_m(2\lambda) = 2T_m(\lambda)$.

The structure and factorization of $\Delta \pm 2$ will tell us about the spectrum of $L(Q)$, $Q \in (3.26)$ in terms of the spectrum of $L(V)$, $V \in (3.2)$, (3.5). But first let us do the case $m = 2$ where

$$c^2 + 1/c^2 + 2 = (c + 1/c)^2 = \delta^2, \tag{3.30}$$

$$c^2 + 1/c^2 - 2 = \delta^2 - 4 = (\delta - 2)(\delta + 2). \tag{3.31}$$

These simple formulae help us to describe spectra of $L_{Per^{\pm}}(Q)$, $m = 2$, i.e.

$$p_2(t) = 2p(2t), \quad q_2(t) = 2q(2t) \tag{3.32}$$

if we know spectra $L_{Per^{\pm}}(V)$.

Indeed, by (3.30)

$$\Delta(2\lambda) + 2 = \delta^2(\lambda). \tag{3.33}$$

It means that $\mu = 2\lambda$ is an antiperiodic eigenvalue of $L(Q)$ if and only if

$$\delta(\lambda) = 0. \tag{3.34}$$

Such λ is not a point of $\sigma_{Per^{\pm}}(V)$. Moreover, by (3.33) and (3.34)

$$2\Delta'(2\lambda) = 2\delta(\lambda)\delta'(\lambda) = 0. \tag{3.35}$$

Therefore, all eigenvalues $\sigma(L_{Per^-}(Q))$ are of multiplicity 2.

Next, by (3.31)

$$\Delta(2\lambda) - 2 = (\delta - 2)(\delta + 2). \tag{3.36}$$

It means that $\mu = 2\lambda$ is a periodic eigenvalue, i.e. $2\lambda \in \sigma_{Per^+}(L(Q))$, if and only if

$$\delta(\lambda) = 2, \quad \text{or} \quad \delta(\lambda) = -2, \tag{3.37}$$

or if

$$\lambda \in \sigma_{Per^+}(L(V)) \cup \sigma_{Per^-}(L(V)) \tag{3.38}$$

Multiplicities are preserved because ± 2 are simple roots of the polynomial on the right-hand side of (3.36). Indeed, like in (3.35)

$$2\Delta'(2\lambda) = 2\delta(\lambda)\delta'(\lambda), \tag{3.39}$$

but now $\delta(\lambda) = \pm 2$ and

$$\Delta'(2\lambda) = \pm 2\delta'(\lambda). \tag{3.40}$$

Therefore, $\Delta'(2\lambda) = 0$ at roots of (3.36) if and only if $\delta'(\lambda) = 0$. It explains that multiplicity is the same. It leads us to the following statement.

Proposition 12. *Let $V(x+1) = V(x)$ and $Q \in$ (3.26, $m = 2$) or (3.32). Then*

(a) all antiperiodic eigenvalues of $L(Q)$ are of multiplicity 2, and gaps

$$\gamma_{2k+1}(Q) = 0 \tag{3.41}$$

are closed;

(b) periodic eigenvalues of $L(Q)$ are 2-multiples of both periodic and antiperiodic eigenvalues of $L(V)$ of the same multiplicity, so

$$\sigma_{Per^+}(L(Q)) = \{2\mu | \quad \mu \in \sigma_{Per^+}(L(V)) \cup \sigma_{Per^-}(L(V))\}. \tag{3.42}$$

Even gaps are determined by

$$\gamma_{2k}(Q) = 2\gamma_k(V), \qquad k \in \mathbb{Z}. \tag{3.43}$$

In particular, if all eigenvalues of $L(V)$ are simple then all periodic eigenvalues of $L(Q)$ are simple and v.v.

Proof. Each antiperiodic eigenvalue of $L(Q))$ is a root of the equation

$$\Delta(\mu) + 2 = 0, \tag{3.44}$$

or by (3.33)

$$0 = \Delta(\mu) + 2 = (\delta(\mu/2))^2. \tag{3.45}$$

It happens if and only if $\delta(\mu/2) = 0$, and in this case μ is a double root of (3.44). Of course, the gaps are closed. This proves Part (a).

Each periodic eigenvalue of $L(Q))$ is a root of the equation

$$\Delta(\mu) - 2 = 0, \tag{3.46}$$

or by (3.36)

$$0 = \Delta(\mu) - 2 = (\delta(\mu/2) - 2)(\delta(\mu/2) + 2), \tag{3.47}$$

therefore (3.42) holds. Multiplicities are preserved by (3.39)-(3.40). Counting lemma (Lemma 1) gives a proper enumeration of eigenvalues and spectral gaps. It leads to formula (3.41). □

See more general constructions for $m \geq 2$ in Prop. 20, Sect. 4.1.

