INSTABILITY ZONES OF A PERIODIC 1D DIRAC OPERATOR AND SMOOTHNESS OF ITS POTENTIAL

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Abstract. Let L be the differential operator

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $P(x), Q(x)$ are 1-periodic functions such that $Q(x) = \overline{P(x)}$. The operator L, considered on $[0,1]$ with periodic ($y(0) = y(1)$), or antiperiodic ($y(0) = -y(1)$) boundary conditions, is self-adjoint, and moreover, for large $|n|$ it has, close to $n\pi$, a pair of periodic (if n is even), or antiperiodic (if n is odd) eigenvalues λ_n^+, λ_n^-. We study the relationship between the decay rate of instability zone sequence $\gamma_n = \lambda_n^+ - \lambda_n^-$, $n \to \pm\infty$, and the smoothness of the potential function $P(x)$.

1. Introduction

The operator

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + \begin{pmatrix} 0 & P(x) \\ \overline{P(x} & 0 \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1.1}$$

with periodic function $P(x)$ of period 1, $P \in L^2([0,1])$, is a self-adjoint operator on the real line $\mathbb{R}$. Its spectrum $\sigma(L)$ is absolutely continuous and has "band structure", i.e.,

$$\sigma(L) = \mathbb{R} \setminus \cup_{n \in \mathbb{Z}} (\lambda_n^-, \lambda_n^+),$$

where

$$\cdots < \lambda_n^- \le \lambda_n^+ < \lambda_{n+1}^- \le \lambda_{n+1}^+ < \cdots,$$

and λ_n^-, λ_n^+ is a pair of eigenvalues of the same differential operator L, but considered on the interval $[0,1]$, respectively with periodic (for even n), and antiperiodic (for odd n) boundary conditions (bc):

$$Per^+ : \ \ y(0) = y(1), \qquad Per^- : \ \ y(0) = -y(1).$$

See basic facts and further references on 1-D Dirac operators in [19], [21], [22], [26].

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Let $\gamma_n = \lambda_n^+ - \lambda_n^-$, $n \in \mathbb{Z}$, be the lengths of spectral gaps, or zones of instability, $(\lambda_n^+, \lambda_n^-)$. *What is the relationship between the decay rate of* $\gamma_n, n \to \pm\infty$, *and the smoothness of a potential* p?

In the case of Schrödinger (Hill) operators this question has a long history. Let us remind a few results and steps in understanding of this relation. H. Hochstadt [15] proved that a real-valued L^2-potential v of a Schrödinger operator

$$My = -y'' + v(x)y, \quad v(x+1) = v(x), \quad x \in \mathbb{R},$$

is a C^∞-function if and only if the gap sequence $(\gamma_n)_1^\infty$ decays faster than any power of $1/n$, that is

$$(\gamma_n) \in \ell^a = \{(x_n) \ : \quad \sum_{n=1}^{\infty} |x_n|^2 (1+n^2)^a < \infty\}$$

for every $a > 0$ (see [21]). For H. Hochstadt, it was an important step in analysis of finite-zone potentials; as soon as one knew that such a potential is a C^∞- function, it was possible [12, 15, 16] to use derivatives and derive polynomial identities involving $v, v', v'', \ldots$ to determine v. Further analysis of finite-zone potentials [9, 25] led to Dubrovin equations (see [10]).

The Gelfand-Levitan trace formula [11] and Dubrovin equations [9] have been used by E. Trubowitz [27] to show that a real-valued L^2-potential $v(x)$ is analytic if and only if the gap sequence (γ_n) decays exponentially fast, that is

$$\exists a > 0, C > 0 : \quad \gamma_n \le C \exp(-an) \ \ \forall n \ge 0.$$

In terms of weighted sequence spaces

$$\ell^2_\Omega = \{(x_n) : \quad \sum |x_n|^2 \Omega^2(n) < \infty\},$$

Sobolev or analytic functions v,

$$v(x) = \sum v_k \exp(2\pi i k x),$$

can be characterized as having their Fourier-coefficient sequences in ℓ^2_Ω, where $\Omega = (1+n^2)^{a/2}$, or $\Omega = \exp(an)$, $a > 0$, respectively. T. Kappeler and B. Mityagin [17, 18] raised the general question about the relationship between the two conditions $v \in H(\Omega)$ and $(\gamma_n) \in \ell^2_\Omega$, where

$$H(\Omega) = \{v : \ (v_k) \in \ell^2_\Omega\}, \tag{1.2}$$

for general (submultiplicative) weights. They showed that

$$v \in H(\Omega) \Rightarrow (\gamma_n) \in \ell^2_\Omega. \tag{1.3}$$

The opposite implication

$$(\gamma_n) \in \ell^2_\Omega \Rightarrow v \in H(\Omega) \tag{1.4}$$

required a delicate analysis of special non-linear equations in sequence spaces and a priori estimates of Sobolev norms of their solutions. It has been done in [2, 3, 4] for, roughly speaking, all submultiplicative sequences of subexponential growth, i.e.,

$$\lim \left(\log \Omega_n\right)/n = 0.$$

This is not just a technical restriction. For Ω with superexponential growth like $\exp(|n|^b, b > 1$, the implications (1.3) and (1.4) are not valid, but the proper adjustment can be made, and it is presented in [5]. Analysis of non-selfadjoint Hill operators, i.e., the case of complex-valued potentials, is done in [6]; see further references there.

Let us return to Dirac operators. Surprisingly enough, we could not find in the literature even a Hochschtadt type statement in this case. Still, after [17, 18] the approach developed there for the Schrödinger-Hill case has been used in the Dirac case in [13], [14] to get claims about the decay rate of spectral gaps:

$$p \in H(\Omega) \quad \Longrightarrow \quad \sum_{n\in\mathbb{Z}} \gamma_n^2 \Omega_n^2 < \infty \tag{1.5}$$

for some weights Ω, with rigid and (as we will see) unnecessary restrictions on Ω.

The main goal of the present paper is to show that for subexponential weights Ω the $H(\Omega)$-smoothness of a potential P, i.e., the condition $P \in H(\Omega)$, follows from ℓ^2_Ω-decay of two-sided sequence (γ_n), i.e.,

$$\sum \gamma_n^2 \Omega_n^2 < \infty \quad \Longrightarrow \quad P \in H(\Omega) \tag{1.6}$$

(see Theorem 11, Sect. 4, for accurate formulation). This result has been announced in [7], Thm 2(i). Maybe, it's worth to mention that there is an analogue of this implication (and equivalence) in the non-selfadjoint case (see [7], Thm 2(ii); this result will be given in detail in [8]).

In particular, (1.5) and (1.6) tell us that

(a) (γ_n) decays faster than any power of $1/n$ if and only if $P \in C^\infty$ (compare to [15]).

(b) (γ_n) decays faster than $\exp(-an)$ for some $a > 0$ if and only if P is analytic in a strip around the real axis (compare to [27])

(c) (γ_n) decays faster than $\exp(-an^\beta)$, $\beta \in (0,1)$, for some $a > 0$ if and only if the Fourier coefficients (p_k) of P decay faster than $\exp(-A|k|^\beta)$, for some $A > 0$ (compare [4]).

In the case of Schrödinger - Hill operators we have proven similar statements in [4] and [6]. The general scheme of the present paper is close to the scheme of [4]. However, the technical details and difficulties are quite different, because

(i) Dirac operator is not semibounded;
(ii) its resolvent is not a trace class operator.

We are going to make this point explicit and specific in our proofs and comments below.

The structure of our paper is as follows.

2. Basic Equation and formulae for spectral gaps

1. The Dirac operator

$$L^0 y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{2.1}$$

considered on the interval $[0,1]$ with periodic ($y(0) = y(1)$) or antiperiodic ($y(0) = -y(1)$) boundary conditions, has a discrete spectrum, respectively, $\{2k\pi, k \in \mathbb{Z}\}$ and $\{(2k+1)\pi, k \in \mathbb{Z}\}$. Each eigenvalue $n\pi$, both for periodic (if n is even), or antiperiodic (if n is add) boundary conditions has multiplicity 2, and

$$e_n^1(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-in\pi x}, \quad e_n^2(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{in\pi x} \tag{2.2}$$

are eigenfunctions corresponding to the eigenvalue $n\pi$. Moreover, if the Hilbert space $\mathbb{H} = L^2[0,1] \times L^2[0,1]$ is equipped with the scalar product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \int_0^1 \left(f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)} \right) dx, \tag{2.3}$$

then each of the systems $\{e_{2k}^1, e_{2k}^2, \ k \in \mathbb{Z}\}$ and $\{e_{2k+1}^1, e_{2k+1}^2, \ k \in \mathbb{Z}\}$ is an orthonormal basis in $\mathbb{H}$.

The operator

$$L = L^0 + V, \qquad V = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \tag{2.4}$$

where P and Q are 1-periodic functions, may be considered as a perturbation of L^0. Further we always assume that $P, Q \in L^2[0,1]$; then the operator L, considered with periodic or antiperiodic boundary conditions, has also a discrete spectrum. The following statement is known (see, for example [20, 21, 22, 23], in particular, [24], Thm. 4.1 and Prop. 4.3).

Lemma 1. *There exists $N_0 = N_0(P,Q)$ such that for each $|n| \geq N_0$ the open disc with center πn and radius $\pi/2$ contains exactly two (counted with multiplicity) periodic (if n is even), or antiperiodic (if n is odd) eigenvalues $\{\lambda_n^-, \lambda_n^+\}$ of L, i.e.,*

$$|\lambda_n^\pm - \pi n| < \pi/2, \quad |n| \geq N_0 \tag{2.5}$$

2. Suppose that $\lambda = n\pi + z$, $|n| \geq N_0$, is a periodic (or antiperiodic) eigenvalue of L with $|z| < \pi/2$ and $y \neq 0$ is a corresponding eigenvector. Let $E_n^0 = [e_n^1, e_n^2]$ be the eigenspace of L^0 that corresponds to $n\pi$, and let $\mathbb{H}(n)$ be its orthogonal complement. We denote by P_n^0 and Q_n^0, respectively, the orthogonal projectors on E_n^0 and $\mathbb{H}(n)$. Then the equation $(n\pi + z - L)y = 0$ is equivalent to the following system of two equations:

$$Q_n^0(n\pi + z - L^0 - V)Q_n^0 y + Q_n^0(n\pi + z - L^0 - V)P_n^0 y = 0, \tag{2.6}$$

$$P_n^0(n\pi + z - L^0 - V)Q_n^0 y + P_n^0(n\pi + z - L^0 - V)P_n^0 y = 0. \tag{2.7}$$

Taking into account that $P_n^0 Q_n^0 = Q_n^0 P_n^0 = 0$ and $P_n^0 L^0 Q_n^0 = Q_n^0 L^0 P_n^0 = 0$ we obtain that (2.6) and (2.7) can be written as

$$Q_n^0(n\pi + z - L^0 - V)Q_n^0 y - Q_n^0 V P_n^0 y = 0 \tag{2.8}$$

$$-P_n^0 V Q_n^0 y - P_n^0 V P_n^0 y + z P_n^0 y = 0 \tag{2.9}$$

The operator

$$A = A(n,z) := Q_n^0(n\pi + z - L^0 - V)Q_n^0 \ : \ \mathbb{H}(n) \to \mathbb{H}(n) \tag{2.10}$$

is invertible for large $|n|$ (see below (2.18), (2.26) and (2.27)). So, solving (2.8) for $Q_n^0 y$, we obtain $Q_n^0 y = A^{-1} Q_n^0 V P_n^0 y$, where $P_n^0 y \neq 0$ (otherwise $Q_n^0 y = 0$ which implies $y = P_n^0 y + Q_n^0 y = 0$). Now (2.9) implies (after plugging the above expression for $Q_n^0 y$ in it) that $(S - z)P_n^0 y = 0$, where the operator S is given by

$$S := P_n^0 V A^{-1} Q_n^0 V P_n^0 + P_n^0 V P_n^0 \ : \ E_n^0 \to E_n^0. \tag{2.11}$$

Let $\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}$ be the matrix representation of the two-dimensional operator S with respect to the basis e_n^1, e_n^2; then

$$S^{11} = \langle e_n^1, S e_n^1 \rangle, \quad S^{22} = \langle e_n^2, S e_n^2 \rangle, \quad S^{12} = \langle e_n^1, S e_n^2 \rangle, \quad S^{21} = \langle e_n^2, S e_n^1 \rangle. \tag{2.12}$$

Hence we obtain (since $P_n^0 y \neq 0$)

$$\det \begin{vmatrix} S^{11} - z & S^{12} \\ S^{21} & S^{22} - z \end{vmatrix} = 0. \tag{2.13}$$

In the selfadjoint case ($Q(x) = \overline{P(x)}$), if λ is a double eigenvalue, then there exists another eigenvector $\tilde{y}$ (corresponding to λ), such that y and $\tilde{y}$

are linearly independent. Then $P_n^0 y$ and $P_n^0 \tilde{y}$ are linearly independent also. Indeed, if $P_n^0 y = cP_n^0 \tilde{y}$ then

$$Q_n^0 y = A^{-1} Q_n^0 V P_n^0 y = cA^{-1} Q_n^0 V P_n^0 \tilde{y} = cQ_n^0 \tilde{y},$$

which leads to a contradiction:

$$y = P_n^0 y + Q_n^0 y = c\left(P_n^0 \tilde{y} + Q_n^0 \tilde{y}\right) = c\tilde{y}.$$

Thus $S \equiv 0$, i.e., if $\lambda = \pi n + z$ is a double eigenvalue of a self-adjoint Dirac operator L, then (for large enough n)

$$S^{11} - z = 0, \quad S^{12} = 0, \quad S^{21} = 0, \quad S^{22} - z = 0. \tag{2.14}$$

3. Let $\mathbb{H}^1$ and $\mathbb{H}^2$ be the subspaces of $\mathbb{H}$ generated, respectively, by $\{e^1_m, m \in \mathbb{Z}\}$ and $\{e^2_m, m \in \mathbb{Z}\}$, and let $\mathbb{H}^1(n)$ and $\mathbb{H}^2(n)$ be, respectively, the intersections of these spaces with $\mathbb{H}(n)$. Then $\mathbb{H} = \mathbb{H}^1 \oplus \mathbb{H}^2$, so each operator $B : \mathbb{H} \to \mathbb{H}$ may be identified with a 2×2 operator matrix (B^{ij}), where $B^{ij} : \mathbb{H}^j \to \mathbb{H}^i, \quad i, j = 1, 2$. If we consider the matrix representation of B in the basis $\{e^1_{2k}, e^2_{2k}, \ k \in \mathbb{Z}\}$ (or $\{e^1_{2k+1}, e^2_{2k+1}, \ k \in \mathbb{Z}\}$) then this matrix itself combines the matrix representations of B^{ij}. Of course, similar remark holds for operators acting in $\mathbb{H}(n)$.

Further we always work with one of the bases (2.2) (respectively, using the first basis in the case of periodic boundary conditions, and the second one in the case of antiperiodic boundary conditions). However, we don't specify below which basis is used because the formulas for the matrix representations in these bases are formally the same (with running indices being even in the first case, and odd in the second case).

