The best n-dimensional linear approximation of the Hardy operator and the Sobolev Classes on unit circle and on line.

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Abstract

Let $I = [a, b]$ with $-\infty < a < b < \infty$ and $1 < p < \infty$. Let

$$Tf(x) = \int_a^x f(t)dt, \qquad \text{with } a \le x \le b$$

be the Hardy operator on interval I and let we have the following Sobolev Embeddings on interval I and on unit circle $\mathbb{T}$:

$$\begin{aligned}
E_0 &: W_0^{1,p}(I) \to L^p(I) \\
E_a &: W_a^{1,p}(I) \to L^p(I) \\
E_{mid} &: W_{mid}^{1,p}(I) \to L^p(I) \\
E_1 &: W^{1,p}(I) \to L^p(I) \\
E_2 &: W^{1,p}(\mathbb{T}) \to L^p(\mathbb{T}) \\
E_3 &: W^{1,p}(I)/\operatorname{span}\{1\} \to L^p(I)/\operatorname{span}\{1\} \\
E_4 &: W^{1,p}(\mathbb{T})/\operatorname{span}\{1\} \to L^p(\mathbb{T})/\operatorname{span}\{1\}.
\end{aligned}$$

Exact values of the Approximation numbers and n-widths for the Hardy operator and these Sobolev embeddings are shown.

We also show the optimal n-dimensional linear map for approximation of the Hardy operator and the Sobolev embeddings and the optimal n-dimensional subspace for approximation of the Hardy operator and the Sobolev spaces in L^p, together with corresponding extremal functions.

Keyword: Approximation numbers, Sobolev Embedding, Hardy-type operators, Kolmogorov width, Gel'fand width, Bernstein width, linear width, Integral operators, MSC 47G10, 47B10

1 Introduction.

Let in this paper $I = [a, b]$ be an interval with $-\infty < a < b < \infty$ and $\mathbb{T}$ be the unit circle realized as the interval $[-\pi, \pi]$ with identified points $-\pi$ and π. We also assume that $1 < p < \infty$ and denote $p' = \frac{p}{p-1}$ and by $\|.\|_{p,\mathbb{T}}$ and $\|.\|_{p,I}$ the usual norm on the Lebesque space $L^p(\mathbb{T})$ and on $L^p(I)$.

1.1 Definitions of spaces and embeddings

Let we define on bounded interval $I = [a, b]$ the Hardy operator T_c by

$$T_c f(x) = \int_c^x f(t)dt, \qquad \text{where } a \le x, c \le b.$$

In the case when $c = a$ we can shortly use the notation T instead T_a. By $BL^p(I) = \{f; f \in L^p(I) \text{ and } \|f\|_{p,I} \le 1\}$ we shall mean the unit ball in $L^p(I)$ space and by $T_c BL^p(I) = \{T_c f; f \in BL^p(I)\}$ image of $BL^p(I)$ under T_c. It is well-known that for bounded interval I the operator $T_c : L^p(I) \to L^p(I)$ is a compact bounded operator.

By $W^{1,p}(\mathbb{T})$ (or respectively by $W^{1,p}(I)$) we understand the Sobolev space of functions on $\mathbb{T}$ (or on I) (i.e. the set of all absolutely continuous functions on $\mathbb{T}$ with $\|f'\|_{p,\mathbb{T}} < \infty$, or respectively on I with $\|f'\|_{p,I} < \infty$). Let we remind that $\|f'\|_{p,\mathbb{T}}$ and $\|f'\|_{p,I}$ are pseudonorms on $W^{1,p}(\mathbb{T})$ or respectively on $W^{1,p}(I)$.

By $W_0^{1,p}(I)$ we understand, as usual, the space of all absolutely continuous functions on I with finite norm $\|f'\|_{p,I}$ and 0 boundary value at a and b.

By $W_a^{1,p}(I)$ we mean the space of all absolutely continuous functions on I with finite norm $\|f'\|_{p,I}$ and 0 boundary value at a.

And by $W_{mid}^{1,p}(I)$ we mean the space of all absolutely continuous functions on I with finite norm $\|f'\|_{p,I}$ and 0 value at the middle of interval I.

Let we define unit balls on our Sobolev spaces:

$$BW_0^{1,p}(I) = \{f; f \in W_0^{1,p}(I) \text{ and } \|f'\|_{p,I} \le 1\},$$

$$BW_{mid}^{1,p}(I) = \{f; f \in W_{mid}^{1,p}(I) \text{ and } \|f'\|_{p,I} \le 1\},$$
$$BW_a^{1,p}(I) = \{f; f \in W_a^{1,p}(I) \text{ and } \|f'\|_{p,I} \le 1\},$$
$$BW^{1,p}(I) = \{f; f \in W^{1,p}(I) \text{ and } \|f'\|_{p,I} \le 1\},$$
$$BW^{1,p}(\mathbb{T}) = \{f; f \in W^{1,p}(\mathbb{T}) \text{ and } \|f'\|_{p,\mathbb{T}} \le 1\}.$$

Since each function $g \in BW^{1,p}(I)$ can be express as $g(x) = \int_a^x f(t)dt + c$, where c is a constant then we have

$$BW^{1,p}(I) = TBL^p(I) + const., \tag{1}$$

and also we can see that

$$BW^{1,p}(\mathbb{T}) = BW_0^{1,p}([-\pi, \pi]) + const., \tag{2}$$

$$BW^{1,p}(I) = BW_0^{1,p}(I) + const. + x * const., \tag{3}$$

$$BW_a^{1,p}(I) = TBL^p(I), \tag{4}$$

$$BW_{mid}^{1,p}(I) = T_c BL^p(I) \qquad \text{where } c = (b + a)/2, \tag{5}$$

here "=" is used as equation for sets.

For $1 < p < \infty$ we shall consider in this paper the following Sobolev embeddings

$$E_0 : W_0^{1,p}(I) \to L^p(I),$$
$$E_a : W_a^{1,p}(I) \to L^p(I),$$
$$E_{mid} : W_{mid}^{1,p}(I) \to L^p(I),$$

and also these unbounded Sobolev embeddings:

$$E_1 : W^{1,p}(I) \to L^p(I),$$
$$E_2 : W^{1,p}(\mathbb{T}) \to L^p(\mathbb{T}),$$

and their variations:

$$E_3 : W^{1,p}(I)/\operatorname{span}\{1\} \to L^p(I)/\operatorname{span}\{1\},$$
$$E_4 : W^{1,p}(\mathbb{T})/\operatorname{span}\{1\} \to L^p(\mathbb{T})/\operatorname{span}\{1\}.$$

By $W^{1,p}(I)/\operatorname{span}\{1\}$ we mean the factorization of the space $W^{1,p}(I)$ with respect to constant functions equipped with norm $\|f'\|_p$. Then we have $f \in W^{1,p}(I)/\operatorname{span}\{1\}$ if and only if $\|f\|_{p,I} = \inf_{c\in\mathbf{R}} \|f-c\|_{p,I}$ and $\|f'\|_{p,I} < \infty$. In a similar way we define $L^p(I)/\operatorname{span}\{1\}$, $W^{1,p}(\mathbb{T})/\operatorname{span}\{1\}$ and $L^p(\mathbb{T})/\operatorname{span}\{1\}$.

The norms of E_0, E_a and E_{mid} are defined by

$$\|E_0\| = \sup_{\|f'\|_{p,I}>0, f(a)=f(b)=0} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}},$$

$$\|E_a\| = \sup_{\|f'\|_{p,I}>0, f(a)=0} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}},$$

$$\|E_{mid}\| = \sup_{\|f'\|_{p,I}>0, f((b+a)/2)=0} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}}.$$

For unbounded E_1, E_2 we have

$$\|E_1\| = \sup_{\|f'\|_{p,I}\neq 0} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}} = \infty,$$

$$\|E_2\| = \sup_{\|f'\|_{p,\mathbb{T}}\neq 0} \frac{\|f\|_{p,\mathbb{T}}}{\|f'\|_{p,\mathbb{T}}} = \infty,$$

and next we shall study approximation of these unbounded embeddings by linear maps and by linear subspaces.

The norms of E_3 and E_4 are defined by:

$$\|E_3\| = \sup_{f\in W^{1,p}(I)/\operatorname{span}\{1\}} \frac{\|f\|_{p,I}}{\|f'\|_{p,I}},$$

$$\|E_4\| = \sup_{f\in W^{1,p}(\mathbb{T})/\operatorname{span}\{1\}} \frac{\|f\|_{p,\mathbb{T}}}{\|f'\|_{p,\mathbb{T}}}.$$

Since $|I| < \infty$ it is well-known that all these embeddings are compact (see for example, [EE], Theorem V.4.18).