3. *Gauge transform and shift of spectra.*

Again, as in (1.1), let

$$L = L(V) = iJ\frac{d}{dx} + V, \tag{3.48}$$

and let

$$LF = \lambda F, \qquad F \in Per^+ \text{ (or } Per^-). \tag{3.49}$$

Put

$$M_\beta = \begin{pmatrix} e^{i\pi\beta x} & 0 \\ 0 & e^{-i\pi\beta x} \end{pmatrix}, \quad \beta \in \mathbb{Z}, \tag{3.50}$$

Define $G(x)$ by

$$F = M_\beta G, \quad \text{or} \quad G = M_\beta^{-1} F = M_{-\beta} F. \tag{3.51}$$

Let us notice that

$$M_\beta'(x) = i\pi\beta J M_\beta(x),$$

so

$$F' = M_\beta' G + M_\beta G' = i\pi\beta J F + M_\beta G'$$

and by (3.49)

$$iJ(i\pi\beta J F + M_\beta G') + V M_\beta G = \lambda M_\beta G,$$

or

$$i M_\beta J G' + V M_\beta G = (\lambda + \pi\beta) M_\beta G$$

and

$$iJG' + (M_\beta^{-1} V M_\beta) G = (\lambda + \pi\beta) G.$$

If we consider a new potential

$$U(x) = M_\beta^{-1} V M_\beta = \begin{pmatrix} 0 & p e^{-2\pi i\beta x} \\ q e^{2\pi i \beta x} & 0 \end{pmatrix} \tag{3.52}$$

then G satisfies a differential equation

$$iJG' + U(x) G = (\lambda + \pi\beta) G. \tag{3.53}$$

For any $\beta \in \mathbb{Z}$ the new potential is periodic: $U(x+1) = U(x)$ if the initial potential V is periodic. Now (3.51) shows that if β is even then

$$F \in Per^+ \Leftrightarrow G \in Per^+, \quad F \in Per^- \Leftrightarrow G \in Per^-,$$

but for odd β

$$F \in Per^+ \Leftrightarrow G \in Per^-, \quad F \in Per^- \Leftrightarrow G \in Per^+.$$

Equation (3.53) shows how spectra shift. Our discussion proved the following.

Lemma 13. *If β is even then with notation (3.48) and (3.52)*

$$\sigma_{Per^\pm}(L(U)) = \pi\beta + \sigma_{Per^\pm}(L(V)), \tag{3.54}$$

and λ for $L(V)$ and $\pi\beta + \lambda$ for $L(U)$ have the same multiplicities. If β is odd then

$$\sigma_{Per^\pm}(L(U)) = \pi\beta + \sigma_{Per^\mp}(L(V)), \tag{3.55}$$

and λ for $L(V)$ and $\pi\beta + \lambda$ for $L(U)$, with corresponding bc, have the same multiplicities.

4. We've proven everything by now. Let us collect this information to make claims about the cosine-potential. But first, put

$$V(x) = \begin{pmatrix} 0 & a(1+e^{-2\pi i)x} \\ a(1+e^{2\pi ix}) & 0 \end{pmatrix}. \tag{3.56}$$

By Proposition 2 we know that its periodic and antiperiodic eigenvalues λ are simple. Therefore, by Proposition 12 if we consider the potential

$$Q(x) = \begin{pmatrix} 0 & 2a(1+e^{-4\pi i)x} \\ 2a(1+e^{4\pi ix}) & 0 \end{pmatrix} \tag{3.57}$$

then

$$\textit{all periodic eigenvalues of} \quad L(Q) \quad \textit{are simple,} \tag{3.58}$$

and

$$\textit{all its antiperiodic eigenvalues are double, or of multiplicity } 2. \tag{3.59}$$

If we put $\beta = 1$ (an odd integer) and transform Q as in (3.52), i.e.,

$$U(x) = M_1^{-1} Q M_1 = \begin{pmatrix} 0 & 2a(e^{-2\pi ix} + e^{2\pi ix}) \\ 2a(e^{-2\pi ix} + e^{2\pi ix}) & 0 \end{pmatrix}$$

then

$$U(x) = \begin{pmatrix} 0 & 4a\cos 2\pi x \\ 4a\cos 2\pi x & 0 \end{pmatrix}. \tag{3.60}$$

By (3.55) in Lemma 13

$$\sigma_{Per^+}(L(U)) = \pi + \sigma_{Per^-}(L(W)) \tag{3.61}$$

and

$$\sigma_{Per^-}(L(U)) = \pi + \sigma_{Per^+}(L(W)) \tag{3.62}$$

with multiplicities preserved.