Let

$$P(x) = \sum_{m \in \mathbb{Z}} p(m) e^{im\pi x} \quad \text{and} \quad Q(x) = \sum_{m \in \mathbb{Z}} q(m) e^{im\pi x}, \tag{2.15}$$

where $p(m) = q(m) = 0$ for odd m, be the Fourier expansions of the functions P and Q. It is easy to see that the operator V has the following matrix representation

$$V = \begin{pmatrix} 0 & V^{12} \\ V^{21} & 0 \end{pmatrix}, \qquad V^{12}_{km} = p(-k-m), \quad V^{21}_{km} = q(k+m). \tag{2.16}$$

The diagonal operator $Q_n^0 (n\pi + z - L^0) Q_n^0 : \mathbb{H}(n) \to \mathbb{H}(n)$ is invertible in $\mathbb{H}(n)$ for any z with $|z| \leq \pi/2$. Let D_n denote its inverse operator; then the matrix representation of D_n is

$$D_n = \begin{pmatrix} D_n^{11} & 0 \\ 0 & D_n^{22} \end{pmatrix}, \qquad \left(D_n^{11}\right)_{km} = \left(D_n^{22}\right)_{km} = \frac{\delta_{km}}{\pi(n-k)+z}. \tag{2.17}$$

The operator A defined in (2.10) can be written as

$$A = Q_n^0 (n\pi + z - L^0) Q_n^0 (1 - T_n Q_n^0), \tag{2.18}$$

where

$$T_n = D_n Q_n^0 V \ : \ \mathbb{H} \to \mathbb{H}(n). \tag{2.19}$$

Thus $A = A(n,z)$ is invertible if and only if $1 - T_n Q_n^0$ is invertible in $\mathbb{H}(n)$. By (2.16) and (2.17) one can easily see that the operator (2.19) has a matrix representation

$$T_n = \begin{pmatrix} 0 & T_n^{12} \\ T_n^{21} & 0 \end{pmatrix}, \tag{2.20}$$

where

(2.21)

$$(T_n^{12})_{km} = \frac{p(-k-m)}{\pi(n-k)+z}, \quad (T_n^{21})_{km} = \frac{q(k+m)}{\pi(n-k)+z}, \quad k,m \in \mathbb{Z}, \ k \neq n.$$

We need also the matrix representation of its square T_n^2. From (2.20) and (2.21) it follows that

$$T_n^2 = \begin{pmatrix} T_n^{12} T_n^{21} & 0 \\ 0 & T_n^{21} T_n^{12} \end{pmatrix}, \tag{2.22}$$

where

$$(T_n^{12} T_n^{21})_{km} = \sum_{j \neq n} \frac{p(-k-j) q(j+m)}{[\pi(n-k)+z][\pi(n-j)+z]},$$

(2.23) $$(k,m \in \mathbb{Z}, \quad k \neq n)$$

$$(T_n^{21} T_n^{12})_{km} = \sum_{j \neq n} \frac{q(k+j) p(-j-m)}{[\pi(n-k)+z][\pi(n-j)+z]}.$$

Lemma 2. *The norm of the operator $T_n^2 : \mathbb{H} \to \mathbb{H}(n)$ tends to 0 as $|n| \to \infty$. More precisely, if $|z| < \pi/2$, then*

(2.24)

$$\|T_n^2\| \leq C \frac{\|P\| \|Q\|}{\sqrt{|n|}} + C\|P\| \left(\sum_{|k| \geq |n|} |q(k)|^2 \right)^{1/2} + C\|Q\| \left(\sum_{|k| \geq |n|} |p(k)|^2 \right)^{1/2},$$

where C is an absolute constant.

Proof. The norm of T_n^2 does not exceed its Hilbert-Schmidt norm, so, by (2.22), it is less than the sum of the Hilbert-Schmidt norms of the operators $T_n^{12} T_n^{21}$ and $T_n^{21} T_n^{12}$.

We estimate in detail only the Hilbert-Schmidt norm $\|T_n^{12} T_n^{21}\|_{HS}$ because $\|T_n^{21} T_n^{12}\|_{HS}$ could be estimated in the same way. One can easily see that

$$\frac{1}{|\pi(n-k)+z|} \leq \frac{1}{|n-k|} \qquad \text{for} \quad |z| < \pi/2, \ k \neq n, \tag{2.25}$$

so by (2.23) we have

$$\|T_n^{12}T_n^{21}\|_{HS}^2 \le \sum_{k\neq n}\sum_m \left(\sum_{j\neq n} \frac{|p(-k-j)||q(j+m)|}{|n-k||n-j|}\right)^2 \le \Sigma_1+\Sigma_2+\Sigma_3,$$

where

$$\Sigma_1 = \sum_{|k-n|\ge \frac{|n|}{2}} \sum_m \left(\sum_{j\neq n}\cdots\right)^2, \qquad \Sigma_2 = \sum_{k\neq n}\sum_m \left(\sum_{|j-n|\ge\frac{|n|}{2}} \cdots\right)^2$$

$$\Sigma_3 = \sum_{|k-n|<\frac{|n|}{2}} \sum_m \left(\sum_{|j-n|<\frac{|n|}{2}} \cdots\right)^2.$$

Now we estimate each of these sums separately. By Cauchy inequality we obtain

$$\Sigma_1 \le \sum_{|k-n|\ge\frac{|n|}{2}} \sum_m \left(\sum_{j\neq n} \frac{1}{(n-k)^2(n-j)^2}\right)\left(\sum_{j\neq n} |p(-k-j)|^2|q(j+m)|^2\right)$$

$$\le \frac{\pi^2}{3} \sum_{|k-n|\ge|n|/2} \frac{1}{(n-k)^2} \sum_{j\neq m}\sum_m |p(-k-j)|^2|q(j+m)|^2 \le \frac{C_1}{n}\|P\|^2\|Q\|^2.$$

The sum Σ_2 can be estimated in an analogous way, so

$$\Sigma_2 \le \frac{C_1}{n}\|P\|^2\|Q\|^2.$$

Finally, we obtain that Σ_3 does not exceed

$$\sum_{|k-n|<\frac{|n|}{2}} \sum_m \left(\sum_{|j-n|<\frac{|n|}{2}} \frac{|p(-k-j)|^2|q(j+m)|^2}{(n-k)^2}\right)\left(\sum_{j\neq n}\frac{1}{(n-j)^2}\right)$$

$$\le \frac{\pi^2}{3} \sum_{|k-n|<|n|/2} \frac{1}{(n-k)^2} \sum_{|j-n|<|n|/2} |p(-k-j)|^2 \sum_m |q(j+m)|^2$$

$$\le C_2\|Q\|^2 \sum_{|\nu|\ge|n|} |p(\nu)|^2,$$

which completes the proof. □

By Lemma 2, for each potential matrix V there exists $N_1 = N_1(V)$ such that

$$\|T_n^2\| \le 1/2 \quad \text{for} \quad n \ge N_1. \tag{2.26}$$

Since

$$\|T_n^{2k}\| \leq \|T_n^2\|^k \quad \text{and} \quad \|T_n^{2k+1}\| \leq \|T_n\|\|T_n^2\|^k,$$

the series

$$(1 - T_n Q_n^0)^{-1} = \sum_{\ell=0}^{\infty} T_n^\ell Q_n^0$$

converges. Thus, in view of (2.18), A^{-1} exists and

$$A^{-1} = \sum_{\ell=0}^{\infty} T_n^\ell D_n, \quad n \geq N_1. \tag{2.27}$$

Now, from (2.11) and (2.19) it follows that

$$S = P_n^0 V P_n^0 + \sum_{\ell=0}^{\infty} P_n^0 V T_n^\ell D_n Q_n^0 V P_n^0 = \sum_{k=0}^{\infty} P_n^0 V T_n^k P_n^0, \tag{2.28}$$

so, in view of (2.12),

$$S^{ij} = \left\langle e_n^i, S e_n^j \right\rangle = \sum_{k=0}^{\infty} S_k^{ij}, \tag{2.29}$$

where

$$S_\nu^{ij} = \left\langle e_n^i, V T_n^k e_n^j \right\rangle, \quad k = 0, 1, 2, \ldots. \tag{2.30}$$

From (2.16) and (2.21 - 2.23) it follows that

$$VT_n^{2\nu} = \begin{pmatrix} 0 & V^{12}(T_n^{21}T_n^{12})^\nu \\ V^{21}(T_n^{12}T_n^{21})^\nu & 0 \end{pmatrix}, \tag{2.31}$$

$$VT_n^{2\nu+1} = \begin{pmatrix} V^{12}T_n^{21}(T_n^{12}T_n^{21})^\nu & 0 \\ 0 & V^{21}T_n^{12}(T_n^{21}T_n^{21})^\nu \end{pmatrix}. \tag{2.32}$$

It is easy to see that

$$\left\langle e_n^i, VT_n^{2\nu} e_n^i \right\rangle = 0, \qquad i = 1, 2; \quad \nu = 0, 1, 2, \ldots,$$

therefore by (2.12), (2.28), (2.29) and (2.32) we obtain

$$S^{11} = \sum_{\nu=0}^{\infty} S_{2\nu+1}^{11}, \qquad S^{22} = \sum_{\nu=0}^{\infty} S_{2\nu+1}^{22}, \tag{2.33}$$

where

$$S_{2\nu+1}^{11} = \left\langle e_n^1, VT_n^{2\nu+1} e_n^1 \right\rangle = \left\langle e_n^1, V^{12}T_n^{21}(T_n^{12}T_n^{21})^\nu e_n^1 \right\rangle = \tag{2.34}$$

$$\sum_{j_0, j_1, \ldots, j_{2\nu} \neq n} \frac{p(-n - j_0)q(j_0 + j_1)p(-j_1 - j_2)q(j_2 + j_3) \ldots p(-j_{2\nu-1} - j_{2\nu})q(j_{2\nu} + n)}{[\pi(n - j_0) + z][\pi(n - j_1) + z] \ldots [\pi(n - j_{2\nu-1}) + z][\pi(n - j_{2\nu}) + z]},$$

$$S_{2\nu+1}^{22} = \left\langle e_n^2, VT_n^{2\nu+1} e_n^2 \right\rangle = \left\langle e_n^2, V^{21}T_n^{12}(T_n^{21}T_n^{12})^\nu e_n^2 \right\rangle = \tag{2.35}$$

$$\sum_{i_0,i_1,\dots,i_{2\nu}\neq n} \frac{q(n+i_0)p(-i_0-i_1)q(i_1+i_2)p(-i_2-i_3)\dots q(j_{2\nu-1}+j_{2\nu})p(-j_{2\nu}-n)}{[\pi(n-i_0)+z][\pi(n-i_1)+z]\dots[\pi(n-i_{2\nu-1})+z][\pi(n-i_{2\nu})+z]}.$$

In an analogous way we obtain formulas for S^{12} and S^{21}. Indeed,

$$\langle e_n^1, VT_n^{2\nu+1}e_n^2\rangle = 0, \quad \langle e_n^2, VT_n^{2\nu+1}e_n^1\rangle = 0, \quad \nu = 0,1,2,\dots,$$

and therefore, from (2.12), (2.28), (2.29) and (2.31) it follows

$$S^{12} = \sum_{\nu=0}^{\infty} S^{12}_{2\nu}, \qquad S^{21} = \sum_{\nu=0}^{\infty} S^{21}_{2\nu}, \tag{2.36}$$

where

$$S^{12}_0 = \langle e_n^1, Ve_n^2\rangle = p(-2n), \quad S^{21}_0 = \langle e_n^2, Ve_n^1\rangle = q(2n), \tag{2.37}$$

and for $\nu = 1,2\dots$

$$S^{12}_{2\nu} = \langle e_n^1, VT_n^{2\nu}e_n^2\rangle = \langle e_n^1, V^{12}(T_n^{21}T_n^{12})^{\nu}e_n^2\rangle = \tag{2.38}$$

$$\sum_{j_1,\dots,j_{2\nu}\neq n} \frac{p(-n-j_1)q(j_1+j_2)p(-j_2-j_3)q(j_3+j_4)\dots q(j_{2\nu-1}+j_{2\nu})p(-j_{2\nu}-n)}{[\pi(n-j_1)+z][\pi(n-j_2)+z]\dots[\pi(n-j_{2\nu-1})+z][\pi(n-j_{2\nu})+z]},$$

$$S^{21}_{2\nu} = \langle e_n^2, VT_n^{2\nu}e_n^1\rangle = \langle e_n^2, V^{12}(T_n^{21}T_n^{12})^{\nu}e_n^1\rangle = \tag{2.39}$$

$$\sum_{j_1,\dots,j_{2\nu}\neq n} \frac{q(n+j_1)p(-j_1-j_2)q(j_2+j_3)p(-j_3-j_4)\dots p(-j_{2\nu-1}-j_{2\nu})q(j_{2\nu}+n)}{[\pi(n-j_1)+z][\pi(n-j_2)+z]\dots[\pi(n-j_{2\nu-1})+z][\pi(n-j_{2\nu})+z]}.$$

Lemma 3. *(a) For any potential functions P, Q*

$$S^{11}(n,z) = S^{22}(n,z). \tag{2.40}$$

(b) If $Q(x) = \overline{P(x)}$, then

$$S^{12}(n,z) = \overline{S^{21}(n,\overline{z})}. \tag{2.41}$$

Proof. (a) Changing the summation indices in (2.35) by

$$j_s = i_{2\nu-s}, \quad s = 0,1,\dots,2\nu$$

we obtain (by (2.34) that

$$S^{22}_{2\nu+1} = S^{11}_{2\nu+1}, \quad \nu = 0,1,2,\dots,$$

and therefore, by (2.33), we have $S^{22} = S^{11}$.

(b) If $Q(x) = \overline{P(x)}$, then $q(m) = \overline{p(-m)}$ $\forall m \in \mathbb{Z}$, and therefore, (2.37), (2.38) and (2.39) yield $\overline{S^{21}_{2\nu}(n,\overline{z})} = S^{12}_{2\nu}(n,z)$ for each $\nu = 0,1,2,\dots,$ so (2.36) implies (2.41). □

4. Let us set for convenience

$$\alpha_n(z) := S^{11}(n,z) \qquad \beta_n(z) := S^{21}(n,z). \tag{2.42}$$

Lemma 4. *For each pair $P(x), Q(x)$ of potential functions there exists $N_2 > 0$ such that for $n \geq N_2$ and $|z| \leq \pi/2$, $\alpha_n(z)$ and $\beta_n(z)$ are well defined, differentiable, and*

$$\sup_{|z|\leq\pi/2} |\alpha_n'(z)| \to 0, \qquad \sup_{|z|\leq\pi/2} |\beta_n'(z)| \to 0 \quad \text{as } n \to \infty. \tag{2.43}$$

Proof. By (2.10) and (2.11)

$$\frac{d}{dz} S(n,z) = -P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0,$$

and therefore, in view of (2.12) and (2.42), we have

$$\alpha_n'(z) = -\langle P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0 e_n^1, e_n^1 \rangle, \tag{2.44}$$

$$\beta_n'(z) = -\langle P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0 e_n^1, e_n^2 \rangle. \tag{2.45}$$

By (2.27)

$$A^{-1} Q_n^0 V = \sum_{\ell=0}^{\infty} T_n^{\ell} D_n Q_n^0 V = \sum_{k=1}^{\infty} T_n^k,$$

and therefore,

$$A^{-1}\left(A^{-1} Q_n^0 V\right) = \left(\sum_{\ell=0}^{\infty} T_n^{\ell} D_n\right)\left(\sum_{k=1}^{\infty} T_n^k\right) = D_n T_n + T_n D_n T_n + R, \tag{2.46}$$

where, in view of Lemma 2,

$$\|R\| = O\left(\|T_n^2\|\right) \to 0 \quad \text{as } n \to \infty. \tag{2.47}$$

Thus, by (2.44) and (2.45), we have

$$\alpha_n'(z) = -\langle V D_n T_n e_n^1, e_n^1 \rangle - \langle V T_n D_n T_n e_n^1, e_n^1 \rangle - \langle V R e_n^1, e_n^1 \rangle, \tag{2.48}$$

$$\beta_n'(z) = -\langle V D_n T_n e_n^1, e_n^2 \rangle - \langle V T_n D_n T_n e_n^1, e_n^2 \rangle - \langle V R e_n^1, e_n^2 \rangle. \tag{2.49}$$

We are going to show that all terms on the right of the above formulae go to 0 uniformly in z, $|z| \leq \pi/2$, as $n \to \infty$.