1.2 Definitions of widths and approximation numbers

Let we recall the definitions of n-widths and the approximation numbers (for more see [PI], [EE], [ET] and [T]).

Definition 1.1 *Let X be a normed linear space with norm $\|.\|_X$ and let A be a central symmetric set in X.*

The linear n-width of A with respect to X is given by

$$\delta_n(A,X) = \inf_{P_n} \sup_{x\in A} \|x - P_n(x)\|_X$$

where the infimum is taken over all continuous linear operators P_n of X into X of rank n (i.e. range of P_n is of dimension n). Continuous linear operator P_n of rank at most n, for which $\delta_n(A,X) = \sup_{x\in A} \|x - P_n(x)\|_X$ is called an optimal *linear operator for $\delta_n(A,X)$.*

The Kolmogorov n-width of A with respect to X is given by

$$d_n(A,X) = \inf_{X_n} \sup_{x\in A} \inf_{y\in X_n} \|x-y\|_X$$

where the infimum is taken over all n-dimensional subspaces X_n of X. Subspace X_n of X with dimension at most n, for which $d_n(A,X) = \sup_{x\in A} \inf_{y\in X_n} \|x - y\|_X$ is called an optimal *subspace for $d_n(A,X)$.*

The Gel'fand n-width of A with respect to X is given by

$$d^n(A,X) = \inf_{L_n} \sup_{x\in A\cap L_n} \|x\|$$

where the infimum is taken over all closed subspaces L_n of X of codimension at most n. Subspace L_n of X with codimension at most n, for which $d^n(A,X) = \sup\{\|x\|; x \in A\cap L_n\}$ is called an optimal subspace for $d^n(A,X)$.

The Bernstein n-width of A with respect to X is defined by

$$b_n(A,X) = \sup_{X_{n+1}} \sup\{\lambda \geq 0; X_{n+1} \cap (\lambda\ BX) \subset A\}$$

where BX is the unit ball of X and the outer supremum is taken over all subspaces $X_{n+1} \subset X$ such that $\dim X_{n+1} = n+1$. Subspace $X_{n+1} \subset X$ with dimension $n+1$ for which $X_{n+1} \cap (b_n(A,X)\ BX) \subset A$ is called an optimal subspace for $b_n(A,X)$.

Definition 1.2 *Let X and Y be normed linear spaces with norms $\|.\|_X$ and $\|.\|_Y$ and T be a linear operator (possibly unbounded) from X to Y. The n^{th}-approximation number of T is defined*

$$a_n(T) = \inf \sup_{\|f\|_X\leq 1} \|Tf - Pf\|_Y$$

where the infimum is taken over all bounded linear maps $P : X \to Y$ with rank less that n. Bounded linear operator P from X to Y of rank at most $n-1$ for which $a_n(T) = \sup_{\|f\|_X\leq 1} \|Tf - Pf\|_Y$ is called optimal *linear operator for $a_n(T)$.*

In this definition of approximation numbers we allow an unbounded linear operator, this helps us to deal with unbounded embeddings E_1 and E_2. For more about approximation numbers see [EE].

Let we state the relation between n-widths and the approximation numbers

Observation 1.3 *Let Y, X be linear spaces such that $Y \subset X$ and let $BY = \{y; y \in Y, \|y\|_Y \leq 1\}$ and $T : Y \to X$ be a linear operator (possibly unbounded), then*

$$a_{n+1}(T) = \delta_n(TBY, X) \geq d_n(TBY, X), d^n(TBY, X) \geq b_n(TBY, X)$$

where TBY is the image of unit ball in Y under T.

Proof: For proof see [PI]. □

1.3 Known results

Let we recall some known results about extremal problems for the Hardy operator and n-width and the approximation numbers. From papers [Le], [S], [BS] or [EL] we have the following lemma about extremal problem for the Hardy operator.

Lemma 1.4 *Let $Tf(x) = \int_0^x f(t)dt$ be the Volterra operator and let $T : L^p(0,l) \to L^p(0,l)$ where $p \geq 1$ and $0 < l < \infty$. Then*

$$\|T\| := \sup_{\|f\|_{p,(0,l)}>0} \frac{\|Tf\|_{p,(0,l)}}{\|f\|_{p,(0,l)}} = 2C(p)l$$

where $C(p) = \frac{1}{2}\frac{p'^{1/p}p^{1/p'}}{B(1/p',1/p)} = \frac{1}{2}p'^{1/p}p^{1/p'}\frac{\sin(\pi/p)}{\pi}$, where p' is the dual exponent of p and B is the classical beta function. Extremals are all non-zero multiples of $\cos_p(\frac{\pi_p x}{l})$, where $\pi_p = B(1/p, 1/p') = \frac{\pi}{\sin(\pi/p)}$ and $T(\cos_p)(x) = \sin_p(x)$.

(Historical remark: As far as we can find, this lemma was the first time proved by V.I. Levin [Le] and then in more general form by E. Schmidt [S]. Recently the lemma was independently proved again in [BS] and in [EL].)

In this lemma $\cos_p$ and $\sin_p$ are the p-goniometric functions which are generalizations of the usual sin and cos functions (see [Li2] and [DM] for more). $\sin_p(.)$ is defined as the unique (global) solution to the initial–value problem

$$(|u'|^{p-2}u')' + \frac{2^p}{p'p^{p-1}}|u|^{p-2}u = 0$$

$$u(0) = 0, \qquad u'(0) = 1.$$

Or can be defined by using the following inverse functions.

For $s \in [0, p/2]$ we have

$$\mathrm{arcsin}_p(s) = \frac{p}{2}\int_0^{\frac{2s}{p}} \frac{dt}{(1-t^p)^{1/p}},$$

(note that this integral converges for all $s \in [0, p/2]$).

Note that as $\arcsin_p : [0, p/2] \to [0, \pi_p/2]$ is strictly increasing then its inverse function $\sin_p : [0, \pi_p/2] \to [0, p/2]$ is also strictly increasing.

We extended $\sin_p$ from $[0, \pi_p/2]$ to $\mathbf{R}$ as a $2\pi_p$ periodic function by the usual way as in the $p = 2$ case.

We define $\cos_p$ as $\cos_p(t) := \frac{d}{dt}\sin_p(t)$.

And then we have that

$$\left(\frac{p}{2}\right)^p |\cos_p(t)|^p + |\sin_p(t)|^p = \left(\frac{p}{2}\right)^p \text{ for all } t \in \mathbf{R},$$

and

$$\pi_p = \pi_{p'}.$$

We note that in this paper we are using the definition of π_p, $\sin_p$ and $\cos_p$ functions from the paper [DM] which is slightly different from the definition of π_p and the $\sin_p$ function used in [Li1] and in [Li2].

We can learn more about these functions from historical paper [Lu] or from more recent paper [Li2] or [DM].

$\sin_p$ function plays important rules also in characterization of eigenvalues for p-Laplacian eigenvalue problem as we shall see.

Let us recall the definition of the p-Laplacian eigenvalue problem with Dirichlet and Neumann boundary conditions and results about characterization of their eigenfunctions and eigenvalues.

Definition 1.5 *For $1 < p < \infty$, $\lambda > 0$ and $T > 0$ we define the p-Laplacian eigenvalue problem by this equation:*

$$(|u'|^{p-2}u')' + \lambda |u|^{p-2}u = 0, \text{ on } (0, T), \tag{6}$$

with Neumann boundary condition:

$$u'(0) = 0, \qquad u'(T) = 0, \tag{7}$$

or with Dirichlet boundary condition:

$$u(0) = 0, \qquad u(T) = 0. \tag{8}$$

The set of eigenvalues for p-Laplacian eigenvalue problem with Neumann boundary condition is given by $\lambda_0(T) = \lambda_0 = 0$ and

$$\lambda_n(T) = \lambda_n = \left(\frac{2n\pi_p}{T}\right)^p \frac{1}{p' p^{p-1}} \quad \text{for each } n \in \mathbf{N}.$$

The corresponding eigenfunctions are $u_0(t) = c, \quad c \in \mathbf{R}$ and

$$u_n(t) = \frac{T}{n\pi_p} \sin_p\left(\frac{n\pi_p}{T}\left(t - \frac{T}{2n}\right)\right) \quad \text{for each } n \in \mathbf{N}.$$

The set of eigenvalues for p-Laplacian eigenvalue problem with Dirichlet boundary condition is the same as it is for the problem with Neumann boundary condition:

$$\lambda_n(T) = \lambda_n = \left(\frac{2n\pi_p}{T}\right)^p \frac{1}{p' p^{p-1}} \qquad \text{for each } n \in \mathbf{N}.$$

And the corresponding eigenfunctions for for p-Laplacian eigenvalue problem with Neumann boundary condition are:

$$v_n(t) = \frac{T}{n\pi_p} \sin_p\left(\frac{n\pi_p}{T}t\right) \qquad \text{for each } n \in \mathbf{N}.$$

(for more see [Li1] and [DM])

The following lemma and remark will play important role in next.