Then (3.59) transformed by (3.61) means that all periodic eigenvalues of $L(U)$ are double, i.e., of multiplicity 2, while (3.58) transformed by (3.62) means that all antiperiodic eigenvalues of $L(U)$ are simple. It concludes the proof of our main claim:

Theorem 14. *For real $a \neq 0$, if $U(x) = \begin{pmatrix} 0 & a\cos 2\pi x \\ a\cos 2\pi x & 0 \end{pmatrix}$ then all $\lambda \in \sigma_{Per^+}(L(U))$ are double, and all $\lambda \in \sigma_{Per^-}(L(U))$ are simple.*

4. Links between spectra of Dirac and Hill operators

Results of Section 2, in particular its main Theorem 14, about spectra of Dirac operators lead to information about spectra of Hill operators with potentials induced by a potential of Dirac operators.

1. Let L be a Dirac operator (1.1) with $p = q$ real-valued, and with $bc = Per^+$ or Per^-. We will use Pauli (selfadjoint) matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Their commutation rules are

$$\begin{aligned} &J^2 = K^2 = H^2 = I, \quad JK = -KJ = iH, \\ &JH = -HJ = -iK, \quad KH = -HK = iJ. \end{aligned} \tag{4.1}$$

Now we can write L as

$$L = iJD + pK; \tag{4.2}$$

therefore

$$L^2 = -D^2 + p^2 - p'H. \tag{4.3}$$

Observe that

$$\frac{1}{\sqrt{2}}(1 - iK) \cdot \frac{1}{\sqrt{2}}(1 + iK) = 1 \tag{4.4}$$

and

$$\frac{1}{2}(1 - iK)H(1 + iK) = \frac{1}{2}(1 - iK)^2 H = -iKH = J, \tag{4.5}$$

so L^2 is (unitary) equivalent to

$$M = \frac{1}{2}(1 - iK)L^2(1 + iK) = -D^2 + p^2 - p'J. \tag{4.6}$$

This is a diagonal matrix, and

$$M \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h^- y_1 \\ h^+ y_2 \end{pmatrix}, \tag{4.7}$$

where

$$h^\pm u = -u'' + (p^2 \pm p')u \tag{4.8}$$

can be considered as Hill operators. Boundary conditions $bc = Per^+$, or Per^-, should be chosen the same for h^+ and h^- correspondingly to the boundary conditions of L.

Let us denote

$$E(T,\tau) = \{x \in X : Tx = \tau x\} \tag{4.9}$$

a τ-eigen-subspace of an operator T if $\tau \in \sigma_{disc}(T)$. Put

$$\mu = \lambda^2. \tag{4.10}$$

The operator h is self-adjoint, and therefore its spectrum is discrete. It is easy to see that

$$E(L^2,\mu) = E(L,\lambda) + E(L,-\lambda). \tag{4.11}$$

The Pauli matrix

$$H : \begin{pmatrix} f \\ g \end{pmatrix} \to i \begin{pmatrix} -g \\ f \end{pmatrix} \tag{4.12}$$

gives an (unitary) isomorphism between the spaces $E(L,\lambda)$ and $E(L,-\lambda)$, so their dimensions are equal, and

$$\dim E(L^2,\mu) = 2 \dim E(L,\lambda). \tag{4.13}$$

With J being diagonal, by (4.6) and (4.7), we have

$$M = \begin{pmatrix} h^- & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & h^+ \end{pmatrix} \tag{4.14}$$

and

$$E(M,\mu) = (E(h^-,\mu) \oplus 0) \oplus (0 \oplus E(h^+,\mu)). \tag{4.15}$$

Notice that with even p and odd p' the linear map

$$S : f(x) \to f(1-x) \tag{4.16}$$

gives an isomorphism

$$S : E(h^-,\mu) \to E(h^+,\mu), \tag{4.17}$$

so the two subspaces on the right of (4.15) are isomorphic and their dimensions are equal, i.e.,

$$\dim E(h^-,\mu) = \dim E(h^+,\mu). \tag{4.18}$$

Therefore by (4.15) we have

$$\dim E(M,\mu) = 2 \dim E(h^{\pm},\mu). \tag{4.19}$$

On the other hand, by (4.6), we obtain

$$\dim E(M,\mu) = \dim E(L^2,\mu). \tag{4.20}$$

Comparing (4.20), (4.19), (4.13) we conclude that

$$\dim E(L,\lambda) = \dim E(h^+,\mu). \tag{4.21}$$

This formula proves the following theorem.