From (2.47) it follows that

$$\langle V R e_n^1, e_n^1 \rangle \to 0, \quad \langle V R e_n^1, e_n^1 \rangle \to 0 \quad \text{as } n \to \infty. \tag{2.50}$$

By (2.16), (2.20), (2.21) and (2.25),

$$\left|\langle V D_n T_n e_n^1, e_n^1 \rangle\right| = \left| \sum_{k\neq n} \frac{p(n+k) q(-k-n)}{\pi(n-k)+z} \right| \leq \Sigma_1 + \Sigma_2, \tag{2.51}$$

where

(2.52)

$$\Sigma_1 = \sum_{|n-k|\leq|n|/2} \frac{|p(n+k)||q(-k-n)|}{|n-k|}, \qquad \Sigma_2 = \sum_{|n-k|>|n|/2} \frac{|p(n+k)||q(-k-n)|}{|n-k|}.$$

Let us change the summation index k in Σ_1 to $i = n + k$. Then, since

$$|i| = |2n - (n-k)| \geq 2|n| - |n-k| > |n|,$$

we obtain

(2.53)

$$\Sigma_1 \leq \sum_{|i|>|n|} |p(i)||q(-i)| \leq \left(\sum_{|i|>|n|} |p(i)|^2 \right)^{1/2} \left(\sum_{|i|>|n|} |q(i)|^2 \right)^{1/2} \to 0 \quad \text{as } n \to \infty.$$

The Cauchy inequality yields the following estimate:

$$\Sigma_2 \leq \|P\|\|Q\| \left(\sum_{|n-k|>|n|/2} \frac{1}{(n-k)^2} \right)^{1/2} = O(1/\sqrt{|n|}). \tag{2.54}$$

It is easy to see by (2.16), (2.20) and (2.21), that

$$\langle VT_nD_nT_ne_n^1, e_n^1 \rangle = 0, \qquad \langle VD_nT_ne_n^1, e_n^2 \rangle = 0. \tag{2.55}$$

Next we estimate $\langle VT_nD_nT_ne_n^1, e_n^2 \rangle = 0$. Set $U_n = T_nD_nT_n$; then, by (2.16) and (2.20)–(2.23) the absolute value of each term in the matrix representation of U_n does not exceed the absolute value of the corresponding term in the matrix representation of $(T_n)^2$, and therefore, by the proof of Lemma 2,

$$\|U_n\| = \|T_nD_nT_n\| \to 0 \quad \text{as } n \to \infty. \tag{2.56}$$

Of course, (2.56) implies that

$$\langle VT_nD_nT_ne_n^1, e_n^2 \rangle \to 0 \quad \text{as } n \to \infty. \tag{2.57}$$

Now, in view of (2.48) and (2.49), (2.50)–(2.57) show that (2.43) holds. □

Theorem 5. *Let L be a self-adjoint Dirac operator given by (1.1), and let (γ_n) be the sequence of its spectral gaps. Then there exist $N_2 > 0$ and a sequence of positive numbers (ε_n), $\varepsilon_n \to 0$, such that*

$$2|\beta_n(z)|(1-\varepsilon_n) \leq \gamma_n \leq 2|\beta_n(z)|(1+\varepsilon_n), \quad n \geq N_2, \tag{2.58}$$

where $z = z_n$,

$$|z_n| \leq \pi/2. \tag{2.59}$$

Proof. By Lemma 1, if $n \geq N_0$, then there are exactly two eigenvalues $\lambda_n^\pm = n^2 + z_n^\pm$ of L (periodic for even n and antiperiodic for odd n) such that $|z_n^\pm| < \pi/2$. Moreover, we know (see (2.26) and (2.27) that there exists $N_1 > N_0$ such that for $n \geq N_1$ z_n^- and z_n^+ are roots of the quasi-quadratic equation (2.13). Since the operator L is self-adjoint, z_n^- and z_n^+ are real numbers, $z_n^- \leq z_n^+$, and

$$\gamma_n = z_n^+ - z_n^-. \tag{2.60}$$

By (2.40) and (2.41) in Lemma 3, the quasi-quadratic equation (2.13) becomes

$$(z - \alpha_n(z))^2 - |\beta_n(z)|^2 = 0, \tag{2.61}$$

which splits into two equations

$$z - \alpha_n(z) - |\beta_n(z)| = 0, \tag{2.62}$$

$$z - \alpha_n(z) + |\beta_n(z)| = 0. \tag{2.63}$$

Set

$$\delta_n = \sup_{|z| \leq \pi/2} |\alpha'(n,z)| + \sup_{|z| \leq \pi/2} |\beta'(n,z)|. \tag{2.64}$$

By Lemma 4, $\delta_n \to 0$. Choose $N_2 > N_1$ so that

$$\delta_n < 1/8 \qquad \text{for } \ |n| \geq N_2. \tag{2.65}$$

Fix an $n \geq N_2$. If $\gamma_n = 0$, then $\lambda_n^- = \lambda_n^+$ is a double eigenvalue of L, so (2.14) and (2.42) yield (2.58).

If $z_n^- < z_n^+$, set

$$\zeta_n^+ = z_n^+ - \alpha_n(z_n^+), \qquad \zeta_n^- = z_n^- - \alpha_n(z_n^-). \tag{2.66}$$

Then, by (2.62) and (2.63),

$$|\zeta_n^+| = |\beta_n(z_n^+)|, \qquad |\zeta_n^-| = |\beta_n(z_n^-)|. \tag{2.67}$$

By (2.66),

$$\zeta_n^+ - \zeta_n^+ = \int_{z_n^-}^{z_n^+} (1 - \alpha_n'(z))\, dz.$$

Thus, in view of Lemma 4,

$$(z_n^+ - z_n^-)(1 - \delta_n) \leq |\zeta_n^+ - \zeta_n^-| \leq (z_n^+ - z_n^-)(1 + \delta_n), \tag{2.68}$$

which yields (since $\delta_n < 1/8$)

$$|\zeta_n^+ - \zeta_n^-|\,(1 - \delta_n) \leq z_n^+ - z_n^- \leq |\zeta_n^+ - \zeta_n^-|\,(1 + 2\delta_n) \leq 2|\zeta_n^+ - \zeta_n^-|. \tag{2.69}$$

Since z_n^+ and z_n^- are roots of (2.61), each of these numbers is a root of either (2.62), or (2.63). There are two cases: (i) z_n^+ and z_n^- are roots of different equations; (ii) z_n^+ and z_n^- are roots of one and the same equation.

In Case (i) we have, by (2.62), (2.63) and (2.67), that

$$|\zeta_n^+ - \zeta_n^-| = |\beta_n(z_n^+)| + |\beta_n(z_n^-)| = |\zeta_n^+| + |\zeta_n^-|. \tag{2.70}$$

On the other hand, since $\beta_n(z_n^+) - \beta_n(z_n^-) = \int_{z_n^-}^{z_n^+} \beta_n'(t)dt$, (2.64) and (2.66) imply that

$$|\beta_n(z_n^+) - \beta_n(z_n^-)| \leq (z_n^+ - z_n^-)\delta_n \leq |\zeta_n^+ - \zeta_n^-| \cdot 2\delta_n \tag{2.71}$$

Thus (2.67) and (2.70) yield

$$\big||\zeta_n^+| - |\zeta_n^-|\big| = \big||\beta_n(z_n^+)| - |\beta_n(z_n^-)|\big| \leq \big(|\zeta_n^+| + |\zeta_n^-|\big) \cdot 2\delta_n.$$

so, since $2|\zeta_n^+| = (|\zeta_n^+| + |\zeta_n^-|) + (|\zeta_n^+| - |\zeta_n^-|)$,

$$\big(|\zeta_n^+| + |\zeta_n^-|\big)(1 - 2\delta_n) \leq 2|\zeta_n^+| \leq \big(|\zeta_n^+| + |\zeta_n^-|\big)(1 + 2\delta_n),$$

and therefore, since $\delta_n < 1/8$,

$$2|\zeta_n^+|(1 - 2\delta_n) \leq |\zeta_n^+| + |\zeta_n^-| \leq 2|\zeta_n^+|(1 + 4\delta_n) \tag{2.72}$$

Finally, using again that $\delta_n < 1/8$, we obtain by (2.69), (2.70) and (2.72) that (2.58) holds with $z = z_n^+$ and $\varepsilon_n = 8\delta_n$.

Case (ii), where z_n^+ and z_n^- are simultaneously roots of one of the equations (2.62) and (2.63), is impossible. Indeed, by (2.71) we would have since $\delta_n < 1/8$,

$$|\zeta_n^+ - \zeta_n^-| = \big||\beta_n(z_n^+)| - |\beta_n(z_n^-)|\big| \leq |\zeta_n^+ - \zeta_n^-| \cdot 2\delta_n \leq \frac{1}{4}|\zeta_n^+ - \zeta_n^-|,$$

which implies $\zeta_n^+ = \zeta_n^-$. But then (2.69) yield $z_n^+ = z_n^-$, which is a contradiction to our assumption that $z_n^+ \neq z_n^-$. □

3. Weights and Carlemann sequences

1. A sequence of positive numbers $\Omega(n), n \in \mathbb{Z}$ is called *weight*, or *weight sequence*, if

$$\Omega(-n) = \Omega(n), \quad \Omega(n) \nearrow \infty \quad \text{as} \quad n \nearrow \infty, \ n \geq n_0 > 0. \tag{3.1}$$

Each weight Ω generates a corresponding weighted ℓ^2-space

$$\ell^2(\Omega, \mathbb{Z}) = \{x = (x_n)_{n \in \mathbb{Z}} : \ \|x\|^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 (\Omega(n))^2 < \infty\}.$$

We say that two weights Ω_1 and Ω_2 are *equivalent* if

$$\exists\, C > 0 \ : \quad C^{-1}\Omega_1(n) \leq \Omega_2(n) \leq C\Omega_1(n), \quad n \in \mathbb{Z}. \tag{3.2}$$

Obviously equivalent weights yield equivalent norms, so they generate one and the same weighted ℓ^2-space.

A weight Ω is called *submultiplicative* if

$$\exists\, C > 0 \ : \quad \Omega(n + m) \leq C\Omega(n)\Omega(m), \quad n, m \in \mathbb{Z}. \tag{3.3}$$

Of course, if Ω_1 and Ω_2 are equivalent weights, then whenever one of them is submultiplicative, the other one is submultiplicative also. Obviously, if Ω satisfies (3.3) then $\tilde{\Omega} = C\Omega$ satisfies (3.3) with $C = 1$. Therefore, we may assume that (3.3) holds with $C = 1$ by passing to an equivalent weight. Moreover, it is easy to see that if (3.3) holds for $|n|, |m| \geq n_0$, then it holds for all $n, m \in \mathbb{Z}$, maybe with another constant.

A weight Ω is said to be *slowly increasing* if

$$\sup_n \Omega(2n)/\Omega(n) < \infty. \tag{3.4}$$

It is easy to see that (3.4) implies

$$\exists\, m > 0,\, C > 0: \quad \Omega(n) \leq C|n|^m, \quad \text{for } |n| \geq 1. \tag{3.5}$$

Indeed, if $M = \sup_{n\geq 1} \Omega(2n)/\Omega(n)$, then (3.4) implies that

$$\Omega(2^k) \leq \Omega(1)M^k = \Omega(1)(2^k)^m, \quad m = \log_2(M).$$

Now (3.5) follows (since Ω is monotone increasing for $n \geq n_0$) : if $n_0 \leq 2^k \leq n < 2^{k+1}$ then

$$\Omega(n) \leq \Omega(2^{k+1}) \leq M\Omega(2^k) \leq M\Omega(1)(2^k)^m \leq M\Omega(1)n^m.$$

Further we consider weights of the form

$$\Omega(n) = \exp(h(|n|)), \quad |n| \geq n_0 > 0, \tag{3.6}$$

or

$$\Omega(n) = \exp(\varphi(\log|n|)), \quad |n| \geq n_0 > 0, \tag{3.7}$$

and characterize some properties of Ω in terms of h and φ.

Remark. Observe that in (3.6) or (3.7) we don't care to define Ω for all n because our main object is the corresponding weighted ℓ^2-space. Therefore weights are important only "up to equivalence" and the values of $\Omega(n)$ for $|n| < n_0$ may be chosen in an arbitrary way since the corresponding ℓ^2-spaces will coincide.

Of course, with the formulae

$$\varphi(t) = h(e^t), \qquad h(n) = \varphi(\log(n)),$$

one can easily pass from representation (3.6) to (3.7), and back.

It is more convenient to give concrete weights in the form (3.6). For example,

$$\Omega_m(n) = |n|^m, \quad m > 0, \tag{3.8}$$

are known as Sobolev weights, and

$$\Omega_{a,b}(n) = \exp(a|n|^b), \quad a > 0, b \in (0,1), \tag{3.9}$$

are the Gevrey weights.

Lemma 6. *A weight of the form (3.6) is submultiplicative if h is an increasing concave function.*

Proof. Indeed, one can easily see that if $h : [n_0, \infty) \to \mathbb{R}$ is an increasing concave function, then there exists an increasing concave function $h_1 : [0, \infty \to [0, \infty)$ such that $h_1(n) = h(n) + C$ for $n \geq n_0,\ n \in \mathbb{N}$.

Then the weight Ω is equivalent to the weight $\Omega_1(\cdot) = \exp(h_1(\cdot))$, so it is enough to show that Ω_1 is submultiplicative. On the other hand, since h_1 is concave we have for $m, n > 0$

$$h_1(0) + h_1(m+n) \leq h_1(m) + h_1(n),$$

which implies (in view of (3.3) and (3.6)) that the weight Ω_1 is submultiplicative. □

2. The next lemma characterize a class of rapidly increasing submultiplicative weights of the form (3.7). In particular, this class contains Gevrey weights (3.9).