Lemma 1.6 *Suppose the same condition as in Lemma 1.4 then*

$$\sup_{\|f\|_{p,(0,l)}>0} \inf_{c\in\mathbf{R}} \frac{\|Tf - c\|_{p,(0,l)}}{\|f\|_{p,(0,l)}} = C(p)l = \sup_{\|f\|_{p,(0,l)}>0} \frac{\|Tf - T_d f\|_{p,(0,l)}}{\|f\|_{p,(0,l)}}$$

where $C(p)$ *is as in Lemma 1.4,* $d = l/2$ *and supremum is reached for* $f(x) = \cos_p\left(\frac{\pi_p}{l}(x - \frac{l}{2})\right)$ *with* $c = \sin_p\left(\frac{\pi_p}{2}\right)$.

Proof: The first identity can be obtained from Lemma 1.4, see [BS] and [EL] for the exact proof. The proof of the second identity can be found in [EHL]. □

The following remark is obvious consequence of Lemma 1.6.

Remark 1.7 *Let* $I = [a,b]$*,* $-\infty < a < b < \infty$ *and* $1 < p < \infty$ *then for* $\phi(x) = \sin\left(\frac{x-a}{b-a}\right)$ *we have*

$$\frac{\|\phi\|_{p,I}}{\|\phi'\|_{p,I}} = C(p)|I|,$$

where $C(p)$ *is as in Lemma 1.4.*

In paper [TB] we have the following lemma about exact value of the Kolmogorov n-width,

Lemma 1.8 *Let* $1 \leq p < \infty$*, then*

$$d_n(BW^{1,p}(0,1), L^p(0,1)) = C(p)\frac{1}{n},$$

where $C(p)$ *is as in Lemma 1.4.*

Let us introduce different partitions of the interval $[a, b]$ into subintervals for our next use.

$$\begin{aligned}
I(n) &= \left\{ I_i; I_0 = \left[a, a + \frac{b-a}{2n}\right], I_n = \left[b - \frac{b-a}{2n}, b\right], \right. \qquad (9) \\
&\qquad \left. I_i = \left[a + \frac{(b-a)}{2n}(2i-1), a + \frac{(b-a)}{2n}(2i+1)\right] \text{ for } 0 < i < n \right\} \\
J(n) &= \left\{ J_i; J_0 = \left[a, a + \frac{b-a}{2n+1}\right], \right. \\
&\qquad \left. J_i = \left[a + \frac{(2i-1)(b-a)}{2n+1}, a + \frac{(2i+1)(b-a)}{2n+1}\right] \text{ for } 0 < i \leq n \right\} \\
S(n) &= \left\{ S_i; S_i = \left[a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}\right] \text{ for } 1 \leq i \leq n \right\} \\
K(n) &= \left\{ K_i; K_n = \left[b - \frac{b-a}{2n+1}, b\right], \right. \\
&\qquad \left. K_i = \left[a + \frac{2i(b-a)}{2n+1}, a + \frac{2(i+1)(b-a)}{2n+1}\right] \text{ for } 0 \leq i \leq n-1 \right\}
\end{aligned}$$

From the main theorem of [EL] or from combination of papers [EHS] and [BS] we have the following lemma about precise behaviour of approximation numbers for embeddings E_0, E_1 and E_3.

Lemma 1.9 *Let $I = [a, b]$ be an bounded interval, $n \in \mathbf{N}$.*

(i) Then

$$a_n(E_0) = \frac{|I|}{n} \, . \, C(p) = \frac{|I|}{n} \, . \, p'^{1/p} p^{1/p'} \, . \, \frac{\sin(\pi/p)}{2\pi} = \frac{1}{\lambda_n^{1/p}}.$$

Moreover, the bounded linear operator

$$P_0 f(x) = \sum_{i=1}^{n-1} f(c_i)\chi_{I_i}(x) + 0\chi_{I_0 \cup I_n}(x), \qquad (10)$$

where $\{I_i\}_{i=0}^n = I(n)$ (see (9)) and c_i is the middle point of I_i, is the optimal *linear operator for $a_n(E_0)$.*

(ii) Then

$$a_{n+1}(E_1) = \frac{|I|}{n} \, . \, C(p) = \frac{|I|}{n} \, . \, p'^{1/p} p^{1/p'} \, . \, \frac{\sin(\pi/p)}{2\pi} = \frac{1}{\lambda_n^{1/p}}.$$

Moreover, the bounded linear operator

$$P_1 f(x) = \sum_{i=1}^{n} f(d_i)\chi_{S_i}(x), \qquad (11)$$

where $\{S_i\}_{i=1}^n = S(n)$ (see (9)) and d_i is the middle point of S_i, is the optimal *linear operator for $a_{n+1}(E_1)$.*

(iii) Then

$$a_n(E_3) = \frac{|I|}{n} \ . \ C(p) = \frac{|I|}{n} \ . \ {p'}^{1/p} p^{1/p'} \ . \ \frac{\sin(\pi/p)}{2\pi} = \frac{1}{\lambda_n^{1/p}}.$$

Moreover, the bounded linear operator

$$P_1 f(x) = \sum_{i=1}^{n} f(d_i) \chi_{S_i}(x), \tag{12}$$

where $\{S_i\}_{i=1}^n = S(n)$ *(see (9)) and* d_i *is the middle point of* S_i*, is the* optimal *linear operator for* $a_n(E_3)$*.*

Here λ_n *corresponds to* n*-th eigenvalue of* p*-Laplacian problem (6) on interval* I *and* $C(p)$ *is as in Lemma 1.4.*

In [BMN] we can find the following result about partial characterization of the Bernstein n-widths for E_2 (see the Main Theorem in [BMN] for $p = q$).

Lemma 1.10 *Let* $n \in \mathbf{N}$ *and* $1 < p < \infty$*. Then*

$$b_{2n-1}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = \frac{\nu(p)}{n}$$

where $\nu(p) = \left(\frac{2}{\pi}\right) \lambda(p)$ *and* $\lambda(p)$ *is the value of the extremal problem*

$$\|x(.)\|_{p,(0,1)} \to \sup \qquad x(.) \in BW^{1,p}([0,1]) \qquad x(\frac{1}{2}) = 0.$$

(From Lemma 1.6 or from [S] we have that $\lambda(p) = C(p)$, where $C(p)$ is as in Lemma 1.4.)

And in [DJ] we can find the next statement about partial characterization of the Kolmogorov, Gel'fand and Linear n-widths for E_2 (see Theorem 4.1 in [DJ] with $Q(x) = x$ and with corresponding $G(x) = \chi_{\mathbf{R}_-}(x)$).

Lemma 1.11 *Let* $n \in \mathbf{N}$*. Then for* $1 < p < \infty$

$$d_{2n}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = d^{2n}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) =$$

$$= \delta_{2n}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = \lambda_n(p, G) \le b_{2n-1}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})).$$

Where $G(x) = \chi_{\mathbf{R}_-}(x)$ *and* $\lambda_n(p, G) := \sup\{\|G * h\|_{p,\mathbb{T}}; h \in D_{n,p}\}$*, where* $D_{n,p}$ *is the class of functions* $h(x)$ *such that* $\|h\|_{p,\mathbb{T}} \le 1$ *and*

$$h(x + \frac{\pi}{n}) = -h(x), \quad \textit{for } x \in \mathbb{T}$$

$$h(x) \ge 0, \quad \textit{for } x \in \left[-\pi, \frac{\pi}{n} - \pi\right).$$

2 Preparation.

2.1 Approximation numbers

In this section we shall extend results from [EL] and [EHS] for the Approximation numbers. At first we shall look at approximation numbers (or linear widths) of the Hardy operator and the Sobolev embedding E_a.