Theorem 15. *Let L be a Dirac operator (1.1) with $p = q$, p real-valued, even, and $bc = Per^+$ or Per^-, and let $h^\pm$ be the Hill operators (4.8) with $bc = Per^+$ (or correspondingly Per^-). Then*

$$\sigma(h^+) = \sigma(h^-) = \{\mu = \lambda^2 : \lambda \in \sigma(L)\}, \tag{4.22}$$

and for each $\mu = \lambda^2 \in \sigma(h^\pm)$ its multiplicity, i.e., $\dim E(h^\pm, \mu)$ is the same as the multiplicity of λ, an eigenvalue of L, i.e., $\dim E(L, \lambda)$.

2. This Theorem 15 helps us to transform statements of Theorem 14 into claims about spectra of Hill operators with potentials

$$v^\pm(x) = \pm p' + p^2(x), \quad p(x) = a \cos 2\pi x, \tag{4.23}$$

or

$$v(x) = b \cos 4\pi x + c \sin 2\pi x, \quad b = a^2/2, \ c = 2\pi a. \tag{4.24}$$

Proposition 16. *Let*

$$hy = -y'' + v(x) y, \quad x \in [0, 1], \tag{4.25}$$

where v is defined in (4.24), $a \in \mathbb{R} \setminus 0$. Then

(i) all periodic eigenvalues, i.e., $\mu \in \sigma_{Per^+}(h)$, are double, so all even spectral gaps are closed;

(ii) all antiperiodic eigenvalues, i.e., $\mu \in \sigma_{Per^-}(h)$, are simple, so all odd spectral gaps are open.

It should be mentioned that these statements are known. It has been proven by Magnus and Winkler [12], Thm 7.9, in more general form. They give an analogous statement if in (4.24) we have $8bt^2 = (c/\pi)^2$, t being an integer. See more details in Example 1, Sect. 4.2, below.

Corollary 17. *A real-valued trig polynomial*

$$v(x) = b \cos 4\pi x + c \sin 2\pi x, \tag{4.26}$$

$$8b = (c/\pi)^2, \quad c \neq 0, \tag{4.27}$$

has a minimal period 1, but all even zones of instability are closed, i.e., $\gamma_n = 0$ for every even n.

Indeed, this statement immediately follows from Proposition 16 because for $a \neq 0$ the conditions (4.27) and (4.24) on b and c are equivalent.

See further discussion of these questions and related Grigis' results ([8], Cor. 4.3) in Sect. 5.2.

3. Of course, analysis of this section gives information about the *size* of spectral gaps of Hill operators with potential (4.23) if we will use

our result in [4], mentioned in the introduction; see (1.15). By Thm 15, formula (4.22),

$$\sigma(h) = \{\mu = \lambda^2 : \quad \lambda \in \sigma(L)\}. \tag{4.28}$$

A pair (μ_n^-, μ_n^+) close to $\pi^2 n^2$ comes from $(\lambda_n^-, \lambda_n^+)$ close to $\pi n, n > 0$, and

$$\mu^+ - \mu^- = (\lambda^+)^2 - (\lambda^-)^2 = (\lambda^+ + \lambda^-)(\lambda^+ - \lambda^-). \tag{4.29}$$

By Lemma 1, (1.7),

$$(\lambda_n^-, \lambda_n^+) \subset [(n - 1/3)\pi, (n + 1/3)\pi] \quad \text{if} \quad n > M(a),$$

so

$$\lambda_n^+ + \lambda_n^- = 2\pi n(1 + 0(1/n)). \tag{4.30}$$

It is an easy part. But we know, by [4], Thm. 1, (26), a sharp asymptotic [see (1.15) in Introduction] for $\lambda_n^+ - \lambda_n^- = \gamma_n(L)$, n odd, as well. If we combine (1.15) and (4.29), (4.30) we come to the following.

Proposition 18. *Under assumptions of Prop. 16, for n odd we have*

$$\gamma_n(h) = 4\pi|a|n \left(\frac{a}{4\pi}\right)^{n-1} \left[\left(\frac{n-1}{2}\right)!\right]^{-2} \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

If n is even, then Prop. 16 tells us that $\gamma_n(h) = 0$. Prop. 18 is a quantitative addition to Prop. 16 (ii).