Lemma 7. *Suppose* $\varphi : [0, \infty) \to [0, \infty),\ \varphi(0) = 0$, *is a twice differentiable function such that the following conditions hold:*

$$\varphi'(t) \nearrow \infty \quad \text{as} \quad t \nearrow \infty; \tag{3.10}$$

$$e^t/\varphi'(t) \nearrow \infty \quad \text{as} \quad t \nearrow \infty; \tag{3.11}$$

(a) Let $\psi(s)$ *be the Young dual function of* φ, *i.e.*

$$\psi(s) = \sup_{t \geq 0} [st - \varphi(t)], \quad s \geq 0. \tag{3.12}$$

Then

$$e_k := \frac{1}{k} \exp(\psi'(k)) \nearrow \infty \quad \text{as} \ k \nearrow \infty. \tag{3.13}$$

(b) In addition, if

$$\liminf_{t \to \infty} \frac{\varphi'(t) - \varphi''(t)}{\log \varphi'(t)} > 1, \tag{3.14}$$

then

$$\exists p \in \mathbb{N},\, \tau > 1 \ : \quad k^\tau \left(\frac{e_k}{e_{pk}} \right)^k \leq 1 \quad \text{for} \quad k \geq k_0. \tag{3.15}$$

Proof. (a) Since $(st - \varphi(t))'_t = s - \varphi'(t)$ one can easily see, by (3.10), that the expression $st - \varphi(t)$ achives its maximum at the point

$$t(s) = (\varphi')^{-1}(s), \tag{3.16}$$

thus

$$\psi(s) = st(s) - \varphi(t(s)). \tag{3.17}$$

The function $s \to t(s)$ is increasing because φ' is increasing. From $\varphi'(t(s)) = s$ and (3.17) it follows that

$$\psi'(s) = t(s) + st'(s) - \varphi'(t(s))t'(s) = t(s), \tag{3.18}$$

$$\psi''(s) = t'(s) = 1/\varphi''(t(s)). \tag{3.19}$$

Therefore, (3.11) implies that

$$e_k = e^{\psi'(k)}/k = e^{t(k)}/\varphi'(t(k)) \nearrow \infty.$$

(b) One can easily see that (3.15) is equivalent to

$$\exists p \in \mathbb{N} \ : \quad \liminf_k \frac{k}{\log k} \log\left(\frac{e_{pk}}{e_k}\right) > 1. \tag{3.20}$$

By (3.13) we have $\log e_k = \psi'(k) - \log k$, and therefore, (3.16) and (3.19) imply that

$$\log\left(\frac{e_{pk}}{e_k}\right) = [\psi'(pk) - \log(pk)] - [\psi'(k) - \log k]$$

$$= k\int_1^p \left(\psi''(uk) - \frac{1}{uk}\right) du = k\int_1^p \left(\frac{1}{\varphi''[t(uk)]} - \frac{1}{\varphi'[t(uk)]}\right) du.$$

For large enough k it follows from (3.14) and (3.16) that $uk = \varphi'[t(uk)] > \varphi''[t(uk)]$, so

$$\frac{1}{\varphi''[t(uk)]} - \frac{1}{\varphi'[t(uk)]} > \frac{\varphi'[t(uk)] - \varphi''[t(uk)]}{u^2k^2}$$

Thus (again by (3.14)) we obtain that

$$\frac{k}{\log k}\log\left(\frac{e_{pk}}{e_k}\right) > \int_1^p \frac{\varphi'[t(uk)] - \varphi''[t(uk)]}{\log \varphi'[t(uk)]} \cdot \frac{1}{u^2}du \tag{3.21}$$

for large enough k. Let $\ell > 1$ be the *liminf* in (3.14). Choose $p \in \mathbb{N}$ so that $\frac{\ell+1}{2}(1 - 1/p) > 1$. Since $(\ell + 1)/2 < \ell$ there exists k_0 such that for $k \geq k_0$ (3.21) holds and the integral there is greater than

$$\int_1^p \frac{\ell+1}{2} \cdot \frac{1}{u^2}du = \frac{\ell+1}{2}(1 - 1/p) > 1.$$

This completes the proof of the lemma.

□

Remark. Obviously, if φ satisfies

$$\lim_{t\to\infty} \frac{\varphi'(t) - \varphi''(t)}{\log \varphi'(t)} = \infty \tag{3.22}$$

then (3.14) holds. One can easily see that Gevrey weights (3.9) satisfy (3.22). Now we present a family of weights that satisfy (3.14) but don't satisfy (3.22).

Consider weights (3.7) generated by

$$\varphi(t) = \int_0^t e^{\omega(u)} du,$$

where

$$\omega(u) = \beta u - (1-\beta)\cos u + \alpha u e^{-\beta u}, \quad \alpha > 1, \ \beta \in (0,1).$$

Then

$$\varphi'(t) = e^{\omega(t)}, \quad \varphi''(t) = e^{\omega(t)} \left[\beta + (1-\beta)\sin t + \alpha(1-\beta t)e^{-\beta t}\right],$$

so

$$\frac{\varphi'(t) - \varphi''(t)}{\log \varphi'(t)} = \frac{e^{\omega(t)}}{\omega(t)} \left[(1-\beta)(1-\sin t) + \alpha(\beta t - 1)e^{-\beta t}\right] \tag{3.23}$$

which is greater than

$$\alpha \frac{e^{\omega(t)}}{\omega(t)} \left[(\beta t - 1)e^{-\beta t}\right] = \alpha \frac{\beta t - 1}{\omega(t)} \exp[(\beta - 1)\cos t + \alpha t e^{-\beta t}]. \tag{3.24}$$

Let (t_k) be a sequence of positive numbers such that $t_k \to \infty$. Observe that if $\liminf_k (1 - \sin(t_k)) > 0$ then the expression (3.23) with $t = t_k$ goes to ∞ as $k \to \infty$, while whenever $\lim_k (1 - \sin(t_k)) = 0$ the expression (3.24) with $t = t_k$ tends to α. On the other hand for $t = t_k = (4k+1)\pi/2, \ k = 1, 2, \ldots$ the expressions (3.23) and (3.24) coincide. By these observations it is easy to see that

$$\liminf_{t \to \infty} [\varphi'(t) - \varphi''(t)] / \log \varphi'(t) = \alpha.$$

Thus (3.14) holds, since $\alpha > 1$, while (3.22) fails.

3. We say that a sequence of positive numbers $(M_k)_{k=0}^{\infty}$ is a Carlemann sequence if

$$M_0 = 1, \quad M_k / (k M_{k-1}) \nearrow \infty. \tag{3.25}$$

We attach to any Carlemann sequence (M_k) the following sequences:

$$m_0 = 1, \quad m_k = M_k / M_{k-1}, \qquad e_0 = 1, \quad e_k = m_k / k, \ k \geq 1. \tag{3.26}$$

We set also

$$E_0 := e_0, \quad E_k = e_1 \ldots e_k = M_k / k!, \quad k \geq 1. \tag{3.27}$$

Observe that if a sequence $(e_k)_{k=0}^{\infty}$ satisfies the condition $e_k \nearrow \infty$, then it generates a corresponding Carlemann sequence $M_k = k! E_k$ with E_k defined by (3.27).

Suppose $\Omega_\varphi \in (3.7)$ is a weight that grows faster then any power of n. For a technical reason we need to characterize the relation $x = (x_n) \in \ell^2(\Omega)$ by the sequence of ℓ^1 norms

$$\|x\|_k = \|x_0\| + \sum |x_n||n|^k, \qquad k = 1, 2, \ldots. \tag{3.28}$$

It turns out that this can be done in terms of an appropriate Carlemann sequence generated by the function φ.

For every function φ such that (3.10), (3.11) and (3.14) hold we denote by $(M_k(\varphi))$ the Carlemann sequence generated by

$$m_k(\varphi) = \exp(\psi'(k)), \quad k = 1, 2, \ldots,$$

that is

$$M_k(\varphi) = \exp(\psi'(1) + \cdots + \psi'(k)), \quad k = 1, 2, \ldots,$$

where ψ is the Young dual function of φ.

We may assume without loss of generality that the function φ is defined on $[0, \infty)$, and moreover, that the condition $\varphi'(t) - \varphi''(t) > 0$ holds for $t \geq 0$ (since otherwise one can consider an equivalent weight generated by a suitable function $\tilde{\varphi}$). Moreover, the condition (3.11) implies that the weight Ω_φ is submultiplicative. Indeed, since

$$\Omega_\varphi(n) = \exp[\varphi(\log|n|)] = e^{\psi(|n|)} \quad \text{with} \quad \psi(s) = \varphi(\log s),$$

we obtain, in view of (3.11), that the derivative

$$\psi'(s) = \frac{\varphi(\log s)}{s} = \frac{\varphi(\log s)}{e^{\log s}}$$

is decreasing, so $\psi(s)$ is a concave function. Thus, by Lemma 6, the weight Ω_φ is submultiplicative.

Lemma 8. *If φ satisfies (3.10), (3.11), (3.14) and $\Omega_\varphi(|n|) = \exp(\varphi(\log|n|))$ then*

$$(a) \qquad x = (x_n) \in \ell^2(\Omega_\varphi) \Rightarrow \|x\|_k \leq C M_k(\varphi_1), \quad \varphi_1(t) = \varphi(t) - t; \tag{3.29}$$

$$(b) \qquad \|x\|_k \leq C M_k(\varphi) \Rightarrow x = (x_n) \in \ell^2(\Omega_{\varphi_2}), \quad \varphi_2(t) = \varphi(t) - 4t. \tag{3.30}$$

Proof. (a) Observe that we have (with $\psi(0) = 0$)

$$\psi(k) \leq \psi'(1) + \cdots + \psi'(k) \leq \psi(k+1).$$

Therefore

$$\sup_n \frac{|n|^k}{\Omega_\varphi(n)} = \exp\left(\sup_n [k \log|n| - \varphi(\log|n|)]\right)$$
$$\leq \exp(\psi(k)) \leq M_k(\varphi) = \exp(\psi'(1) + \cdots + \psi'(k)).$$

If $x = (x_n) \in \ell^2(\Omega_\varphi)$ then we obtain with $\Omega = \Omega_\varphi$ by Cauchy inequality

$$\|x\|_k = \sum |x_n||n|^k = \sum (|x_n|\Omega_\varphi(n)) \left(|n|^k/\Omega_\varphi(n)\right)$$

$$\leq \|x\|_{\ell^2(\Omega_\varphi)} \left(\sum \frac{1}{n^2} \cdot \left(\frac{|n|^{k+1}}{\Omega_\varphi(n)} \right)^2 \right)^{1/2} \leq C \sup_n \left(\frac{|n|^k}{\Omega_\varphi(n)/n} \right) \leq C M_k(\varphi_1),$$

where $\varphi_1(t) = \varphi(t) - t$.

(b) Suppose that $\|x\|_k \leq C M_k(\varphi)$; then for large $|n|$ we have

$$C \geq \sup_k \frac{\|x\|_k}{M_k} \geq |x_n| \sup_k \frac{|n|^k}{M_k}$$

$$\geq |x_n| \exp \left(\sup_k \left[k \log |n| - (\psi'(1) + \cdots + \psi'(k)) \right] \right)$$

$$\geq |x_n| \exp \left(\sup_k \left[k \log |n| - \psi(k+1) \right] \right)$$

$$\geq |x_n||n|^{-2} \exp \left(\sup_{s>0} \left[s \log |n| - \psi(s) \right] \right) = |x_n||n|^{-2} \exp(\varphi(\log |n|)),$$

that is $|x_n||n|^{-2}\Omega_\varphi(n) \leq C$. Therefore

$$\sum_n |x_n| \frac{\Omega_\varphi(n)}{n^4} \leq C \sum_n \frac{1}{n^2} < \infty$$

which implies that

$$x = (x_n) \in \ell^1(\Omega_{\varphi_2}) \subset \ell^2(\Omega_{\varphi_2})$$

with $\varphi_2(t) = \varphi(t) - 4t$.

□

Lemma 9. *Suppose* $(e_k)_{k=1}^\infty$ *is a sequence of positive numbers such that* $e_k \uparrow \infty$ *and let*

$$E_0 = 1, \quad E_k = \prod_1^k e_j, \quad k \geq 1.$$

Then the following implications hold:

(3.31)
$$\exists\, p \in \mathbb{N}, \tau > 0 \ : \quad \sup_k k^\tau (e_k/e_{pk})^k < \infty \quad \Rightarrow \quad \sup_k k^\tau (E_k)^2/E_{2k} < \infty;$$

$$\sum_{k=1}^\infty (e_k/e_{pk})^k < \infty \quad \Rightarrow \quad \sum_{k=1}^\infty (E_k)^2/E_{2k} < \infty; \tag{3.32}$$

$$\sum_{k=1}^{\infty}(E_k)^2/E_{2k} < \infty \quad \Rightarrow \quad Q := \sup_m \sum_{j=0}^{m} \frac{E_j E_{m-j}}{E_m} < \infty, \tag{3.33}$$

and moreover,

$$\sup_m \sum_{s_0+\cdots+s_\mu=m} \frac{E_{s_0}\dots E_{s_\mu}}{E_m} < Q^\mu, \quad \mu = 1,2,\dots. \tag{3.34}$$

Remark. This lemma is a "multidimensional" version of the statements on p. 164 in [2]. It improves Lemma 5 on p. 251 in [3], where we can now omit the factor k^{p-2} in the hypothesis (5.7).

Proof. If $k = p\nu + r$ with $0 \le r < p$ then we have

$$\frac{(E_k)^2}{E_{2k}} = \frac{e_1 \dots e_\nu}{e_{k+1}\dots e_{k+\nu}} \cdot \frac{e_{\nu+1}\dots e_k}{e_{k+\nu+1}\dots e_{2k}} \le \frac{e_1\dots e_\nu}{e_{k+1}\dots e_{k+\nu}} \le \left(\frac{e_\nu}{e_{p\nu}}\right)^\nu$$

(because $e_i < e_j$ for $i < j$). Thus (3.31) and (3.32) hold.

To prove (3.33), (3.34) consider the sums

$$T_m = \sum_{j=0}^{m} \frac{E_j E_{m-j}}{E_m}.$$

Then

$$\frac{E_j E_{m+1-j}}{E_{m+1}} \le \frac{E_j E_{m-j}}{E_m}, \qquad 0 \le j \le m, \tag{3.35}$$

because (3.35) is equivalent to

$$e_{m+1-j} \le e_{m+1},$$

which holds since the sequence (e_k) is increasing.

By symmetry

$$T_m = 2\sum_{0\le j\le m/2} \frac{E_j E_{m-j}}{E_m} - \delta_m,$$

where

$$\delta_m = \begin{cases} 0 & , \quad m = 2n+1 \\ E_n^2/E_{2n} & , \quad m = 2n. \end{cases}$$

The next sum is

$$T_{m+1} = 2\sum_{0\le j\le m/2} \frac{E_j E_{m+1-j}}{E_{m+1}} + \delta_{m+1},$$

and (3.35) implies that

$$T_{m+1} \le T_m + \delta_m + \delta_{m+1}.$$

Therefore we obtain

$$T_m \leq 2 + \sum_{k=1}^{\infty} (\delta_k + \delta_{k+1}) = 2 + 2 \sum_{n=1}^{\infty} E_n^2 / E_{2n} < \infty,$$

thus (3.33) holds.