Lemma 2.1 *Let $n \in \mathbf{N}$ and $I = [a, b]$ be an bounded interval. Then*

$$a_{n+1}(T_a) = a_{n+1}(E_a) = C(p)\frac{|I|}{n + 1/2},$$

where $C(p)$ is as in Lemma 1.4. Moreover, bounded linear operators

$$P_T f(x) = \sum_{i=1}^{n} (\int_a^{e_i} f(t)dt)\chi_{J_i}(x) + 0\chi_{J_0}(x), \tag{13}$$

or

$$P_a f(x) = \sum_{i=1}^{n} f(e_i)\chi_{J_i}(x) + 0\chi_{J_0}(x), \tag{14}$$

where $\{J_i\}_0^n = J(n)$ is a partition of I (see (9)) and e_i is the middle point of J_i, are optimal linear operators for the Hardy operator T_a or respectively for the Sobolev embedding E_a.

Proof: In this proof we shall write T instead T_a. From the definition of T and $W^{1,p}_a(I)$ we can see that $a_{n+1}(T) = a_{n+1}(E_a)$ (see (4)), then to finish the proof is enough to study $a_n(T)$.

Let us mention that rank of P_T is equal to n. Take $J(n) = \{J_i\}_{i=0}^n$ partition of interval I (see (9)) and denote $[a_i, b_i] = J_i$, then $|[a_i, e_i]| = |[e_i, b_i]| = \frac{b-a}{2n+1}$ for $i > 0$. Let $f \in L^p(I)$ by Lemma 1.4 we have:

$$\|Tf(.)\|_{p,(a_0,b_0)} \le \left(\frac{b-a}{2n+1}\right) 2C(p)\|f\|_{p,(a_0,b_0)},$$

$$\|Tf(.) - Tf(e_i)\|_{p,(a_i,e_i)} \le \left(\frac{b-a}{2n+1}\right) 2C(p)\|f\|_{p,(a_i,e_i)}$$

and

$$\|Tf(.) - Tf(e_i)\|_{p,(e_i,b_i)} \le \left(\frac{b-a}{2n+1}\right) 2C(p)\|f\|_{p,(e_i,b_i)}$$

for $0 < i \leq n$. From this we obtain

$$\begin{aligned}
\|Tf &- P_T f\|_{p,I}^p = \sum_{i=1}^n \|f - (P_1 f)(e_i)\|_{p,J_i}^p \\
&= \sum_{i=1}^n \Big(\|Tf(.) - Tf(e_i)\|_{p,(a_i,e_i)}^p + \|Tf(.) - Tf(e_i)\|_{p,(e_i,b_i)}^p \Big) \\
&\qquad + \|Tf\|_{p,(a_0,b_0)}^p \\
&\leq \left[\left(\frac{(b-a)}{2n+1} \right) 2C(p) \right]^p \left[\sum_{i=1}^n \Big(\|f\|_{p,(a_i,e_i)}^p + \|f\|_{p,(e_i,b_i)}^p \Big) + \|f\|_{p,(a_0,b_0)}^p \right] \\
&\leq \left[\left(\frac{(b-a)}{2n+1} \right) 2C(p) \right]^p \|f\|_{p,I}^p.
\end{aligned}$$

Then $a_{n+1}(T) \leq \sup_{f \in L^p(I)} (\|Tf - P_T f\|_{p,I} / \|f\|_{p,I}) \leq \frac{b-a}{n+1/2} C(p)$.

Let us prove the other inequality for a_{n+1}. Let us have a partition of I, $\{J_i\}_{i=0}^n = J(n)$ where $J_i = [a_i, b_i]$ and $b_i - a_i = (b-a)/(n+1/2)$ for $1 < i \leq n$ and $b_0 - a_0 = (b-a)/(2n+1)$. Take $0 < \gamma < 1$, then from Lemma 1.4 and Lemma 1.6 there exist functions $\phi_i(x)$, only non-zero on J_i, such that:

$$\frac{\inf_{c \in \mathbf{R}} \|T\phi_i - c\|_{p,J_i}}{\|\phi_i\|_{p,J_i}} \geq \gamma C(p)|J_i| \text{ for } 1 < i \leq n$$

and

$$\frac{\|T\phi_0\|_{p,J_0}}{\|\phi_0\|_{p,J_0}} \geq \gamma 2C(p)|J_0|.$$

Let P_n be a bounded linear operator mapping $L^p(I)$ onto with rank equal to n, then there exist constants $\{\lambda_i\}_{i=0}^n$, not all equal zero, such that for $g = \sum_{i=0}^n \lambda_i \phi_i$ we have $P_n g = 0$. Then we have:

$$\begin{aligned}
\|Tg &- P_n g\|_{p,I}^p = \|Tg\|_{p,I}^p \\
&= \sum_{i=0}^n \|Tg\|_{p,J_i}^p = \|\int_a^\cdot \lambda_0 \phi_0\|_{p,J_0}^p + \sum_{i=1}^n \|\int_{a_i}^\cdot \lambda_i \phi_i(t)dt + \int_a^{a_i} g(t)dt\|_{p,J_i}^p \\
&\geq \|\int_a^\cdot \lambda_0 \phi_0\|_{p,J_0}^p + \sum_{i=1}^n \inf_{c \in \mathbf{R}} \|\int_{a_i}^\cdot \lambda_i \phi_i(t)dt - c\|_{p,J_i}^p \\
&\geq \gamma^p 2^p C(p)^p \left(\frac{|I|}{2n+1} \right)^p |\lambda_0|^p \|\phi_0\|_{p,J_0}^p \\
&\qquad + \sum_{i=1}^n \gamma^p C(p)^p \left(\frac{|I|}{n+1/2} \right)^p |\lambda_i|^p \|\phi_i\|_{p,J_i}^p \\
&= \gamma^p C(p)^p \left(\frac{|I|}{n+1/2} \right)^p \|g\|_{p,I}^p.
\end{aligned}$$

From this we obtain that $a_{n+1}(T) \geq \frac{b-a}{n+1/2} C(p)$. And by this we prove the theorem. □

Now we shall prove a version of the previous lemma for the Sobolev embedding E_{mid}.

Lemma 2.2 *Let n be an odd integer, $I = [a, b]$ be an bounded interval and let $c = (b + a)/2$. Then*

$$a_{n+1}(T_c) = a_{n+1}(E_{mid}) = a_n(T_c) = a_n(E_{mid}) = C(p)\frac{|I|}{n},$$

where $C(p)$ is as in Lemma 1.4. Moreover, for n odd, the bounded linear operator

$$P_{T_c} f(x) = \sum_{i=1; i\neq \frac{n+1}{2}}^{n} \left(\int_c^{d_i} f(t)dt \right) \chi_{S_i}(x) + 0\chi_{S_{(n+1)/2}}(x), \tag{15}$$

or

$$P_c f(x) = \sum_{i=1; i\neq \frac{n+1}{2}}^{n} f(d_i)\chi_{S_i}(x) + 0\chi_{S_{\frac{n+1}{2}}}(x), \tag{16}$$

where $\{S_i\}_1^n = S(n)$ is a partition of I (see (9)) and d_i is the middle point of S_i, are optimal linear operators for the Hardy operator T_c or respectively for the Sobolev embedding E_{mid} between all n and $n - 1$ dimensional linear operators.

Proof: From the definition of T_c and $W^{1,p}_{mid}(I)$ we have that $a_{n+1}(T_c) = a_{n+1}(E_{mid})$ (see 5). Then it is enough to study only $a_n(T_c)$.