5. Comments

1. Proposition 12 suffices in our dealing with the cosine-potential. But to get more examples let us state a general elementary fact about polynomial roots of unity and polynomial representation of

$$\Delta = c^m + 1/c^m \tag{5.1}$$

in terms of $\delta = c + 1/c$.

Lemma 19. *If $m = 2n$ is even then*

$$\Delta - 2 = (\delta - 2)(\delta + 2) \prod_{k=1}^{n-1} \left(\delta - 2\cos\frac{k\pi}{n}\right)^2, \tag{5.2}$$

$$\Delta + 2 = \prod_{k=0}^{n-1} \left(\delta - 2\cos\frac{2k+1}{2n}\pi\right)^2. \tag{5.3}$$

If $m = 2n + 1$ is odd then

$$\Delta - 2 = (\delta - 2) \prod_{k=1}^{n} \left(\delta - 2\cos\frac{2k\pi}{m}\right)^2, \tag{5.4}$$

$$\Delta + 2 = (\delta + 2) \prod_{k=1}^{n} \left(\delta + 2 \cos \frac{2k\pi}{m} \right)^2. \tag{5.5}$$

Proof. These formulas are elementary (see, e.g. [6], pp 146-147). They follow from (5.1). Let us explain (5.3); others can be done in the same way. We have

$$\Delta + 2 = c^m + 1/c^m + 2 = \frac{(c^m + 1)^2}{c^m}. \tag{5.6}$$

Put

$$\omega = e^{i2\pi/m} = e^{i\pi/n} \tag{5.7}$$

and $\tau = e^{\pi/2n}$, so $\tau^2 = \omega$, and

$$\overline{\tau\omega^k} = \tau^{-1}\omega^{-k} = \tau^{-1}\omega^{2n-k} = \tau\omega^{2n-1-k}. \tag{5.8}$$

Therefore,

$$c^m + 1 = \prod_{0}^{m-1} (c - \tau\omega^k) = \prod_{k=0}^{n-1} \cdot \prod_{k=n}^{2n-1} \cdots$$

$$= \prod_{k=0}^{n-1} (c - \tau\omega^k)(c - \tau\omega)^{2n-1-k}$$

$$= \prod_{k=0}^{n-1} (c^2 - 2cRe(\tau\omega^k) + 1)$$

$$= \prod_{k=0}^{n-1} \left(c^2 - 2c \cos \frac{2k+1}{m}\pi + 1 \right)$$

and

$$\frac{(c^m + 1)^2}{c^m} = \prod_{k=0}^{n-1} \left(\frac{c^2 - 2c\cos\frac{2k+1}{m}\pi + 1}{c} \right)^2$$

$$= \prod_{k=0}^{n-1} \left(\delta - 2 \cos \frac{2k+1}{m}\pi \right)^2.$$

Observe that (5.5) follows from (5.4) - and v.v. - if we replace c by $-c$. □

These formulae can be used - in the same way as we've proven Proposition 12 - to show that the following statement holds.

Proposition 20. *Let $V(x+1) = V(x)$, and $Q \in (3.26)$, $m \geq 2$.*

(A) Let m be even, $m = 2n$. Then all antiperiodic eigenvalues of $L(Q)$ are of multiplicity 2, and gaps are closed, i.e.

$$\gamma_{2k+1}(Q) = 0. \tag{5.9}$$

More precisely,

$$\sigma\left(L_{Per^-}(Q)\right) = \{m\lambda|\ \delta(\lambda) = 2\cos\frac{k+1}{n}\pi, \quad 0 \leq k \leq n-1.\},$$

and each $\mu = m\lambda$ in this spectrum is of multiplicity 2.

(B) Let m be even, $m = 2n$. Then

$$\sigma\left(L_{Per^+}(Q)\right) = S^0 \cup S^1,$$

where

$$S^0 = \{m\lambda|\ \delta(\lambda) = 2 \quad \text{or} \quad \delta(\lambda) = -2\}$$

and

$$S^1 = \bigcup_{k=1}^{n-1}\{m\lambda\ |\delta(\lambda) = 2\cos\frac{k\pi}{n}\},$$

with $\mu = m\lambda \in S^0$ being of multiplicity 2, and $\mu = m\lambda \in S^0$ having the same multiplicity as $\lambda \in \sigma(L(V))$.

(C) Let $m = 2n+1$ be odd. Then

$$\sigma\left(L_{Per^-}(Q)\right) = T^0 \cup T^1,$$

where

$$T^0 = \{m\lambda\ |\delta(\lambda) = -2\}, \quad T^1 = \bigcup_{k=1}^{n}\{m\lambda:\ \delta(\lambda) = -2\cos\frac{k\pi}{n}\},$$

with $\mu \in T^1$ being of multiplicity 2, and $\mu = m\lambda \in T^0$ having the same multiplicity as $\lambda \in \sigma\left(L_{Per^-}(Q)\right)$.