Now we prove (3.34) by induction in μ. Let us denote by $S_\mu(m)$ the set of all $(\mu+1)$-tuples of integers $s = (s_0, \ldots, s_\mu)$ such that $0 \leq s_i \leq m$ and $|s| = s_0 + \cdots + s_\mu = m$, i.e.

$$S_\mu(m) = \{s = (s_0, \ldots, s_\mu) : \quad 0 \leq s_i \leq m, \ |s| = m\}. \tag{3.36}$$

By (3.33) it holds for $\mu = 1$. Assume that (3.34) holds for some $\mu \geq 1$. Then we have

$$\sum_{s \in S_{\mu+1}(m)} \frac{E_{s_0} \ldots E_{s_{\mu+1}}}{E_m} = \sum_{s_{\mu+1}=0}^{m} \left(\sum_{s \in S_\mu(m - s_{\mu+1})} \frac{E_{s_0} \ldots E_{s_\mu}}{E_{m - s_{\mu+1}}} \right) \frac{E_{s_{\mu+1}} E_{m - s_{\mu+1}}}{E_m}$$

$$\leq Q^\mu \sum_{s_{\mu+1}=0}^{m} \frac{E_{s_{\mu+1}} E_{m - s_{\mu+1}}}{E_m} \leq Q^{\mu+1},$$

which proves (3.34). □

4. The next statement (Lemma 10) has as its prototypes Lemma 6 in [2] and Theorem 3 in [3] (see also the proof of Prop. 4 there). But, influenced by Lemma 1.1 in [1], now we use "maxima" instead of "sums" in the formulation, which makes the lemma more convenient for applications. The proof of Lemma 10 uses the same idea that was used to prove its prototypes, but it is more simple.

Lemma 10. *Let $(f_k)_{k=1}^\infty$ be a sequence of positive numbers such that*

$$f_k \nearrow \infty, \tag{3.37}$$

and let

$$F_0 = 1, \quad F_k = \prod_{j=1}^{k} f_j, \quad k = 1, 2, \ldots. \tag{3.38}$$

If $T > 0$ and $(X_k)_{k=0}^\infty$ is a sequence of positive numbers such that $X_0 = 1$ and

$$X_k \leq \max \left(T, \sup_\mu \max_{s_i < |s| = m} \frac{F_{s_0} \ldots F_{s_\mu}}{F_k} X_{s_0} \ldots X_{s_\mu} \right), \quad k \geq 2, \tag{3.39}$$

where $s = (s_0, \ldots, s_\mu)$ and $|s| = s_0 + \cdots + s_\mu$, then the sequence (X_k) is bounded.

Proof. For convenience the proof is divided into 3 steps.

Step 1. We may assume without loss of generality that

$$X_1 \le T/F_1, \quad 1 \le T/F_2 \tag{3.40}$$

(otherwise T could be replaced by a larger constant). Let

$$k_0 = \min\{k \ge 2 : \ T > T^{k+1}/F_{k+1}\}. \tag{3.41}$$

It is easy to see by (3.37) and (3.38) that $T^k/F_k \to 0$ as $k \to \infty$, thus k_0 is well defined, and moreover, in view of (3.40) and (3.41) we have

$$T \le \frac{T^k}{F_k} \quad \text{for} \quad 2 \le k \le k_0, \quad T > \frac{T^k}{F_k} \quad \text{for} \quad k \ge k_0 + 1, \tag{3.42}$$

and

$$f_k > T \quad \text{for} \quad k > k_0. \tag{3.43}$$

Step 2. *Claim. The following inequalities hold:*

$$X_k \le T^k/F_k, \quad k = 0, 1, \dots, k_0. \tag{3.44}$$

We prove (3.44) by induction. In view of (3.38) and (3.40) our claim holds for $k = 0, 1$.

Let

$$P_s = \frac{F_{s_0} \dots F_{s_\mu}}{F_{|s|}} X_{s_0} \dots X_{s_\mu}, \quad s = (s_0, \dots, s_\mu). \tag{3.45}$$

Assume that (3.44) holds for $k = 1, \dots, m$ for some m with $1 \le m < k_0$. Then, for each μ and for each $(\mu+1)$-tuple $s = (s_0, \dots, s_\mu) \in S_\mu(m+1)$, we have by (3.44)

$$P_s \le \frac{F_{s_0} \dots F_{s_\mu}}{F_{m+1}} \cdot \frac{T^{s_0}}{F_{s_0}} \cdots \frac{T^{s_\mu}}{F_{s_\mu}} = \frac{T^{m+1}}{F_{m+1}}.$$

By (3.42) $T^{m+1}/F_{m+1} \ge T$, thus (3.39) implies that

$$X_{m+1} \le T^{m+1}/F_{m+1},$$

i.e., (3.44) holds for $k = m+1$. The claim is proven.

Step 3. Here we show that

$$X_k \le T \quad \text{for} \quad k \ge k_0 + 1. \tag{3.46}$$

For a technical reason we prove also that

$$P_s < T \quad \text{for} \quad s = (s_0, \dots, s_\mu) \quad \text{with} \quad s_j < |s| = k, \quad k \ge k_0 + 1. \tag{3.47}$$

Observe that in view of (3.39), if the inequalities (3.47) hold for some k, then (3.46) holds for the same k also.

We are proving (3.46) and (3.47) by induction for $k \geq k_0 + 1$. Let $k = k_0 + 1$. For each $(\mu+1)$-tuple $s = (s_0, \dots, s_\mu) \in S_\mu(k_0+1)$ with $s_j < k_0 + 1$ we obtain by (3.44) and (3.42) that

$$P_s \leq \frac{F_{s_0} \dots F_{s_\mu}}{F_{k+1}} \cdot \frac{T^{s_0}}{F_{s_0}} \cdots \frac{T^{s_\mu}}{F_{s_\mu}} = \frac{T^{k_0+1}}{F_{k_0+1}} < T.$$

Thus (3.47), and of course (3.46), hold for $k = k_0 + 1$.

Let $m \geq k_0 + 1$; assume that (3.46) and (3.47) hold for every $k = k_0 + 1, \dots, m$. Then, we claim that (3.46) and (3.47) hold for $k = m+1$. Indeed, fix any $(\mu+1)$-tuple $s = (s_0, \dots, s_\mu)$ with $|s| = m+1$ and $s_j < m+1$. There are several cases:

(a) If $s_j \leq k_0$ for every $j = 0, \dots, \mu$, then the numbers X_{s_j} satisfy the estimates (3.44). Thus one can easily see (as in the proof for $k = k_0 + 1$) that

$$P_s \leq T^{m+1}/F_{m+1} < T,$$

so in this case (3.47) holds.

(b) Suppose that there exists j with $s_j > k_0$, say $j = 0$. (Since a transposition of $s_0, \dots, s_\mu$ does not change P_s one may assume without loss of generality that $j = 0$.) Then we have two subcases: (b1) where $m+1-s_0 \leq k_0$, and (b2) where $m+1-s_0 > k_0$.

In the subcase (b1) we estimate P_s by using (3.46) for X_{s_0} and (3.44) for $X_{s_1}, \dots, X_{s_\mu}$. Since $T < f_k$ for $k > k_0$ by (3.43), we obtain that

$$P_s \leq \frac{F_{s_0} F_{s_1} \dots F_{s_\mu}}{F_{m+1}} \cdot T \cdot \frac{T^{s_1}}{F_{s_1}} \cdots \frac{T^{s_\mu}}{F_{s_\mu}} = T \cdot \frac{T^{m+1-s_0}}{f_{s_0+1} \dots f_{m+1}} < T,$$

thus (3.47) holds for $k = m+1$..

In the case (b2) we have

$$P_s = \frac{X_{s_0} F_{s_0} F_{m+1-s_0}}{F_{m+1}} \left(\frac{F_{s_1} \dots F_{s_\mu}}{F_{m+1-s_0}} X_{s_1} \dots X_{s_\mu} \right).$$

The expression in the brackets equals $P_{\tilde{s}}$ with $\tilde{s} = (s_1, \dots, s_\mu)$, $|\tilde{s}| = m + 1 - s_0$, so by the inductive assumption $P_{\tilde{s}} < T$. Since $X_{s_0} < T$ (by (3.46) with $k = s_0$) we have

$$P_s < \left[\frac{T F_{s_0} F_{m+1-s_0}}{F_{m+1}} \right] \cdot T,$$

so it remains to show that the expression in the square brackets does not exceed 1. By (3.42) $F_{k_0} \leq T^{k_0-1}$, and therefore,

$$F_{s_0} = F_{k_0} f_{k_0+1} \dots f_{s_0} \leq T^{k_0-1} f_{k_0+1} \dots f_{s_0}.$$

Thus

$$\frac{TF_{s_0}F_{m+1-s_0}}{F_{m+1}} \leq \frac{T^{k_0}f_{k_0+1}\cdots f_{s_0}}{f_{m+2-s_0}\cdots f_{m+1}} < 1,$$

because due to (3.37) and (3.43) each factor in the numerator of the latter fraction is strictly less than the corresponding factor in the denominator. □

4. Basic results; estimates on the smoothness of the potential in terms of the decay rate of spectral gaps

1. Our main result is the following statement.

Theorem 11. *Let L be a selfadjoint Dirac operator given by (1.1), with a potential function $P \in L^2([0,1]))$, $p(x) = \sum p(2n)e^{i2nx}$. Let Ω be a submultiplicative weight (see (3.1) and (3.3)) such that either Ω slowly increasing (i.e., (3.4) holds), or Ω is a rapidly increasing weight of the form $\Omega(n) = \exp(\varphi(\log|n|))$, where φ has the properties (3.10), (3.11) and (3.14). Then*

$$\sum_{n\in\mathbb{Z}} |\lambda_n^+ - \lambda_n^-|^2(\Omega(2n))^2 < \infty \quad \Longrightarrow \quad \sum_{n\in\mathbb{Z}} |p(2n)\Omega(2n)|^2 < \infty. \tag{4.1}$$

An implication into the opposite direction is given by Theorem 12 below; see further comments in Sect. 5.1.

Theorem 12. *Let L be a selfadjoint Dirac operator given by (1.1), with a potential function $P \in L^2([0,1]))$, $p(x) = \sum p(2n)e^{i2nx}$. If Ω is a submultiplicative weight, then*

$$\sum |p(2n)|^2(\Omega(2n))^2 < \infty \quad \Longrightarrow \quad \sum |\lambda_n^+ - \lambda_n^-|^2(\Omega(2n))^2 < \infty. \tag{4.2}$$

Proof. By Theorem 5, for large enough n,

$$\gamma_n = \lambda_n^+ - \lambda_n^- \asymp 2|\beta_n(z_n)| \quad \text{with} \quad |z_n| \leq \pi/2, \tag{4.3}$$

where, in view of (2.42) and (2.36)–(2.38),

$$\beta_n(n, z_n) = p(-2n) + \sum_{\nu=1}^{\infty} S_{2\nu}^{21}(n, z_n). \tag{4.4}$$

Therefore, by (2.25), we have

$$|\beta_n(z_n)| \leq |p(-2n)| + \sum_{\nu=1}^{\infty} \left|S_{2\nu}^{21}(n, z_n)\right| \leq |r(2n)| + \sum_{\nu=1}^{\infty} \sigma_\nu(n, r), \tag{4.5}$$

where

$$r = (r(m))_{m\in\mathbb{Z}}, \quad r(m) = \max(|p(m)|, |p(-m)|). \tag{4.6}$$

and
(4.7)
$$\sigma_\nu(n,r) = \sum_{j_1,\dots,j_{2\nu}\neq n} \frac{r(n+j_1)r(-j_1-j_2)r(j_2+j_3)\dots r(-j_{2\nu-1}-j_{2\nu})r(j_{2\nu}+n)}{|n-j_1||n-j_2|\dots|n-j_{2\nu}|}.$$

Consider the operator

$$\sigma : \quad r = (r(m)) \in \ell^2(\mathbb{Z}) \to (\sigma(n,r)) \in \ell^2(\mathbb{Z}), \tag{4.8}$$

where

$$\sigma(n,r) = \sum_{\nu=1}^{\infty} \sigma_\nu(n,r). \tag{4.9}$$

So, in view of (4.3)–(4.9), the following statement completes the proof of Theorem 12.

Proposition 13. *If Ω is a submultiplicative weight, then for each sequence of non-negative numbers,*

$$\sum |r(2n)|^2(\Omega(2n))^2 < \infty \quad \Longrightarrow \quad \sum |\sigma(n,r)|^2(\Omega(2n))^2 < \infty. \tag{4.10}$$

Proposition 13 is proven in Sect. 4.2 as a corollary of some basic properties of the operator σ. $\square$

The proof of Theorem 11 follows from the properties of the operator σ also, but it is much more complicated. Set

$$\zeta(n) = |\beta(n, z_n)|;$$

then Theorem 11 will be proven if we show that

$$\sum |\zeta(n)|^2(\Omega(2n))^2 < \infty \quad \Longrightarrow \quad \sum |p(2n)|^2(\Omega(2n))^2 < \infty. \tag{4.11}$$

Under the above notations we have, by (4.4), that

$$|p(-2n)| \leq |\beta_n(z_n)| + \sum_{\nu=1}^{\infty} \left|S^{21}_{2\nu}(n,z_n)\right| \leq |\zeta(n)| + \sum_{\nu=1}^{\infty} \sigma_\nu(n,r). \tag{4.12}$$

In the same way, changing n to $-n$ one can see that

$$|p(2n)| \leq |\beta_{-n}(z_{-n})| + \sum_{\nu=1}^{\infty} \left|S^{21}_{2\nu}(-n,z_{-n})\right| \leq |\zeta(-n)| + \sum_{\nu=1}^{\infty} \sigma_\nu(n,r), \tag{4.13}$$

Thus, by (4.12) and (4.13), we obtain, with $\xi(n) = \max(\zeta(n), \zeta(-n))$,

$$r(2n) \leq \xi(n) + \sum_{\nu=1}^{\infty} \sigma_\nu(n,r) = \xi(n) + \sigma(n,r).$$

Thus, in view of the above discussion, Theorem 11 would be proven if we prove the following statement.

Theorem 14. *Let Ω be a submultiplicative weight (see (3.1) and (3.3)) such that either Ω slowly increasing (i.e., (3.4) holds), or Ω is a rapidly increasing weight of the form $\Omega(n) = \exp(\varphi(\log n))$, where φ has the properties (3.10), (3.11) and (3.10). If $\xi = (\xi(m))_{m\in\mathbb{Z}}$ and $r = (r(m))_{m\in\mathbb{Z}}$ are two sequences of non-negative numbers such that*

$$r \in \ell^2(\mathbb{Z}), \quad r(m) = 0 \ \text{for odd}\, m, \tag{4.14}$$

$$r(2n) \leq \xi(n) + \sigma(n,r), \qquad |n| \geq n_*, \tag{4.15}$$

then

$$\sum_{n\in\mathbb{Z}} |\xi(n)\Omega(2n)|^2 < \infty \Longrightarrow \sum_{n\in\mathbb{Z}} |r(2n)\Omega(2n)|^2 < \infty. \tag{4.16}$$

The remaining part of this section is devoted to the proof of Theorem 14. Some of the steps of this proof are interesting by themselves (e.g., Lemma 15 and Proposition 16 give a proof of Proposition 13). Therefore, the claims that follow below are formulated and proven as independent statements, although they are steps in the proof of Theorem 12.

2. Throughout the paper we assume that the weights are submultiplicative. The following property of the operator $\sigma(n,r)$ reveals why this assumption is so important.