Let n be odd. Take a partition of I, $S(n) = \{S_i\}_{i=1}^n$ (see (9)) and denote $[a_i, b_i] = S_i$ and $d_i = (a_i + b_i)/2$ (note that $|S_i| = \frac{|I|}{n}$) . Define

$$P_{T_c} f(x) = \sum_{i=1; i\neq \frac{n+1}{2}}^{n} \left(\int_c^{d_i} f(t)dt \right) \chi_{S_i}(x) + 0\chi_{S_{\left(\frac{n+1}{2}\right)}}(x).$$

Rank of P_{T_c} is equal $n - 1$. Let $f \in L^p(I)$ and by Lemma 1.4 we have for $i \neq (n + 1)/2$:

$$\| \int_{d_i}^{\cdot} f(t)dt\|_{p,(d_i,b_i)} \leq \left(\frac{b-a}{n}\right) C(p)\|f\|_{p,(d_i,b_i)},$$

$$\| \int_{d_i}^{\cdot} f(t)dt\|_{p,(a_i,d_i)} \leq \left(\frac{b-a}{n}\right) C(p)\|f\|_{p,(a_i,d_i)},$$

and for $i = (n + 1)/2$:

$$\|T_c f(.)\|_{p,(d_i,b_i)} \leq \left(\frac{b-a}{n}\right) C(p)\|f\|_{p,(d_i,b_i)},$$

$$\|T_c f(.)\|_{p,(a_i,d_i)} \leq \left(\frac{b-a}{n}\right) C(p)\|f\|_{p,(a_i,d_i)}.$$

From this we obtain as in the previous lemma:

$$\begin{aligned}
\|T_c f \quad - \quad P_{T_c} f\|^p_{p,I} &= \sum_{i=1}^{n} \| \int_{d_i}^{\cdot} f(t)dt\|^p_{p,S_i} \\
= \quad & \sum_{i=1; i\neq \frac{n+1}{2}}^{n} \left(\| \int_{d_i}^{\cdot} f(t)dt\|^p_{p,(a_i,d_i)} + \| \int_{d_i}^{\cdot} f(t)dt\|^p_{p,(d_i,b_i)} \right) \\
& + \|T_c f\|^p_{p,\left(a_{\frac{n+1}{2}}, b_{\frac{n+1}{2}}\right)} \\
\leq \quad & \left[\left(\frac{(b-a)}{n} \right) C(p) \right]^p \left[\sum_{i=1}^{n} \left(\|f\|^p_{p,(a_i,d_i)} + \|f\|^p_{p,(d_i,b_i)} \right) \right] \\
\leq \quad & \left[\left(\frac{(b-a)}{n} \right) C(p) \right]^p \|f\|^p_{p,I}.
\end{aligned}$$

Then for odd n we have $a_n(T_c) \leq C(p)\frac{|I|}{n}$.

Now we shall prove the other inequality for the approximation numbers. Let n be odd. Let us have a partition of I, $\{S_i\}_{i=1}^n = S(n)$ where $S_i = [a_i, b_i]$, $b_i - a_i = |I|/n$ and d_i is the middle point of S_i, for $1 \leq i \leq n$, and $d_{\frac{n+1}{2}} = c$. Take $0 < \gamma < 1$, then from Lemma 1.4 and Lemma 1.6 for each $i = 1, ..., n$; $i \neq (n+1)/2$, there exist functions $\phi_i(x) \in L^p(I)$, only non-zero on S_i, such that:

$$\frac{\inf_{\alpha\in\mathbf{R}} \|T_c\phi_i - \alpha\|_{p,S_i}}{\|\phi_i\|_{p,S_i}} \geq \gamma C(p)|S_i|$$

and there exist functions $\phi_-(x), \phi_+(x) \in L^p(I)$ non-zero on $(a_{\frac{n+1}{2}}, c)$ and on $(c, b_{\frac{n+1}{2}})$ respectively such that:

$$\frac{\|T_c\phi_-\|_{p,(a_{\frac{n+1}{2}},c)}}{\|\phi_-\|_{p,(a_{\frac{n+1}{2}},c)}} \geq \gamma C(p)|S_i|$$

and

$$\frac{\|T_c\phi_+\|_{p,(c,b_{\frac{n+1}{2}})}}{\|\phi_+\|_{p,(c,b_{\frac{n+1}{2}})}} \geq \gamma C(p)|S_i|.$$

Let P_n be a bounded linear operator mapping $L^p(I)$ onto with rank equal to n, then there exist constants, not all zero, $\{\lambda_i\}_{i\neq(n+1)/2}$ and λ_-, λ_+ such that for $g = \sum_{i\neq(n+1)/2} \lambda_i\phi_i + \lambda_-\phi_- + \lambda_+\phi_+$ we have $P_n g = 0$.

Then we have:

$$
\begin{aligned}
\|T_c g \quad - \quad P_n g\|_{p,I}^p &= \|T_c g\|_{p,I}^p \\
= \quad & \sum_{i=1;i\neq\frac{n+1}{2}}^{n} \|T_c g\|_{p,J_i}^p + \|T_c g\|_{p,(a_{\frac{n+1}{2}},c)}^p + \|T_c g\|_{p,(c,b_{\frac{n+1}{2}})}^p \\
= \quad & \sum_{i=1;i\neq\frac{n+1}{2}}^{n} \| \int_{a_i}^{\cdot} \lambda_i \phi_i(t)dt + \int_c^{a_i} g(t)dt\|_{p,J_i}^p \\
& +\| \int_c^{\cdot} \lambda_- \phi_- \|_{p,(a_{\frac{n+1}{2}},c)}^p + \| \int_c^{\cdot} \lambda_+ \phi_+ \|_{p,(c,b_{\frac{n+1}{2}})}^p \\
\geq \quad & \sum_{i=1;i\neq\frac{n+1}{2}}^{n} \inf_{\alpha\in\mathbf{R}} \| \int_{a_i}^{\cdot} \lambda_i \phi_i(t)dt + \alpha\|_{p,J_i}^p \\
& +\| \int_c^{\cdot} \lambda_- \phi_- \|_{p,(a_{\frac{n+1}{2}},c)}^p + \| \int_c^{\cdot} \lambda_+ \phi_+ \|_{p,(c,b_{\frac{n+1}{2}})}^p \\
\geq \quad & \sum_{i=1;i\neq\frac{n+1}{2}}^{n} \gamma^p C(p)^p \left(\frac{|I|}{n}\right)^p |\lambda_i|^p \|\phi_i\|_{p,S_i}^p \\
& +\gamma^p C(p)^p \left(\frac{|I|}{n}\right)^p |\lambda_-|^p \|\phi_-\|_{p,(a_{\frac{n+1}{2}},c)}^p \\
& +\gamma^p C(p)^p \left(\frac{|I|}{n}\right)^p |\lambda_+|^p \|\phi_+\|_{p,(c,b_{\frac{n+1}{2}})}^p \\
= \quad & \gamma^p C(p)^p \left(\frac{|I|}{n}\right)^p \|g\|_{p,I}^p .
\end{aligned}
$$

From this we obtain that for odd n we have $a_{n+1}(T_c) \geq \frac{|I|}{n} C(p)$. Then we have from monotonicity of a_n that for odd n: $C(p)\frac{|I|}{n} \leq a_{n+1}(T_c) \leq a_n(T_c) \leq C(p)\frac{|I|}{n}$ and by this theorem is proved. □

Next we shall focus our interest on the Approximation numbers for the Sobolev embeddings on $\mathbb{T}$.

Lemma 2.3 *Let n be an odd integer, then*

$$a_{n+1}(E_2) = C(p)\frac{2\pi}{n+1},$$

where $C(p)$ is as in Lemma 1.4. Moreover, for given odd n, the bounded linear operator

$$P_{\mathbb{T}} f(x) = \sum_{i=1}^{n} \frac{[f(a_i) + f(b_i)]}{2} \chi_{S_i}(x) + \left[\sum_{i=1}^{n} [f(a_i) + f(b_i)](-1)^i \frac{1}{2} \right] \chi_{S_n}(x), \quad (17)$$

where $\{S_i\}_1^{n+1} = S(n+1)$ be a partition of $I = [a,b] = \mathbb{T} = [-\pi, \pi]$ (see (9) with $S_i = [a_i, b_i]$ with $a_0 = b_n$, and $a_{i+1} = b_i$) is an optimal linear operator for the Sobolev embedding E_2 between all n dimensional linear operators with rank $\leq n$.

Proof: Let n be an odd integer. Take $\{S_i\}_1^{n+1} = S(n+1)$ as a partition of $[-\pi, \pi] = \mathbb{T} = I = [a, b]$ and denote $[a_i, b_i] = S_i$. Since $f \in W^{1,p}(\mathbb{T})$ then $f(a_1) = f(b_{n+1})$.