(D) Let $m = 2n+1$ be odd. Then

$$\sigma\left(L_{Per^+}(Q)\right) = T^0 \cup T^1,$$

where

$$T^0 = \{m\lambda\ |\delta(\lambda) = 2\}, \quad T^1 = \bigcup_{k=1}^{n}\{m\lambda:\ \delta(\lambda) = 2\cos\frac{k\pi}{n}\},$$

with $\mu \in T^1$ being of multiplicity 2, and $\mu = m\lambda \in T^0$ has the same multiplicity as $\lambda \in \sigma\left(L_{Per^+}(Q)\right)$.

Proof. As in the proof of Prop. 11 we need to interpret the formulae of Lemma 19, the analogues of (3.33) and (3.36). Then (5.3) leads to (A), (5.2) leads to (B), (5.4) leads to (C) and (5.5) leads to (D). Proposition 20 is proven. $\square$

This proposition tells us not just about eigenvalues of $L(Q)$; it explains their positions in comparison with eigenvalues of $L(V)$ and gives us a proper count and enumeration. We omit explicit statements which would follow case by case the lines of Proposition 20.

2. *Asymptotics of spectral gaps of Dirac and Hill operators with trig polynomial potentials*

This paper concerns on whether zones of instability are open or closed, i.e., whether

$$\gamma_n = 0, \quad \text{or} \quad \gamma_n > 0, \tag{5.10}$$

without special interest in the size of γ_n if it is positive. (Our Letter [4] was about asymptotics of spectral gaps.) However, even if our concern is (5.10),asymptotic formulas could help to claim that $\gamma_n > 0$ *for* n *large enough.* In this context the following A. Grigis' result is very interesting.

Proposition 21. *(*[8]*, Corollary 4.3). Let*

$$v(x) = b\cos 2\pi N x + \sum_{|k| \geq N_0} c_k e^{2\pi i k x},$$

where $b > 0, c_{-k} = \overline{c_k}$ *for* $|k| \leq N_0$, $c_0 \neq 0, 0 < N_0 < N$ *and the integers* N_0 *and* N *are relatively prime. If* $(c_{N_0})^N$ *is not a negative number, then all zones of instability* (μ_n^+, μ_n^-) *of the Hill operator*

$$My = y'' \tag{5.11}$$

are open for n *large enough.*

Example 1. (see Sect. 3.3, Prop. 16).

$$v(x) = b\cos 4\pi x + c\sin 2\pi x \tag{5.12}$$

In this case $N = 2, N_0 = 1, c = -ic/2$ and

$$c_1^2 = -c^2/4 < 0, \tag{5.13}$$

so Prop. 21 cannot be applied. By Thm 7.9, [12], if in (5.12) $8bt^2 = (c/\pi)^2$, then for t being odd integer, all but finitely many ($\leq |t| + 1$) even zones of instability are closed, and all odd zones of instability are open. If t is an even integer, then all but finitely many ($\leq |t| + 1$) are closed, and all even zones of instability are open. If t is not an integer, then all gaps are open.

In Prop. 18 we used our results [4] on asymptotics of spectral gaps of a Dirac operator to get such an asymptotics for Hill operator with the potential (4.24). But we can go to the opposite direction by using our constructions of Sect. 3.1-2 together with Grigis' Cor. 4.3 in [8] to get statements on spectral gaps of Dirac operator (1.1) with

$$p(x) = q(x) = a \cos 2K\pi x + \sum_{|k| \le K_0} a_k e^{2\pi i k x}, \tag{5.14}$$

where $a_{K_0} \neq 0,\ a_{-k} = a_k = \overline{a_k}$. Its twin Hill operator (see Thm 15) has potential

$$v(x) = p'(x) + p^2(x) = \frac{1}{2} a^2 \cos 2K\pi x + \sum_{|k| \le K_0} c_k e^{2\pi i k x} \tag{5.15}$$

where

$$N = 2K, \quad N_0 = K + K_0, \quad c_{N_0} = \frac{1}{2} a a_{K_0}, \tag{5.16}$$

so $(c_{N_0})^N = \left(\frac{1}{2} a a_{K_0}\right)^{2K}$ is negative if and only if $(a_{K_0})^{2K}$ is negative.