Lemma 15. *If Ω is a submultiplicative weight such that*

$$\Omega(k+m) \leq \Omega(k)\Omega(m) \quad \forall k,m \in \mathbb{Z},$$

(i.e., (3.3) holds with $C=1$) then, for each sequence of non-negative numbers $r = (r(m))_{m\in\mathbb{Z}}$,

$$\sigma(n,r)\Omega(2n) \leq \sigma(n,\tilde{r}) \quad \textit{where} \quad \tilde{r} = (r(m)\Omega(m))_{m\in\mathbb{Z}}. \tag{4.17}$$

Proof. Since the weight Ω is submultiplicative, we have, for each 2ν-tuple $(j_1,\dots,j_{2\nu})$, that

$$\Omega(2n) \leq \Omega(n+j_1)\Omega(-j_1-j_2)\Omega(j_2+j_3)\cdots\Omega(-j_{2\nu+1}-j_{2\nu})\Omega(j_{2\nu}+n),$$

and therefore,

$$r(n+j_1)r(-j_1-j_2)\cdots r_(j_{2\nu}+n)\Omega(2n) \leq \tilde{r}(n+j_1)\tilde{r}(-j_1-j_2)\cdots\tilde{r}_(j_{2\nu}+n)$$

Thus, in view of (4.7), we obtain

$$\sigma_\nu(n,r)\Omega(2n) \leq \sigma_\nu(n,\tilde{r}) \qquad \nu = 1,2,\dots,$$

so, by (4.9),

$$\sigma(n,r)\Omega(2n) = \sum_{\nu=1}^{\infty} \sigma_\nu(n,r)\Omega(2n) \leq \sum_{\nu=1}^{\infty} \sigma_\nu(n,\tilde{r}) = \sigma(n,\tilde{r}).$$

$\square$

Next we use the properties of the operator σ to prove the following crucial estimate.

Proposition 16. *Under the above notations*

$$\sum_{|n|\geq N} |\sigma(n,r)|^2 \leq \frac{2}{N} + (R(N))^2, \quad N > N^*, \tag{4.18}$$

where

$$R(N) := \sum_{|n|\geq N} |r(n)|^2. \tag{4.19}$$

Proof. By (4.7) the sequence $(\sigma(n,r))$ is the sum of the sequences $(\sigma_\nu(n,r))$, and therefore, by the triangle inequality for ℓ^2-norms, we obtain

$$\left(\sum_{|n|\geq N} |\sigma(n,r)|^2\right)^{1/2} \leq \sum_{\nu=1}^{\infty} \left(\sum_{|n|\geq N} |\sigma_\nu(n,r)|^2\right)^{1/2}. \tag{4.20}$$

To estimate $\sum_{|n|\geq N} |\sigma_\nu(n,r)|^2$, for fixed $\nu \in \mathbb{N}$, we divide the set of summation indices in (4.7)

$$J(n) = \{j = (j_1, \dots, j_{2\nu}) : \quad j_1, \dots, j_{2\nu} \neq n\}$$

into several subsets by setting

$$a = \{\alpha = (\alpha_1, \dots, \alpha_{2\nu}) : \ \alpha_s \in \{0,1\}\}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_{2\nu},$$

and

$$J^\alpha(n) = \left\{ (j_1, \dots, j_{2\nu}) \in J(n) : \quad \middle| \quad \begin{matrix} |n - j_s| \leq |n|/2 & \text{if } \alpha_s = 0 \\ |n - j_s| > |n|/2 & \text{if } \alpha_s = 1 \end{matrix} \right\}.$$

Then

$$J(n) = \bigcup_{\alpha \in a} J^\alpha(n),$$

so

$$\sum_{J(n)} \cdots = \sum_{\alpha \in a} \sum_{j \in J^\alpha(n)} \cdots,$$

and therefore, the triangle inequality implies that

$$\left(\sum_{|n|\geq N} |\sigma_\nu(n)|^2\right)^{1/2} \leq \sum_{\alpha \in a} \left(\sum_{|n|\geq N} \left| \sum_{j \in J^\alpha(n)} \cdots \right|^2\right)^{1/2}. \tag{4.21}$$

By the Cauchy inequality,

$$\sum_{|n|\geq N} \left(\sum_{j \in J^\alpha(n)} \cdots\right)^2 \leq \sum_{|n|\geq N} A_\alpha(n) B_\alpha(n), \tag{4.22}$$

where

$$A_\alpha(n) = \sum_{j\in J^\alpha(n)} \frac{1}{(n-j_1)^2\dots(n-j_{2\nu})^2} \tag{4.23}$$

$$\le \left(\sum_{|n-k|\le|n|/2} \frac{1}{(n-k)^2}\right)^{2\nu-|\alpha|} \left(\sum_{|n-k|>|n|/2} \frac{1}{(n-k)^2}\right)^{|\alpha|} \le \left(\frac{\pi^2}{3}\right)^{2\nu-|\alpha|} \left(\frac{4}{N}\right)^{|\alpha|},$$

and

$$B_\alpha(n) = \sum_{j\in J^\alpha(n)} |r(n+j_1)|^2|r(-j_1-j_2)|^2\cdots|r(j_{2\nu}+n)|^2. \tag{4.24}$$

In order to estimate $\sum_{|n|\ge N} B_\alpha(n)$ we change the indices of summation to

$$i_1 = n+j_1,\ i_2 = -j_1-j_2,\dots,i_{2\nu} = -j_{2\nu-1}-j_{2\nu},\ i_{2\nu+1} = j_{2\nu}+n.$$

Then

$$\sum_{|n|\ge N} B_\alpha(n) \le \sum_{i\in I^\alpha} |r(i_1)|^2\cdots|r(i_{2\nu+1})|^2, \tag{4.25}$$

where $I^\alpha = I^\alpha(N)$ is the set of indices $i=(i_1,\dots,i_{2\nu+1})$ given by

$$I^\alpha = I_1(\alpha)\times\cdots\times I_{2\nu+1}(\alpha),$$

where

$$I_s(\alpha) = \begin{cases} \mathbb{Z} & \text{if} \quad \alpha_s = 1 \\ \{i_s:\ |i_s|\ge N\} & \text{if} \quad \alpha_s = 0 \end{cases} \quad \text{for} \quad s = 1, 2\nu+1$$

and

$$I_s(\alpha) = \begin{cases} \mathbb{Z} & \text{if} \quad \alpha_{s-1} = 1 \quad \text{or} \quad \alpha_s = 1 \\ \{i_s:\ |i_s|\ge N\} & \text{if} \quad \alpha_{s-1} = 0 \quad \text{and} \quad \alpha_s = 0 \end{cases}, \quad 2\le s\le 2\nu.$$

Indeed, $\alpha_1 = 0$ (or $\alpha_{2\nu+1} = 0$) means that $|n-j_1|\le|n|/2$ (respectively, $|n-j_{2\nu}|\le|n|/2$). Thus

$$|i_1| = |n+j_1| = |2n-(n-j_1)| \ge |2n|-|n|/2 > |n| \ge N,$$

and the same argument shows that $|i_{2\nu+1}|\ge N$. Fix an s such that $2\le s\le 2\nu$. If $\alpha_{s-1}=\alpha_s=0$ then

$$|n-j_{s-1}|\le|n|/2, \quad |n-j_s|\le|n|/2,$$

thus

$$|i_s| = |j_{s-1}+j_s| = |2n-(n-j_{s-1})-(n-j_s)| \ge |2n|-2(|n|/2) \ge |n| \ge N.$$

Now we have

$$\sum_{|n|\ge N} B_\alpha(n) \le \prod_{s=1}^{2\nu+1} \sum_{i_s\in I_s(\alpha)} |r(i_s)|^2 \le (R(N))^{\gamma(\alpha)}(\|r\|^2)^{2\nu+1-\gamma(\alpha)}, \tag{4.26}$$

where

$$\gamma(\alpha) := card\{s : I_s(\alpha) \neq \mathbb{Z}\} \geq 2\nu + 1 - 2|\alpha|. \tag{4.27}$$

Indeed, one can easily see by the definition of $I_s(\alpha)$ that

$$\gamma(\alpha) = (1-\alpha_1) + (1-\alpha_{2\nu}) + \sum_{s=2}^{2\nu}(1-\alpha_s)(1-\alpha_{s-1})$$

$$\geq (1-\alpha_1) + (1-\alpha_{2\nu}) + \sum_{s=2}^{2\nu}(1-\alpha_s-\alpha_{s-1}) = 2\nu+1-2|\alpha|.$$

Taking into account (4.22), (4.23), (4.26) and (4.27), we obtain
(4.28)

$$\sum_{|n|\geq N}\left(\sum_{j\in J^\alpha(n)}\cdots\right)^2 \leq \left(\frac{\pi^2}{3}\right)^{2\nu}\left(\frac{2}{\sqrt{N}}\right)^{2|\alpha|}(R(N))^{\gamma(\alpha)}\left(\|r\|^2\right)^{2\nu+1-\gamma(\alpha)}$$

$$\leq K^{2\nu+1}(\rho(N))^{2|\alpha|+\gamma(\alpha))} \leq K^{2\nu+1}(\rho(N))^{2\nu+1},$$

where

$$\rho(N) = \frac{2}{\sqrt{N}} + R(N), \quad K = \frac{\pi^2}{3}(\|r\|^2+1).$$

Obviously $\rho(N) \to 0$ as $N \to \infty$, so there is N^* such that

$$\rho(N) < (8K)^{-1} \quad \text{for} \quad N \geq N^*. \tag{4.29}$$

Since $card(a) = 2^{2\nu}$, the inequalities (4.21), (4.28) and (4.29) imply, for $N \geq N^*$, that

$$\sum_{\nu=1}^{\infty}\left(\sum_{|n|\geq N}|\sigma_\nu(n,r)|^2\right)^{1/2} \leq \sum_{\nu=1}^{\infty}4^\nu\,(K\rho(N))^{\nu+1/2}$$

$$\leq 4(K\rho(N))^{3/2}\sum_{\nu=0}^{\infty}2^{-\nu} \leq 8(K\rho(N))^{3/2} \leq \frac{1}{2}\rho(N).$$

Thus, by (4.20),

$$\sum_{|n|\geq N}|\sigma(n.r)|^2 \leq \frac{1}{4}(\rho(N))^2 \leq \frac{2}{N} + (R(N))^2,$$

which completes the proof. □

Proof of Proposition 13. Suppose that Ω is a submultiplicative weight (we may assume that (3.3) holds with $C=1$) and $r = (r(n))_{n\in\mathbb{Z}}$ is a sequence of non-negative numbers such that $r(m) = 0$ for odd m and

$$\sum (r(2n)\Omega(2n))^2 < \infty. \tag{4.30}$$

Lemma 15 implies that

$$\sigma(n,r)\Omega(2n) \le \sigma(n,\tilde{r}), \quad \text{where} \quad \tilde{r} = (r(m)\Omega(m)) \,. \tag{4.31}$$

Therefore, in view of (4.30), we have $\tilde{r} \in \ell^2(\mathbb{Z})$, so, by Proposition 16, there exists $N_* > 0$ such that

$$\sum_{|n|\ge N_*} (\sigma(n,\tilde{r}))^2 \le \frac{2}{N} + \left(\sum_{|n|\ge N_*} |\tilde{r}(n)|^2 \right)^2 < \infty.$$

Thus, by (4.31), we obtain that

$$\sum (\sigma(n,r)\Omega(2n))^2 \le \sum (\sigma(n,\tilde{r}))^2 < \infty,$$

which proves Proposition 13.

3. Two elementary lemmas.

Lemma 17. *If $(B(n))_1^\infty$ and $(R(n))_1^\infty$ are decreasing sequences of positive real numbers such that*

$$B(n) \searrow 0, \qquad R(n) \searrow 0, \tag{4.32}$$

$$R(2n) \le C_1 B(n) + C_1 (R(n))^2, \quad C_1 > 0, \; n = 1,2,\ldots, \tag{4.33}$$

and

$$B(n) \le C_2 B(2n), \quad C_2 > 0, \; n = 1,2,\ldots, \tag{4.34}$$

then there exists a constant $C > 0$ such that

$$R(2n) \le C B(n), \quad n = 1,2,\ldots. \tag{4.35}$$

Proof. By (4.32) there exists n_1 such that

$$R(n) < \frac{1}{2C_1C_2} \quad \text{for} \quad n \ge n_1.$$

Therefore, by (4.33) and (4.34), we obtain

$$\frac{R(4n)}{B(2n)} \le C_1 + \frac{1}{2C_2}\frac{R(2n)}{B(2n)} \le C_1 + \frac{1}{2}\frac{R(2n)}{B(n)}, \quad n \ge n_1. \tag{4.36}$$

Consider the sequence

$$X_k = R(2^{k+1}n_1)/B(2^k n_1), \quad k = 1,2,\ldots. \tag{4.37}$$

From (4.36) it follows that

$$X_{k+1} \le C_1 + \frac{1}{2} X_k, \quad k = 1,2,\ldots. \tag{4.38}$$

One can easily derive from (4.38), by induction, that

$$X_{k+1} \le C_1 \sum_{j=0}^{k} 2^{-j} + 2^{-k} X_1,$$

thus the sequence (X_k) is bounded:

$$X_k \le 2C_1 + X_1. \tag{4.39}$$

Fix an arbitrary $n \ge n_1$. Then we have

$$2^k n_1 \le n < 2^{k+1} n_1$$

for some $k \ge 0$. Since $R(m)$ is decreasing, (4.34), (4.37) and (4.39) yield

$$R(2n) \le R(2^{k+1}n_1) = X_k B(2^k n_1) \le X_k C_2 B(2^{k+1} n_1) \le CB(n),$$

with $C = C_2(2C_1 + X_1)$, i.e., (4.35) holds. □

The next lemma explains that, due to Abel's transform, a sequence $(x(n)) \in \ell^1$ belongs to a weighted ℓ^1-space generated by a weight $T(n)$ if and only if the sequence $(X(N))$,

$$X(N) = \sum_{|n| \ge N} |x(n)|,$$

belongs to the weighted ℓ^1-space generated by the weight $T(N) - T(N-1)$.

Lemma 18. *If $(T(n))$, $n \in \mathbb{Z}$ is a weight sequence then the following conditions are equivalent:*

$$(i) \qquad \sum_n |x(n)| T(n) < \infty;$$

$$(ii) \quad \begin{array}{lll} (ii.a) & X(N)T(N) \to 0 \quad as \quad N \to \infty, & X(N) = \sum_{|n|\ge N} |x(n)|; \\ (ii.b) & \sum_n X(n)[T(n) - T(n-1)] < \infty. & \end{array}$$

Proof. $(i) \Rightarrow (ii)$. If (i) holds, then

$$X(N)T(N) \le \sum_{|n| \ge N} |x(n)| T(n) \to 0,$$

thus part (a) of (ii) holds. Moreover, if $0 < M < N$ then

$$\sum_{M \le |n| \le N} |x(n)|\, T(n) = \sum_{n=M}^{N} (|x(-n)| + |x(n)|) T(n) \tag{4.40}$$

$$= X(M)T(M) - X(N+1)T(N) + \sum_{n=M+1}^{N} X(n) \left(T(n) - T(n-1)\right).$$

By (i) the left-hand side of the above identity goes to 0 as $M \to \infty$. Since

$$X(N+1)T(N) \le X(N+1)T(N+1) \to 0,$$

we obtain, by part (a) of (ii), that

$$\sum_{n=M+1}^{N} X(n)\,(T(n) - T(n-1)) \to 0 \quad \text{as} \quad M \to \infty,$$

thus the series in (ii.b) satisfies the Cauchy convergence condition.

The implication $(ii) \Rightarrow (i)$ follows from (4.40) also. Indeed, by part (ii.a) $X(M)T(M) \to 0$ as $M \to \infty$, thus the Cauchy convergence condition for the series in (ii.b) implies the Cauchy convergence condition for the series in (i).