We can see that rank $P_{\mathbb{T}} = n$. Since n is odd then we have

$$\sum_{i=1}^{n} [f(a_i) + f(b_i)](-1)^{i+1} = f(a_{n+1}) + f(b_{n+1})$$

and we can rewrite $P_{\mathbb{T}}$ in the following form:

$$P_{\mathbb{T}} f(x) = \sum_{i=1}^{n+1} \frac{f(a_i) + f(b_i)}{2} \chi_{S_i}(x). \tag{18}$$

Let $f \in W^{1,p}(\mathbb{T})$ then

$$\|f - P_{\mathbb{T}} f\|_{p,\mathbb{T}}^p = \sum_{i=1}^{n+1} \|f - P_{\mathbb{T}} f\|_{p,S_i}^p = \sum_{i=1}^{n+1} \|f - \frac{f(a_i) + f(b_i)}{2}\|_{p,S_i}^p.$$

From [S] we have

$$\|f - \frac{f(a_i) + f(b_i)}{2}\|_{p,S_i}^p \leq (C(p))^p |I_i|^p \|f'\|_{p,S_i}^p \quad \text{for } 1 \leq i \leq n+1.$$

Then from (18) follows:

$$\|f - P_{\mathbb{T}} f\|_{p,\mathbb{T}}^p \leq (C(p))^p \left(\frac{2\pi}{n+1}\right)^p \|f'\|_{p,\mathbb{T}}^p$$

which means

$$a_{n+1}(E_2) \leq C(p) \frac{2\pi}{n+1}.$$

As in the first part of this proof n is an odd integer and we have partition $\{S_i\}_1^{n+1} = S(n+1)$ of $[-\pi, \pi]$. From Lemma 1.4 follows that for any fix $0 < \gamma < 1$ there exist functions ϕ_i, $i = 1, ..., n+1$ such that $\phi_i(a_i) = \phi_i(b_i) = 0$, $\phi_i(x) \geq 0$ for $x \in S_i$ and $\phi_i(x) = 0$ for $x \notin S_i$ and

$$\|\phi_i\|_{p,\mathbb{T}}^p \geq \left(\gamma C(p) \frac{2\pi}{n+1}\right)^p \|\phi_i'\|_{p,\mathbb{T}}^p.$$

Let $P : W^{1,p}(\mathbb{T}) \to L^p(\mathbb{T})$ be a linear operator of rank n. Then there exist $g = \sum_{i=1}^{n+1} \lambda_i \phi_i$ where λ_i are constants, not all equal to 0, such that $P(g) = 0$. Then:

$$\begin{aligned} \|g - Pg\|_{p,\mathbb{T}}^p &= \|\sum_{i=1}^{n+1} \lambda_i \phi_i\|_{p,\mathbb{T}}^p = \sum_{i=1}^{n+1} \|\lambda_i \phi_i\|_{p,S_i}^p \\ &\geq \sum_{i=1}^{n+1} \left(\gamma |\lambda_i| C(p) \frac{2\pi}{n+1}\right)^p \|\phi_i'\|_{p,S_i}^p \\ &= \left(\gamma C(p) \frac{2\pi}{n+1}\right)^p \sum_{i=1}^{n+1} \|\lambda_i \phi_i'\|_{p,S_i}^p = \left(\gamma C(p) \frac{2\pi}{n+1}\right)^p \|g'\|_{p,\mathbb{T}}^p \end{aligned}$$

which give us

$$C(p)\frac{2\pi}{n+1} \leq a_{n+1}(E_2).$$

□

Lemma 2.4 *Let n be an odd integer, then*

$$a_n(E_4) = C(p)\frac{2\pi}{n+1},$$

where $C(p)$ is as in Lemma 1.4. Moreover, for given odd n, the bounded linear operator $P_{\mathbb{T}} : W^{1,p}(\mathbb{T})/\operatorname{span}\{1\} \to L^p(\mathbb{T})/\operatorname{span}\{1\}$ defined by as in Lemma 2.3 is an optimal operator for the Sobolev embedding E_3, among all linear operators with rank $\leq n-1$.

Proof: Let n be an odd integer and $\{S_i\}_{i=1}^{n+1} = S(n+1)$ be a partition of $[-\pi,\pi] = \mathbb{T} = I = [a,b]$ as in the proof of Lemma 2.3. We can rewrite $P_{\mathbb{T}}$ operator from Lemma 2.3 in the following way:

$$\begin{aligned} P_{\mathbb{T}}f(x) &= \frac{f(a_1)+f(b_1)}{2}\chi_{\mathbb{T}}(x) \\ &+ \sum_{i=2}^{n}\left(\frac{[f(a_i)+f(b_i)]}{2} - \frac{[f(a_1)+f(b_1)]}{2}\right)\chi_{S_i}(x) \\ &+ \left(\left[\sum_{i=1}^{n}[f(a_i)+f(b_i)](-1)^i\frac{1}{2}\right] - \frac{f(a_1)+f(b_1)}{2}\right)\chi_{S_{n+1}}(x). \end{aligned}$$

From this we can see that rank of $P_{\mathbb{T}}$ as an linear operator from $W^{1,p}(\mathbb{T})/\operatorname{span}\{1\}$ into $L^p(\mathbb{T})/\operatorname{span}\{1\}$ is equal $n-1$. Let $f \in W^{1,p}(\mathbb{T})/\operatorname{span}\{1\}$ then

$$\inf_{c\in\mathbf{R}} \|f - P_{\mathbb{T}}f - c\|_{p,\mathbb{T}}^p \leq \|f - P_{\mathbb{T}}f\|_{p,\mathbb{T}}^p = \sum_{i=1}^{n+1} \|f - \frac{f(a_i)+f(b_i)}{2}\|_{p,S_i}^p.$$

From [S] and Lemma 1.6 we have for any $1 \leq i \leq n+1$:

$$\begin{aligned} \sup_{\|f\|_{W^{1,p}(S_i)}\leq 1} \|f &- \frac{f(a_i)+f(b_i)}{2}\|_{p,S_i}^p = \sup_{\|f\|_{W^{1,p}(S_i)}\leq 1} \inf_{c\in\mathbf{R}} \|f-c\|_{p,S_i}^p \\ &= \sup_{\|f\|_{W^{1,p}(S_i)}\leq 1} \inf_{c\in\mathbf{R}} \|f - \frac{f(a_i)+f(b_i)}{2} - c\|_{p,S_i}^p \\ &= \sup_{\|f\|_{W^{1,p}(S_i)}\leq 1} (C(p)|S_i|)^p \, \|f'\|_{p,S_i}^p \end{aligned}$$

and then

$$\|f - P_{\mathbb{T}}f\|_{L^p(\mathbb{T})/\operatorname{span}\{1\}} \leq C(p)\left(\frac{2\pi}{n+1}\right)\|f'\|_{p,\mathbb{T}}.$$

And we have

$$a_n(E_3) \le C(p)\frac{2\pi}{n+1}.$$

Now we shall prove the other inequality for approximation numbers. From Lemma 1.4 follows for any $1 > \gamma > 0$ existence of functions ϕ_i, $i = 1, ..., n+1$; such that $\phi_i(a_i) = \phi_i(b_i) = 0$, $\phi_i(x) \ge 0$ for $x \in S_i$ and $\phi_i(x) = 0$ for $x \notin S_i$ and

$$\|\phi_i\|_{p,\mathbb{T}}^p = \left(\gamma C(p)\frac{2\pi}{n+1}\right)^p \|\phi_i'\|_{p,\mathbb{T}}^p.$$

Let us define functions ψ_i, $i = 1, ..., n$; by $\psi_i(x) = \phi_i(x) + \alpha_i\phi_n(x)$ such that $\inf_{c\in\mathbf{R}} \|\psi_i(x) - c\|_{p,\mathbb{T}} = \|\psi_i(x)\|_{p,\mathbb{T}}$.

Let $P : W^{1,p}(\mathbb{T})/\operatorname{span}\{1\} \to L^p(\mathbb{T})/\operatorname{span}\{1\}$ be a linear operator of rank equal $n-1$. Then there exists $g(x) = \sum_{i=0}^{n-1} \lambda_i\psi_i(x) \in L^p(\mathbb{T})/\operatorname{span}\{1\}$ with not all λ_i equal to 0, such that $P(g) = 0$. Then

$$\begin{aligned}
\|g - Pg\|_{p,\mathbb{T}}^p &= \|\sum_{i=1}^n \lambda_i\psi_i(x)\|_{p,\mathbb{T}}^p \\
&= \sum_{i=1}^n \|\lambda_i\phi_i\|_{p,S_i}^p + \|\sum_{i=1}^n \lambda_i\alpha_i\phi_n(x)\|_{p,S_{n+1}}^p \\
&= \left[\gamma C(p)\frac{2\pi}{n+1}\right]^p \sum_{i=1}^n |\lambda_i|^p \|\phi_i'\|_{p,S_i}^p \\
&\quad + \left[\gamma C(p)\frac{2\pi}{n+1}\right]^p \|\sum_{i=1}^n \lambda_i\alpha_i\phi_i'(x)\|_{p,S_{n+1}}^p \\
&= \left[\gamma C(p)\frac{2\pi}{n+1}\right]^p \|g'\|_{p,\mathbb{T}}^p
\end{aligned}$$

which gives us

$$C(p)\frac{2\pi}{n+1} \le a_n(E_3).$$

□

2.2 n-widths

In this section we shall obtain lower bound for n-widths. We start with the Hardy operator and then with Sobolev spaces on intervals.