Corollary 22. *Let $p = q$ be of the form*

$$p(x) = a \cos 2\pi K x + \sum_{k=1}^{K_0} a_k \cos 2\pi k x, \tag{5.17}$$

where

$$a > 0, \quad a_{K_0} \neq 0, \quad 0 < K_0 < K, \tag{5.18}$$

and

$$2K, K + K_0 \quad \textit{be relatively prime.} \tag{5.19}$$

Then for $|n| \ge N_*$ large enough, the zones of instability of Dirac operator (1.1) are open.

Example 2. Certainly, (5.19) holds if $K > 1, K_0 = K - 1$. Therefore, any trig-polynomial

$$p(x) = a \cos 2\pi K x + b \cos 2\pi (K-1) x + \sum_{k=1}^{K-2} a_k \cos 2\pi k x \tag{5.20}$$

with all coefficients real, and $a > 0, b \neq 0$, satisfies the hypotheses of Corollary 22. Its Dirac operator (1.1), with $p = q$, has open zones of instability for $|n|$ large enough.

Example 3. As an excercize in junior high school trigonometry, we can write a series of polynomials by using Corollary 22, which give Dirac operators with all, but may be finitely many, *open* gaps:

(i) $a \cos 2\pi K x + b \cos 2\pi (K-1) x, \quad a > 0, b \neq 0$;

(ii) $a \cos 10\pi x + b \cos 4\pi x + c \cos 2k\pi x$;

(iii) $a \cos 14\pi x + b \cos 8\pi x + \sum_1^3 c_k \cos 2k\pi x$, where c_k are real.

Is it true that *all* of its zones of instability are open in the case of Dirac operator with potential (5.20)?

Of course, the main Grigis' result [8], Thm. 2 could also be rewritten, after our Theorem 15, for Dirac operators.

Proposition 23. *Let L be a Dirac operator with $q = p \in (5.14)-(5.19)$ of Corollary 22. Then there exists a polynomial $Q(t) = \sum_{j=1}^{N-1} \lambda_j t^j$ with coefficients depending algebraically on $a, (a_k)_1^{K_0}$ in (5.17), such that with a notation*

(5.21)
$$A_k(n) = \exp\left[\frac{2ink\pi}{N} + 2nQ\left(\left(\frac{a^2}{4n^2}\right)^{1/N} e^{2ikn/N}\right)\right], \quad N = 2K,$$

the following holds:

$$\gamma_n(L) = 2\left(\frac{a^2}{2\pi^2}\cdot\frac{e^2}{n^2}\right)^{n/2K}\left|\sum_{k=0}^{2K-1} A_k(n)\left(1 + O(\log n/n)\right)\right|, \tag{5.22}$$

$$\gamma_{-n}(L) = \gamma_n(L). \tag{5.23}$$

Remark. As before, we readjust formulas from [8] for the interval $[0,\pi]$ to the interval $[0,1]$.

Proof. In essence, we rewrite Thm. 2, [8], p.643, with understanding of Thm 15 and Prop. 16 (see Sect. 3.2-3.3), that

$$\sigma(h) = \{\mu = \lambda^2 : \quad \lambda \in \sigma(L)\}, \tag{5.24}$$

where h is the Hill operator (4.8) with a potential

$$v(x) = p^2 + p', \quad p \in (5.17) - (5.19).$$

Conditions imposed on p imply that $v(x)$ satisfies Hypotheses of Thm. 2, [8], and therefore by (1.11), [8], p. 643, we have an asymptotics for

$$\gamma_n(h) = \mu_n^+ - \mu_n^-. \tag{5.25}$$

But by (4.29), Sect. 3.4 above,

$$\lambda^+ - \lambda^- = \frac{1}{\lambda^+ + \lambda^-}\cdot(\mu^+ - \mu^-), \tag{5.26}$$

and

$$\gamma_n(L) = \lambda_n^+ - \lambda_n^- = \frac{1}{2\pi n}\gamma_n(h)\cdot\left(1 + O(\log n/n)\right). \tag{5.27}$$

Substitution of Grigis' formula (1.11), [8], p. 643, for $\gamma_n(h)$ into (5.27) on the right gives us the statement (5.22) of Prop. 23. It completes its proof. □

Remark. Let us notice that in the case of two term potential $v(x) = b\cos 4\pi x + c\sin 2\pi x$ we found an explicit sharp asymptotics of spectral gaps. These results will be published elsewhere.