□

4. Now we prove Theorem 14 in the case where Ω is a slowly increasing submultiplicative weight.

Proposition 19. *Suppose Ω is a slowly increasing (i.e. $\Omega \in$ (3.4)) submultiplicative weight. If $r = (r(n))_{n\in\mathbb{Z}}$ and $\xi = (\xi(n))_{n\in\mathbb{Z}}$ are two sequences of non-negative numbers such that*

$$r(n) = 0 \quad \textit{for odd} \quad n, \qquad r \in \ell^2,$$

and

$$r(2n) \le \xi(n) + \sigma(n, r), \tag{4.41}$$

where σ is the operator defined by (4.8), (4.9) and (4.7), then

$$\sum_n |\xi(n)|^2 \,(\Omega(2n))^2 < \infty \quad \Rightarrow \quad \sum_n |r(2n)|^2 \,(\Omega(2n))^2 < \infty. \tag{4.42}$$

Proof. Since the weight Ω is slowly increasing we have, by (3.5), that

$$\exists a > 0 : \quad \Omega(m) \le |m|^a \quad \text{for } |m| > 1. \tag{4.43}$$

For convenience the proof is divided into two steps.

Step 1. *Proof of the claim in the case where $a < 1/4$.*

By (4.41),

$$\sum_{|n|\ge N} |r(2n)|^2 \le 2 \sum_{|n|\ge N} |\xi(n)|^2 + 2 \sum_{|n|\ge N} |\sigma(n, r)|^2, \tag{4.44}$$

and therefore, by Proposition 16, we obtain for $N \ge 4$ (since $(\Omega(N))^2 \le N^{1/2} \le N/2$)

$$R(2N) \le 2X(N) + \frac{2}{(\Omega(N))^2} + 2(R(N))^2, \tag{4.45}$$

where

$$X(n) = \sum_{|n| \geq N} |\xi(n)|^2. \tag{4.46}$$

On the other hand we have

$$\varepsilon_N := X(N)(\Omega(N))^2 \leq \sum_{|n| \geq N} |\xi(n)|^2 (\Omega(n))^2 \to 0,$$

and therefore,

$$X(N) = \varepsilon_N / (\Omega(N))^2 \quad \text{with} \quad \varepsilon_N \to 0. \tag{4.47}$$

Consider the sequence $(B(N))$ given by

$$B(N) := X(N) + \frac{1}{(\Omega(N))^2}. \tag{4.48}$$

Since the weight Ω is slowly increasing, (4.47) and (4.48) imply that

$$\sup_N B(N)/B(2N) < \infty, \qquad B(N) \leq \tilde{C}/(\Omega(N))^2. \tag{4.49}$$

By (4.45) we have

$$R(2N) \leq B(N) + 2(R(N))^2,$$

so, in view of (4.49), Lemma 17 gives us that

$$R(2N) \leq C_1 B(N) \leq C_1 \frac{\tilde{C}}{(\Omega(N))^2}. \tag{4.50}$$

On the other hand, by (4.44) and Proposition 16, we obtain

$$R(2N) \leq 2X(N) + 4/N + 2(R(N))^2. \tag{4.51}$$

Notice that (4.43) with $a < 1/4$ implies $(\Omega(N))^4/N \to 0$. So, since Ω is slowly increasing weight, (4.50) and (4.51 yield

$$R(2N) \leq 2X(N) + \frac{C_2}{(\Omega(N))^4}. \tag{4.52}$$

Now (4.47) and (4.52) imply that

$$R(2N)\Omega(N)^2 \to 0 \quad \text{as} \quad N \to \infty. \tag{4.53}$$

Moreover, (4.52) implies

$$\sum_N R(2N) \left((\Omega(N))^2 - (\Omega(N-1))^2 \right) < \infty. \tag{4.54}$$

Indeed, by Lemma 18 (since $(\xi_n) \in \ell^2(\Omega)$) we have that

$$\sum_N X(N) \left((\Omega(N))^2 - (\Omega(N-1))^2 \right) < \infty.$$

On the other hand

$$\sum_N \frac{((\Omega(N))^2-(\Omega(N-1))^2)}{(\Omega(N))^4} \le \sum_N \left(\frac{1}{(\Omega(N-1))^2}-\frac{1}{(\Omega(N))^2}\right) < \infty,$$

thus (4.54) holds. So, in view of (4.53) and (4.54), by Lemma 18 $\sum |r(m)|^2(\Omega(m))^2 < \infty$, i.e., (4.42) holds if $a < 1/4$.

Step 2. Proof of the claim in the case where $a \ge 1/4$.

If $\Omega(m) \le |m|^a$ with $a \ge 1/4$, then we choose k_0 so that $a/k_0 < 1/4$, set

$$\Omega_k(m) = (\Omega(m))^{k/k_0}, \quad k = 1, \dots, k_0, \tag{4.55}$$

and prove that the claim holds for Ω_k by induction in k. Since $\Omega_1(m) < |m|^{1/4}$, by Step 1 the claim holds for $k = 1$.

Assume that $r = (r(m)) \in \ell^2(\Omega_k)$ for some k, $1 \le k < k_0$. Multiplying both sides of (4.41) by $\Omega_k(2n)$ and using that Ω_k is submultiplicative, we obtain

$$\tilde{r}(2n) \le \tilde{\xi}(n) + \sigma(n, \tilde{r}), \tag{4.56}$$

where

$$\tilde{r}(m) = r(m)\Omega_k(m), \quad \tilde{\xi}(m) = \xi(m)\Omega_k(2m), \quad m \in \mathbb{Z}.$$

Since $(r(m)) \in \ell^2(\Omega_k)$ and $(\xi_m) \in \ell^2(\Omega)$ we have that

$$\tilde{r} = (\tilde{r}(m)) \in \ell^2, \quad (\tilde{\xi}(m)) \in \ell^2(\Omega_{k_0-k}) \subset \ell^2(\Omega_1).$$

From this by Step 1 it follows that $(\tilde{r}(m)) \in \ell^2(\Omega_1)$, thus $r = (r(m)) \in \ell^2(\Omega_{k+1})$.

Hence $r = (r(m)) \in \ell^2(\Omega_k)$ for $k = 1, \dots, k_0$. By (4.55), $\Omega_{k_0} = \Omega$. This proves Proposition 19.

□

5. Finally, we prove Theorem 14 for rapidly increasing weights of the form $\Omega_\varphi(|n|)) = \exp(\varphi(\log |n|))$.

Proposition 20. *Suppose* $(M_k)_{k=0}^\infty$ *is a Carlemann sequence (see (3.25) - (3.27)) such that* $M_k = k!E_k$ *with*

$$\sqrt{k}(E_k)^2/E_{2k} \to 0, \tag{4.57}$$

$$\exists \tau \in (0,1) \ : \ \sum_{k=1}^\infty \left((E_k)^2/E_{2k}\right)^\tau \to 0. \tag{4.58}$$

If $r = (r(n))_{n\in\mathbb{Z}}$ *and* $\xi = (\xi_n)_{n\in\mathbb{Z}}$ *are sequences of non-negative numbers such that*

$$r(n) = 0 \quad \textit{for odd } n, \qquad r \in \ell^2$$

and

$$r(2n) \le \xi(n) + \sigma(n,r), \quad |n| \ge n_*, \tag{4.59}$$

where σ is the operator defined by (4.8) - (4.7), then

(4.60)

$$|\xi|_k := \sum_n |\xi_n||2n|^k \le CM_k \quad \forall k \implies \|r\|_k = \sum_n |r(2n)||2n|^k \le \tilde{C}M_k \quad \forall k.$$

Proof. By Proposition 19,

$$|\xi|_k < \infty \quad \implies \quad \|r\|_k < \infty, \qquad \forall k \in \mathbb{N}.$$

Set

$$\|r\|_k = X_k M_k \|r\|_0, \qquad k = 0, 1, 2, \ldots. \tag{4.61}$$

The lemma will be proven if we show that the sequence (X_k) is bounded.

Multiplying (4.59) by $|2n|^{k+1}$ and summing for $|n| \ge N_* > n_*$ (now $N_* > n_*$ is arbitrary, but later it will be chosen large enough) we obtain

(4.62)

$$\|r\|_{k+1} \le \sum_{|n|<N_*} |r(2n)||2n|^{k+1} + \sum_{|n|\ge N_*} |\xi(n)||2n|^{k+1} + \sum_{|n|\ge N_*} \sigma(n,r)|2n|^{k+1}$$

$$\le C_1(2N_*)^{k+1} + |\xi|_{k+1} + \sum_{\nu=1}^{\infty} \sum_{|n|\ge N_*} |2n|^{k+1}\sigma_\nu(n,r),$$

where $C_1 = \max_m |r(m)|$.

Next we fix $\nu \in \mathbb{N}$ and estimate the sum

$$S_\nu := \sum_{|n|\ge N_*} |2n|^{k+1}\sigma_\nu(n,r).$$

Observe that by (4.7)

$$S_\nu = \sum_{|n|\ge N_*} |2n|^{k+1} \sum_{j_1,\ldots,j_{2\nu}\ne n} \frac{r(n+j_1)r(-j_1-j_2)\ldots r(j_{2\nu}+n)}{|n-j_1||n-j_2|\ldots|n-j_{2\nu}|}. \tag{4.63}$$

As in the proof of Proposition 16 we divide the set of indices in the above sum into subsets

$$J(n) = \{j = (j_1,\ldots,j_{2\nu}) : \quad j_1,\ldots,j_{2\nu} \ne n\} = \sum_{\alpha\in a} J^\alpha(n),$$

where a is the set of all 2ν-tuples $\alpha = (\alpha_1,\ldots,\alpha_{2\nu})$ with $\alpha_i \in \{0,1\}$, and

$$J^\alpha(n) = \left\{ (j_1,\ldots,j_{2\nu}) \in J(n) : \quad \Bigg| \quad \begin{array}{ll} |n-j_s| \le |n|/2 & \text{if } \alpha_s = 0 \\ |n-j_s| > |n|/2 & \text{if } \alpha_s = 1 \end{array} \right\}.$$

By the definition of $J^\alpha(n)$ we have

$$|n-j_1|\ldots|n-j_{2\nu}| \ge (|n|/2)^{|\alpha|} \ge (N_*/2)^{|\alpha|} \quad \text{for} \quad j \in J^\alpha(n).$$

With this estimate for the denominator in (4.63) we obtain

$$S_\nu \le \sum_{\alpha\in a} (N_*/2)^{-|\alpha|} \sum_{|n|\ge N_*} \sum_{J^\alpha(n)} |2n|^{k+1} r(n+j_1 \dots r(j_{2\nu}+n). \tag{4.64}$$

Set

$$a' = \{\alpha \in a: \ |\alpha| = |\alpha_1 + \cdots \alpha_{2\nu}| \ge \nu\}, \quad a'' = a \setminus a', \tag{4.65}$$

and split the sum in (4.64) into two subsums:

$$\sum_{\alpha\in a} \cdots = \sum_{\alpha\in a'} \cdots + \sum_{\alpha\in a''} \cdots \tag{4.66}$$

First we estimate $\sum_{\alpha\in a'}$. Taking into account that $card\,[a] = 2^{2\nu}$ we obtain

$$\sum_{\alpha\in a'} \cdots \le (8/N_*)^\nu \sum_n \sum_{j_1,\dots,j_{2\nu}} |2n|^{k+1} r(n+j_1)\dots r(j_{2\nu}+n). \tag{4.67}$$

By the binomial formula

$$\begin{aligned} |2n|^{k+1} &= |(n+j_1) + (-j_1-j_2) + (j_2+j_3) + \cdots + (j_{2\nu}+n)|^{k+1} \\ &\le \sum_{|s|=k+1} \binom{k+1}{s} |n+j_1|^{s_0} \cdot |-j_1-j_2|^{s_1} \cdots |j_{2\nu}+n|^{s_{2\nu}}, \end{aligned}$$

thus (4.67) implies

$$\sum_{\alpha\in a'} \cdots \le \left(\frac{8}{N_*}\right)^\nu \sum_{|s|=k+1} \binom{k+1}{s} \|r\|_{s_0}\|r\|_{s_1}\dots\|r\|_{s_{2\nu}}$$

$$= \left(\frac{8}{N_*}\right)^\nu (2\nu+1)\|r\|_0^{2\nu}\|r\|_{k+1} + \left(\frac{8}{N_*}\right)^\nu \sum_{\substack{|s|=k+1 \\ s_i<k+1}} \binom{k+1}{s} \|r\|_{s_0}\|r\|_{s_1}\dots\|r\|_{s_{2\nu}}.$$

Obviously there exists $N_1 > 0$ such that

$$\sum_{\nu=1}^\infty (8/N_*)^\nu (2\nu+1)\|r\|_0^{2\nu} < 1/2 \quad \text{for } N_* > N_1.$$

Thus we have

$$\sum_{\nu=1}^\infty \sum_{\alpha\in a'_\nu} \le \frac12 \|r\|_{k+1} + \sum_{\nu=1}^\infty \left(\frac{8}{N_*}\right)^\nu \sum_{\substack{|s|=k+1 \\ s_i<k+1}} \binom{k+1}{s} \|r\|_{s_0}\|r\|_{s_1}\dots\|r\|_{s_{2\nu}} \tag{4.68}$$

(where the notation a'_ν is used to show the dependence of a on ν).

Next we estimate $\sum_{\alpha\in a''}$. Consider the expressions

$$i_1 = n + j_1, \quad i_2 = -j_1 - j_2, \ldots, i_{2\nu} = -j_{2\nu-1} - j_{2\nu}, \quad i_{2\nu+1} = j_{2\nu} + n \tag{4.69}$$

(these formulas are used in the proof of Proposition 16 to change the indices of summation). It is easy to check by the definition of $J^\alpha(n)$ that if $j \in J^\alpha(n)$ then

$$\alpha_1 = 0 \Rightarrow |i_1| = |n + j_1| > |n|, \quad \alpha_{2\nu} = 0 \Rightarrow |i_{2\nu+1}| = |j_{2\nu} + n| > |n|$$

and

$$\alpha_{s-1} = \alpha_s = 0 \Rightarrow |i_s| = |j_{s-1} + j_s| > |n|, \quad 2 \le s \le 2\nu$$

(see for details the proof of Proposition 16).

Let $\gamma(\alpha)$ denotes the number of expressions i_s in (4.69) such that $|i_s| > n$ for $j \in J^\alpha(n)$. Of course, $\gamma(\alpha)$ is the same function that is used in the proof of Proposition 16, so, by (4.27), we have

$$\gamma(\alpha) \ge 2\nu + 1 - 2|\alpha|.$$

In particular, since $|\alpha| \le \nu - 1$ for $\alpha \in a''$, we obtain

$$\gamma(\alpha) \ge 3 \quad \text{for } \alpha \in a''.$$

Choose indices s_1, s_2 so that the corresponding expressions i_{s_1} and i_{s_2} in (4.69) satisfy

$$|i_{s_1}| > |n|, \quad |i_{s_2}| > |n| \quad \text{for } j = (j_1, \ldots, j_{2\nu}) \in J^\alpha(n).$$

Set

$$k_1 = k_2 = \frac{k+1}{2} \ \text{for odd } k, \quad k_1 = \frac{k}{2}, \ \ k_2 = 1 + \frac{k}{2} \ \text{for even } k. \tag{4.70}$$

Then

$$|n|^{k+1} \le |i_{s_1}|^{k_1} |i_{s_2}|^{k_2} \quad \text{for } j \in J^\alpha(n).$$

Thus, changing the indices of summation by formulas (4.69) we obtain by (4.64)

$$\sum_{\alpha\in a''} \cdots \le \sum_{\alpha\in a''} \left(\frac{2}{N_*}\right)^{|\alpha|} \sum_{|n|\ge N_*} \sum_{J^\alpha(n)} 2^{k+1} |i_{s_1}|^{k_1} |i_{s_2}|^{k_2} r(i_1)\ldots r(i_{2\nu+1}), \tag{4.71}$$

$$\le \sum_{\alpha\in a''} \left(\sqrt{\frac{2}{N_*}}\right)^{2|\alpha|} 2^{k+1} \sum_{I^\alpha(N_*)} |i_{s_1}|^{k_1} |i_{s_2}|^{k_2} r(i_1)\ldots r(i_{2\nu+1}),$$

where $I^\alpha(N_*) = I_1^\alpha \times \cdots \times I_{2\nu+1}^\alpha$ with

$$I_s^\alpha = \{m \in \mathbb{Z} : \quad |m| > N_*\} \quad \text{if } |i_s| > n \ \ \forall j \in J^\alpha(n),$$

and $I_s^\alpha = \mathbb{Z}$ otherwise.