Lemma 2.5 *Let $n \in \mathbf{N}$, $I = [a, b]$ and $1 < p < \infty$ then*

$$b_n(T_a BL^p(I), L^p(I)) = b_n(BW_a^{1,p}(I), L^p(I)) \ge C(p)\frac{|I|}{n+1/2},$$

where $C(p)$ is as in Lemma 1.4.

Proof: The first equation follows from (4). Next we shall prove the right-hand inequality. Let $n \in \mathbf{N}$ and let us have for interval $I = [a,b]$ partition $K(n) = \{K_i\}_{i=0}^n$ with $[a_i, b_i] = K_i$ (see (9)). Define on K_i functions $\phi_i(x) = \sin_p(\frac{x-a_i}{b_i-a_i}\pi_p).\chi_{[a_i,b_i]}(x)$ for $0 \le i \le n-1$ and $\phi_n(x) = \sin_p(\frac{x-a_i}{2b_i-2a_i}\pi_p).\chi_{[a_n,b_n]}(x)$. Put

$$X_{n+1} = \operatorname{span}\{\phi_i, 0 \le i \le n\} \tag{19}$$

then we have $X_{n+1} \subset W_a^{1,p}(I)$.

Let $f \in X_{n+1}$ then $f(x) = \sum_{i=0}^n \lambda_i \phi_i(x)$. From Remark 1.7 we have

$$\|f\|_{p,I}^p = \sum_{i=0}^n |\lambda_i|^p \|\phi_i\|_{p,K_i}^p = \left(\frac{C(p)|I|}{n+1/2}\right)^p \sum_{i=0}^n |\lambda_i|^p \|\phi_i'\|_{p,K_i}^p =$$

$$= \left(\frac{C(p)|I|}{n+1/2}\right)^p \|f'\|_{p,I}^p.$$

Let $f \in X_{n+1} \subset W_a^{1,p}(I)$ and $\|f'\|_{p,I} \le 1$ then $\|f\|_{p,I} \le C(p)\frac{|I|}{n+1/2}$. From that we have

$$X_{n+1} \cap \left(\left(C(p)\frac{|I|}{n+1/2} \right) . BX \right) \subset BW_a^{1,p}$$

and then $b_n(BW_a^{1,p}(I), L^p(I)) \ge C(p)\frac{|I|}{n+1/2}$.□

Lemma 2.6 *Let n be an odd number, $I = [a,b]$, $c = \frac{a+b}{2}$ and $1 < p < \infty$ then*

$$b_n(T_c BL^p(I), L^p(I)) = b_n(BW_{mid}^{1,p}(I), L^p(I)) \ge C(p)\frac{|I|}{n},$$

where $C(p)$ is as in Lemma 1.4.

Proof: Let n be an odd integer and $\{I_i\}_{i=0}^n = I(n)$ be a partition of the interval I (see (9)) with $[a_i, b_i] = I_i$ (we can see that $b_{(n+1)/2} = a_{(n+1)/2+1} = c$). Define on I_i functions $\phi_i(x) = \sin_p\left(\frac{x-a_i}{b_i-a_i}\pi_p\right).\chi_{[a_i,b_i]}(x)$ for $1 \le i \le n-1$. For $i = 0$ we put $\phi_0(x) = \sin_p\left(\frac{x-2a_0+b_0}{2b_0-2a_0}\pi_p\right).\chi_{[a_0,b_0]}(x)$ and for $i = n$ we put $\phi_n(x) = \sin_p\left(\frac{x-a_n}{2b_n-2a_n}\pi_p\right).\chi_{[a_n,b_n]}(x)$. Put

$$X_{n+1} = \operatorname{span}\{\phi_i; 0 \le i \le n\} \subset W_{mid}^{1,p}(I). \tag{20}$$

Then for $f \in X_{n+1}$ we have $f(x) = \sum_{i=0}^n \lambda_i \psi_i(x)$ and from Remark 1.7 and Lemma 1.4 we have:

$$\|f\|_{p,I}^p = \sum_{i=0}^n |\lambda_i|^p \|\phi_i\|_{p,I_i}^p = |\lambda_0|^p \|\phi_0\|_{p,I_0}^p + |\lambda_n|^p \|\phi_n\|_{p,I_n}^p + \sum_{i=1}^{n-1} |\lambda_i|^p \|\phi_i\|_{p,I_i}^p =$$

$$= \left(\frac{C(p)|I|}{n}\right)^p \sum_{i=0}^n |\lambda_i|^p \|\phi_i'\|_{p,I_i}^p = \left(\frac{C(p)|I|}{n}\right)^p \|f'\|_{p,I}^p.$$

Then as in proof of Lemma 2.5 we have:

$$b_n(BW^{1,p}_{mid}(I), L^p(I)) = b_n(E_{mid}) \geq C(p)\frac{|I|}{n} \text{ for } n\text{-odd.}$$

□

Now we shall focus on Sobolev spaces $W^{1,p}_0(I)$ and $W^{1,p}(I)$.

Lemma 2.7 *Let $n \in \mathbf{N}$, $I = [a, b]$ and $1 < p < \infty$ then*

$$b_n(BW^{1,p}_0(I), L^p(I)) \geq \frac{|I|}{n+1}.C(p) = \frac{1}{\lambda^{1/p}_{n+1}},$$

where $C(p)$ is as in Lemma 1.4 and λ_n is n-th eigenvalue of p-Laplacian on I with Dirichlet or Neumann boundary condition.

Proof: Let n be an integer and $I = [a, b]$ with partition $S(n+1) = \{S_i\}^{n+1}_{i=1}$ (see (9)). Denote $S_i = [a_i, b_i]$ for $1 \leq i \leq n+1$ and then define functions $\phi_i(x) = \sin_p\left(\frac{x-a_i}{b_i-a_i}\pi_p\right)\ \chi_{[a_i,b_i]}(x)$. Let us denote

$$X_{n+1} = \text{span}\{\phi_i; 1 \leq i \leq n+1\} \tag{21}$$

then we have $X_{n+1} \subset W^{1,p}_0(I)$ and $\dim X_{n+1} = n+1$.

Let us have $f \in X_{n+1}$ then $f(x) = \sum^{n+1}_{i=1} \lambda_i\phi_i(x)$ where $\lambda_i \in \mathbf{R}$. According Remark 1.7 we have:

$$\|f\|^p_{p,I} = \sum^{n+1}_{i=1} |\lambda_i|^p \|\phi_i\|^p_{p,S_i} = \left(\frac{C(p)|I|}{n+1}\right)^p \sum^{n+1}_{i=1} |\lambda_i|^p \|\phi'_i\|^p_{p,S_i} = \left(\frac{C(p)|I|}{n+1}\right)^p \|f'\|^p_{p,I}.$$

From this and the definition of b_n follows that $b_n(BW^{1,p}_0(I), L^p(I)) \geq \frac{|I|}{n+1}.\frac{C(p)}{2}$.
□

Lemma 2.8 *Let $n \in \mathbf{N}$, $I = [a, b]$ and $1 < p < \infty$ then*

$$b_n(BW^{1,p}(I), L^p(I)) \geq \frac{|I|}{n}C(p) = \frac{1}{\lambda^{1/p}_n},$$

where $C(p)$ is as in Lemma 1.4 and λ_n is n-th eigenvalue of p-Laplacian on I with Dirichlet or Neumann boundary condition.