3. *Hill operator with increased frequency of its potential.*

Maybe, after Propositions 11, 12 and 20 we need to mention how the same scheme works in the case of Hill operator

$$My = -y'' + v(x)y, \qquad x \in I = [0,1], \tag{5.28}$$

where

$$v(x) = v(x+1) \in L^2(I), \tag{5.29}$$

with boundary conditions $bc = Per^+$ or Per^-. Let f be an eigenfunction of M_{bc}, i.e.

$$-f'' + v(x)f = \lambda f, \tag{5.30}$$

$$f(0) = f(1), \quad f'(0) = f'(1) \quad \text{if} \quad bc = Per^+, \tag{5.31}$$

or

$$f(0) = -f(1), \quad f'(0) = -f'(1) \quad \text{if} \quad bc = Per^-. \tag{5.32}$$

If $S = S_\lambda(x)$ is the fundamental 2×2-matrix solution of (5.30), i.e.

$$S_\lambda(x) \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} \tag{5.33}$$

gives the solution of the equation

$$-y'' + (v(x) - \lambda)y = 0, \tag{5.34}$$

with initial data

$$y(0) = y_0, \qquad y'(0) = y_1, \tag{5.35}$$

then (5.29) implies

$$S_\lambda(m) = (S_\lambda(1))^m. \tag{5.36}$$

Put

$$a(x) = y(mx); \tag{5.37}$$

then

$$a'(x) = my'(mx), \quad a(0) = y(0), \quad a'(0) = my'(0).$$

Therefore, if $y \in (5.30)$ it follows

$$-a'' + m^2 v(mx)a = m^2\lambda a(x), \tag{5.38}$$

and by (5.33) and (5.37) we have, with $K = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix}$,

$$\begin{pmatrix} a(x) \\ a'(x) \end{pmatrix} = K \begin{pmatrix} y(mx) \\ y'(mx) \end{pmatrix} = \tag{5.39}$$

$$KS(mx) \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = KS(mx)K^{-1} \begin{pmatrix} a(0) \\ a'(0) \end{pmatrix}.$$

It shows that the matrix function

$$U(x) = KS(mx)K^{-1} \tag{5.40}$$

is the fundamental matrix solution of equation (5.38). Therefore, its monodromy matrix is

$$U(1) = KS(m)K^{-1} = KS(1)^m K^{-1}, \tag{5.41}$$

and the corresponding Lyapunov function is

$$\Delta(m^2\lambda) = Trace\ \left(KS(1)^m K^{-1}\right) = Trace\, S(1)^m = c^m + \frac{1}{c^m}, \tag{5.42}$$

where $c, 1/c$ are roots of quadratic equation

$$z^2 - Tr\,(S(1))\, z + 1 = 0 \tag{5.43}$$

and

$$\delta(\lambda) = c + 1/c$$

is a Lyapunov function for (5.30).

The identity (5.42) and Lemma 19 justifies the analogues of Propositions 11 - 20 for Schrödinger-Hill operators.

Proposition 24. *Let Σ^+ and Σ^- be periodic and antiperiodic spectra of the operator $M \in (5.28)$, and let Σ^+_m, Σ^-_m be periodic and antiperiodic spectra of the operator M_m,*

$$M_m g(x) = -g''(x) + m^2 v(mx) g(x), \quad 0 \le x \le 1.$$

Then for even $m = 2n$ we have

$$\Sigma^+_m = T^0 \cup T^1,$$

where

$$T^0 = \{m^2\lambda \ : \quad \delta(\lambda) = 2 \quad or \quad \delta(\lambda) = -2\}$$

$$T^1 = \bigcup_{k=1}^{n-1} T^1_k, \quad T^1_k = \{m^2\lambda \ : \ \delta(\lambda) = 2\cos\frac{2k}{m}\pi\},$$

and

$$\Sigma^-_m = T^1 = \bigcup_{k=0}^{n-1} T^1_k, \quad T^1_k = \{m^2\lambda \ : \ \delta(\lambda) = 2\cos\frac{2k+1}{m}\pi\}.$$

If $m = 2n+1$ *is odd then* $\Sigma_m^+ = T^0 \cup T^1$ *where*

$$T^0 = \{m^2\lambda \ : \quad \delta(\lambda) = 2\}$$

$$T^1 = \bigcup_{k=1}^{n} T_k^1, \quad T_k^1 = \{m^2\lambda \ : \ \delta(\lambda) = 2\cos\frac{2k}{m}\pi\},$$

$\Sigma_m^- = T^0 \cup T^1$ *where*

$$T^0 = \{m^2\lambda \ : \quad \delta(\lambda) = -2\}$$

$$T^1 = \bigcup_{k=1}^{n} T_k^1, \quad T_k^1 = \{m^2\lambda \ : \ \delta(\lambda) = -2\cos\frac{2k}{m}\pi\}.$$

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