Set

$$R(N_*) = \sum_{|n|>N_*} |r(n)|, \quad \rho(N_*) = \sqrt{2/N_*} + R(N_*). \tag{4.72}$$

With these notations (4.71) implies

$$\sum_{\alpha\in a''} \cdots \le \sum_{\alpha\in a''} \left(\sqrt{\frac{2}{N_*}}\right)^{2|\alpha|} 2^{k+1} \sum_{I^\alpha(N_*)} \|r\|_{k_1}\|r\|_{k_2}| \,(R(N_*))^{\gamma(\alpha)-2} \,\|r\|_0^{2\nu+1-\gamma(\alpha)}$$

$$\le \sum_{\alpha\in a''} (\rho(N_*))^{2|\alpha|+\gamma(\alpha)-2}\, 2^{k+1}\|r\|_{k_1}\|r\|_{k_2}|K^{2\nu-1},$$

where $K = \max(1, \|r\|_0)$. Since

$$card\,[a''] \le 2^\nu, \quad 2|\alpha| + \gamma(\alpha) \ge 2\nu + 1$$

(by (4.27)) we obtain

$$\sum_{\alpha\in a''} \cdots \le (2K\rho(N_*))^{2\nu-1} 2^{k+1}\|r\|_{k_1}\|r\|_{k_2}|. \tag{4.73}$$

From (4.72) it follows that $\rho(N_*) \to 0$ as $N_* \to \infty$, so there exists $N_2 > 0$ such that

$$2K\rho(N_*) < 1/2 \quad \text{for } N_* \ge N_2.$$

Thus (4.73) implies that

$$\sum_{\nu=1}^{\infty}\sum_{\alpha\in a''} \cdots \le 2^{k+1}\|r\|_{k_1}\|r\|_{k_2} \quad \text{for } N_* \ge N_2. \tag{4.74}$$

Now we add together the above estimates. From (4.62), (4.64), (4.68) and (4.74) it follows that for $N_* > \max(N_1, N_2)$

$$\|r\|_{k+1} \le C_1(2N_*)^{k+1} + \|\xi\|_{k+1} + 2^{k+1}\|r\|_{k_1}\|r\|_{k_2}$$

$$+2\sum_{\nu=1}^{\infty}\left(\frac{8}{N_*}\right)^{\nu} \sum_{\substack{|s| = k+1 \\ s_i < k+1}} \binom{k+1}{s} \|r\|_{s_0}\|r\|_{s_1}\ldots\|r\|_{s_{2\nu}}$$

Substituting the norms of r by (4.61), and estimating from above the norm of ξ by (4.60), we obtain (with $M_k = k!E_k$, and after dividing with $(k+1)!E_{k+1}\|r\|_0$):

$$X_{k+1} \le \frac{2C_1}{\|r\|_0}\cdot\frac{(2N_*)^{k+1}}{(k+1)!E_{k+1}} + 2C + 2^{k+2}\frac{k_1!k_2!}{(k+1)!}\cdot\frac{E_{k_1}E_{k_2}}{E_{k+1}} \tag{4.75}$$

$$+2\sum_{\nu=1}^{\infty}\left(\frac{8\|r\|_0^2}{N_*}\right)^{\nu}\sum_{\substack{|s|=k+1\\ s_i<k+1}}\frac{E_{s_0}\dots E_{s_{2\nu}}}{E_{k+1}}X_{s_0}\dots X_{s_{2\nu}},$$

where k_1 and k_2 are given in (4.70).

Obviously the first term in the above estimate of X_{k+1} goes to 0 as $k\to\infty$, so it is bounded. The same is true for the third term. Indeed, if $k+1$ is even, say $k+1=2m$, then $k_1=k_2=m$, and by the Stirling formula we have in view of (4.57) that

$$2^{2m}\frac{m!m!}{(2m)!}\cdot\frac{E_mE_m}{E_{2m}}\asymp\sqrt{m}(E_m)^2/E_{2m}\to 0.$$

If $k+1$ is odd, say $k+1=2m+1$, then $k_1=m,\ k_2=m+1$, and we obtain

$$2^{2m+1}\frac{m!(m+1)!}{(2m+1)!}\cdot\frac{E_mE_{m+1}}{E_{2m+1}}=2^{2m}\frac{m!m!}{(2m)!}\cdot\frac{E_mE_m}{E_{2m}}\left[\frac{2m+2}{2m+1}\cdot\frac{e_{m+1}}{e_{2m+1}}\right]\to 0,$$

because the expression in the square brackets is bounded. Thus we have (4.76)

$$D:=\sup\left(\frac{2C_1}{\|r\|_0}\cdot\frac{(2N_*)^{k+1}}{(k+1)!E_{k+1}}+2C+2^{k+2}\frac{k_1!k_2!}{(k+1)!}\cdot\frac{E_{k_1}E_{k_2}}{E_{k+1}}\right)<\infty.$$

By Lemma 9 the assumption (4.58) implies that

$$\exists Q>0\ :\qquad \sup_k\sum_{|s|=k+1}\left(\frac{E_{s_0}\dots E_{s_{2\nu}}}{E_{k+1}}\right)^{\tau}<Q^{2\nu}.$$

Therefore the double sum in (4.75) does not exceed the expression

$$2\sup_{\nu\geq 1}\max_{\substack{|s|=k+1\\ s_i<k+1}}\left(\left(\frac{E_{s_0}\dots E_{s_{2\nu}}}{E_{k+1}}\right)^{1-\tau}X_{s_0}\dots X_{s_{2\nu}}\right)\cdot\sum_{\nu=1}^{\infty}\left(\frac{8\|r\|_0^2Q^2}{N_*}\right)^{\nu}.$$

If

$$N_*>N_3:=40\|r\|_0^2Q^2,$$

then the sum in the above expression is less than 1/4. so in view of (4.76) we obtain that

$$X_{k+1}\leq\max\left(2D,\sup_{\nu\geq 1}\max_{\substack{|s|=k+1\\ s_i<k+1}}\left(\left(\frac{E_{s_0}\dots E_{s_{2\nu}}}{E_{k+1}}\right)^{1-\tau}X_{s_0}\dots X_{s_{2\nu}}\right)\right).$$

Hence by Lemma 10 (with $T=2D,\ F_k=(E_k)^{1-\tau}$ we obtain that the sequence (X_k) is bounded, which completes the proof of Proposition 20. □

Now we complete the proof of Theorem 14 for weights of the form $\Omega_\varphi(n) = \exp(\varphi(|n|))$, where φ has the properties (3.10), (3.11) and (3.14). Let $\xi = (\xi(m))_{m\in\mathbb{Z}}$ and $r = (r(m))_{m\in\mathbb{Z}}$ be sequences with non-negative terms such that

$$r(2n) \leq \xi(n) + \sigma(n,r), \quad n \geq n_*. \tag{4.77}$$

We have to prove that

$$\sum (\xi(n)\Omega_\varphi(2n))^2 < \infty \quad \Longrightarrow \quad \sum (r(m)\Omega_\varphi(m))^2 < \infty. \tag{4.78}$$

Set

$$\overline{\xi} = (\overline{\xi}(m)), \quad \overline{\xi}(2m) = \xi(m), \quad \overline{\xi}_{2m+1} = 0.$$

By part (a) of Lemma 9,

$$\sum \left(\overline{\xi}(m)\Omega_\varphi(m)\right)^2 < \infty \quad \Longrightarrow \quad \exists C > 0 \ : \quad \|\overline{\xi}\| = |\xi|_k \leq CM_k(\varphi_1),$$

where $\varphi_1(t) = \varphi(t) - t$ and $(M_k(\varphi_1))$ is the Carlemann sequence generated by φ_1 (see the text after (3.4) prior Lemma 8).

By Proposition 20 there exists $\tilde{C} > 0$ such that

$$\|\overline{\xi}\|_k = |\xi|_k \leq CM_k(\varphi_1) \quad \Longrightarrow \quad \|r\|_k \leq \tilde{C}M_k(\varphi_1) \quad k = 0,1,2,\ldots.$$

On the other hand part (b) of Lemma 8 yields

$$\|r\|_k \leq \tilde{C}M_k(\varphi_1) \ \ \forall k \quad \Longrightarrow \quad \sum (r(m)\Omega_{\hat{\varphi}}(m))^2 < \infty, \tag{4.79}$$

with $\hat{\varphi}(t) = \varphi_1(t) - 4t = \varphi(t) - 5t$.

Consider the sequences

$$\hat{r} = (r(k)\Omega_{\hat{\varphi}}(k)), \qquad \hat{\xi} = (\xi(k)\Omega_{\hat{\varphi}}(2k)).$$

Multiplying (4.77) by $\Omega_{\hat{\varphi}}(2n)$, we obtain, by Lemma 15, that

$$\hat{r}(2n) \leq \hat{\xi}(n) + \sigma(n,\hat{r}). \tag{4.80}$$

By (4.79) we have that $\hat{r} \in \ell^2(\mathbb{Z})$. Since $\Omega_\varphi(m) = \Omega_{\hat{\varphi}}(m) \cdot |m|^5$, our hypothesis yields

$$\sum \left(\hat{\xi}(k)|2m|^5\right)^2 = \sum (\xi(m)\Omega_\varphi(2m))^2 < \infty. \tag{4.81}$$

In view of (4.80) and (4.81), Proposition 19 can be applied to the sequences $\hat{r}, \hat{\xi}$ and the weight $\Omega(n) = |n|^5$, so we obtain $\sum (\hat{r}(m)|m|^5)^2 < \infty$. Thus

$$\sum (r(m)\Omega_\varphi(m))^2 = \sum \left(\hat{r}(m)|m|^5\right)^2 < \infty,$$

which completes the proof of Theorem 14. Therefore, Theorem 11 has been proven as well.

5. Conclusions and comments

1. If L is a Dirac operator of the form (2.4), not necessarily selfadjoint, the left side inequality in (2.58), Theorem 5, does not hold. But in any case, $|\lambda_n^+ - \lambda_n^-|$ could be estimated from above if we use the basic equation (2.13) and Lemma 4. More precisely, the following is true.

Lemma 21. *If L is a Dirac operator of the form (2.1), then*

$$|\lambda_n^+ - \lambda_n^-| \leq 2 \max_{|z|\leq \pi/2} \left|S^{12}(n,z)\right| + \max_{|z|\leq \pi/2} \left|S^{21}(n,z)\right|, \quad n \neq n_*. \tag{5.1}$$

Proof. Indeed, with $\alpha_n(z) = S^{11}(n,z) = S^{22}(n,z)$ and

$$\zeta = z - \alpha_n(z), \tag{5.2}$$

the equation (2.13) becomes

$$\zeta^2 = S^{11}(n,z) S^{22}(n,z). \tag{5.3}$$

By Lemma 4, there exists $n_* > N_0$, where N_0 is the constant from Lemma 1, such that

$$\left|\frac{d\alpha_n(z)}{dz}\right| \leq \frac{1}{2} \quad \text{for} \quad |z| \leq \pi/2, \;\; |n| \geq n_*.$$

Thus (5.2) defines in the disc $|z| < \pi/2$ a holomorphic mapping $\zeta(z) = z - \alpha_n(z)$ such that

$$1/2 \leq |d\zeta/dz| \leq 3/2.$$

From here it follows that

$$\frac{1}{2} \leq |z_n^+ - z_n^-| \leq |\zeta(z_n^+) - \zeta(z_n^-)| \leq 2|z_n^+ - z_n^-|,$$

where, in view of Lemma 1, $|z_n^\pm| < \pi/2$ for $|n| \geq n_*$. So, taking into account that $|\lambda_n^+ - \lambda_n^-| = |z_n^+ - z_n^-|$, we obtain that

$$\frac{1}{2}|\lambda_n^+ - \lambda_n^-| \leq |\zeta_n^+ - \zeta_n^-| \leq 2|\lambda_n^+ - \lambda_n^-|,$$

where $\zeta_n^+ = \zeta(z_n^+)$ and $\zeta_n^- = \zeta(z_n^-)$.

On the other hand, (5.3) implies that

$$\left|\zeta_n^\pm\right| = \left|S^{12}(n,z_n^\pm) S^{21}(n,z_n^\pm)\right|^{1/2} \leq \frac{1}{2}\left|S^{12}(n,z_n^\pm)\right| + \frac{1}{2}\left|S^{21}(n,z_n^\pm)\right|.$$

Therefore,

$$|\zeta_n^+ - \zeta_n^-| \leq |\zeta_n^+| + |\zeta_n^-| \leq \max_{|z|\leq \pi/2} \left|S^{12}(n,z)\right| + \max_{|z|\leq \pi/2} \left|S^{21}(n,z)\right|;$$

hence (5.1) holds.

□

Theorem 22. *Let L be a Dirac operator of the form (2.1) with potential $P(x) = \sum p(2n)e^{i2nx}$ and $Q(x) = \sum q(2n)e^{i2nx}$. If Ω is a submultiplicative weight, then*

(5.4)
$$\sum \left(|p(2n)|^2 + |q(2n)|^2\right) (\Omega(2n))^2 < \infty \quad \Longrightarrow \quad \sum |\lambda_n^+ - \lambda_n^-|^2 (\Omega(2n))^2 < \infty.$$

Proof. Set

$$r(m) = \max(|p(-m)|, \ |p(m)|, \ |q(-m)|, \ |q(m)|) \tag{5.5}$$

In view of (2.36) -(2.39) we obtain by (2.25)

$$\max_{|z|\leq \pi/2} |S^{12}(n,z)| \leq \sigma(n,r), \quad \max_{|z|\leq \pi/2} |S^{21}(n,z)| \leq \sigma(n,r), \tag{5.6}$$

where $r = (r(m)$ and $\sigma(n,r)$ is defined by (4.8) and (4.7). Now the claim follows from Proposition 13.

□

Under rigid assumptions on Ω, which, for example, exclude such weights Ω as $\Omega(m) = \exp(a|m|)$, or $\Omega(k) = [\log(e + |k|)]^a$, $a > 0$, the claim (5.4) can be found in [13] or [14].

2. The present paper deals only with the case of subexponential growth of the weight Ω, i.e., $\Omega(m) \leq e^{a|m|}$, $a > 0$. The case of superexponential weights Ω could be analyze as well. We will present such analysis elsewhere. See such analysis of Hill–Schrödinger operators in [5].

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