Proof: Let $n \geq 1$ be an integer and $I = [a, b]$ with partition $I(n) = \{I_i\}^n_{i=0}$ (see 9). Denote $I_i = [a_i, b_i]$ for $0 \leq i \leq n$ and define functions

$$\phi_i(x) = \sin_p\left(\frac{x-a_i}{b_i-a_i}\pi_p\right)\ \chi_{[a_i,b_i]}(x) \quad \text{for } 1 \leq i \leq n-1,$$

$$\phi_0(x) = \sin_p\left(\frac{x-2a_0+b_0}{2b_0-2a_0}\pi_p\right).\chi_{[a_0,b_0]}(x)$$

and

$$\phi_n(x) = \sin_p\left(\frac{x-a_n}{2b_n-2a_n}\pi_p\right).\chi_{[a_n,b_n]}(x).$$

Let us define

$$X_{n+1} = \operatorname{span}\{\phi_i; 0 \le i \le n\} \subset W^{1,p}(I). \tag{22}$$

Then for $f \in X_{n+1}$ we have $f(x) = \sum_{i=0}^{n} \lambda_i\phi_i(x)$,where $\lambda_i \in \mathbf{R}$. From Lemma 1.4 and Remark 1.7 we have:

$$\|f\|_{p,I}^p = \sum_{i=0}^{n} |\lambda_i|^p \|\phi_i\|_{p,I_i}^p = \left(\frac{C(p)|I|}{n}\right)^p \sum_{i=0}^{n} |\lambda_i|^p \|\phi_i'\|_{p,I_i}^p = \left(\frac{C(p)|I|}{n}\right)^p \|f'\|_{p,I}^p.$$

According the definition of b_n we have $b_n(BW^{1,p}(I), L^p(I)) \ge \frac{C(p)|I|}{n}$. □

Lemma 2.9 *Let $n \in \mathbf{N}$ and $1 < p < \infty$ then*

$$b_n(BW^{1,p}(I)/\operatorname{span}\{1\}, L^p(I)/\operatorname{span}\{1\}) \ge C(p)\frac{|I|}{n+1} = \frac{1}{\lambda_{n+1}^{1/p}},$$

where $C(p)$ is as in Lemma 1.4 and λ_n is n-th eigenvalue of p-Laplacian on I with Dirichlet or Neumann boundary condition.

Proof: Let $n \ge 1$ be an integer and $I = [a, b]$ with partition $I(n) = \{I_i\}_{i=0}^n$ (see (9)). Denote $I_i = [a_i, b_i]$ for $0 \le i \le n$ and define functions

$$\phi_i(x) = \sin_p\left(\frac{x-a_i}{b_i-a_i}\pi_p\right) \chi_{[a_i,b_i]}(x) \quad \text{for } 1 \le i \le n-1$$

$$\phi_0(x) = \sin_p\left(\frac{x-2a_0+b_0}{2b_0-2a_0}\pi_p\right) \chi_{[a_0,b_0]}(x)$$

and

$$\phi_n(x) = \sin_p\left(\frac{x-a_n}{2b_n-2a_n}\pi_p\right) \chi_{[a_n,b_n]}(x).$$

For $0 \le i \le n-1$ let us define functions $\psi_i(x) = \phi_i(x) + \beta_i\phi_n(x)$ with $\beta_i \in \mathbf{R}$ taken such that $\|\psi_i\|_{p,I} = \inf_{c\in\mathbf{R}} \|\psi_i - c\|_{p,I}$. Let us define

$$X_n = \operatorname{span}\{\psi_i; 0 \le i \le n-1\} \subset W^{1,p}(I)/\operatorname{span}\{1\}. \tag{23}$$

Then for $f \in X_n \subset W^{1,p}(I)/\operatorname{span}\{1\}$ we have $f(x) = \sum_{i=0}^{n-1} \lambda_i\psi_i(x)$,where $\lambda_i \in \mathbf{R}$. According Lemma 1.4 and Remark 1.7 we have:

$$\begin{aligned}
\|f\|_{p,I}^p &= \sum_{i=0}^{n-1} |\lambda_i|^p \|\psi_i\|_{p,I_i}^p = \sum_{i=0}^{n-1} |\lambda_i|^p \|\phi_i\|_{p,I_i}^p + \sum_{i=0}^{n-1} |\lambda_i|^p |\beta_i|^p \|\phi_n\|_{p,I}^p \\
&= \left(\frac{C(p)|I|}{n}\right)^p \left(\sum_{i=0}^{n-1} |\lambda_i|^p \|\phi_i'\|_{p,I_i}^p + \sum_{i=0}^{n-1} |\lambda_i|^p |\beta_i|^p \|\phi_n'\|_{p,I}^p\right) \\
&= \left(\frac{C(p)|I|}{n}\right)^p \|f'\|_{p,I}^p.
\end{aligned}$$

And then from the definition of b_n we have

$$b_{n-1}(BW^{1,p}(I)/\operatorname{span}\{1\}, L^p(I)/\operatorname{span}\{1\}) \geq C(p)\frac{|I|}{n}.$$

□

Next we shall focus on Sobolev spaces on $\mathbb{T}$.

Lemma 2.10 *Let n be an integer and $1 < p < \infty$ then*

$$b_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) \geq C(p)\frac{2\pi}{n+1},$$

where $C(p)$ is as in Lemma 1.4.

Proof: Let n be an integer and $\mathbb{T} = [-\pi, \pi]$ is an interval with partition $S(n+1) = \{S_i\}_{i=1}^{n+1}$ with $S_i = [a_i, b_i]$ (see (9)). Let us define functions $\phi_i(x) = \sin_p(\frac{x+\pi}{2\pi})\chi_{[a_i,b_i]}(x)$, $i = 1, ..., n+1$. We define

$$X_{n+1} = \operatorname{span}\{\phi_i; 1 \leq i \leq n+1\} \tag{24}$$

and then we have $X_{n+1} \subset W^{1,p}(\mathbb{T})$ and $\dim X_{n+1} = n+1$.

Let $f \in X_{n+1}$ then $f(x) = \sum_{i=1}^{n+1} \lambda_i \phi_i(x)$ where $\lambda_i \in \mathbf{R}$. Then from Remark 1.7 we have

$$\|f\|_{p,I}^p = \sum_{i=1}^{n+1} |\lambda_i|^p \|\phi_i\|_{p,S_i}^p = \left(\frac{C(p)2\pi}{n+1}\right)^p \sum_{i=1}^{n+1} |\lambda_i|^p \|\phi_i'\|_{p,S_i}^p = \left(\frac{C(p)2\pi}{n+1}\right)^p \|f'\|_{p,\mathbb{T}}^p.$$

From this follows the lemma. □

Lemma 2.11 *Let n be an integer and $1 < p < \infty$ then*

$$b_{n-1}(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) \geq C(p)\frac{2\pi}{n+1},$$

where $C(p)$ is as in Lemma 1.4.

Proof: Let n be an integer and $\mathbb{T} = [-\pi, \pi]$ is an interval with partition $S(n+1) = \{S_i\}_{i=1}^{n+1}$ with $S_i = [a_i, b_i]$ (see (9)). Let us define functions $\phi_i(x) = \sin_p(\frac{x+\pi}{2\pi})\chi_{[a_i,b_i]}(x)$, $i = 1, ..., n+1$. We introduce functions $\psi_i(x) = \phi_i(x) + \beta_i\phi_{n+1}(x)$ for $1 \leq i \leq n$, take β_i such that $\|\psi_i\|_{p,\mathbb{T}} = \inf_{c\in\mathbf{R}} \|\psi_i - c\|_{p,\mathbb{T}}$. Let us define

$$X_n = \operatorname{span}\{\psi_i; 1 \leq i \leq n\} \tag{25}$$

and then we can see that $X_n \subset W^{1,p}(\mathbb{T})$ and $\dim X_n = n$.

Let $f \in X_n$ then $f(x) = \sum_{i=1}^{n+1} \lambda_i \psi_i(x)$ where $\lambda_i \in \mathbf{R}$. Then from Remark 1.7 we have

$$\|f\|_{p,I}^p = \sum_{i=1}^{n} |\lambda_i|^p \|\psi_i\|_{p,S_i\cup S_{n+1}}^p = \left(\frac{C(p)2\pi}{n+1}\right)^p \sum_{i=1}^{n} |\lambda_i|^p \|\psi_i'\|_{p,S_i\cup S_{n+1}}^p =$$

$$= \left(\frac{C(p)2\pi}{n+1}\right)^p \|f'\|_{p,\mathbb{T}}^p.$$

From this follows the lemma. □

3 Main results

In this section we shall use results from previous sections to formulate our main results about approximation numbers and n-widths for our embeddings and Sobolev spaces.

At first we shall provide results for embeddings and Sobolev spaces on interval $I = [a, b]$ and then on unit circle $\mathbb{T}$.

Theorem 3.1 *Let $I = [a, b]$ be a bounded interval, $1 < p < \infty$ and let us denote by σ_n any of the following n-widths $d_n,\ d^n,\ \delta_n,\ b_n$, then we have:*

(1) *For every $n \in \mathbf{N}$:*

$$a_{n+1}(E_0) = \sigma_n(BW_0^{1,p}(I), L^p(I)) = C(p)\frac{|I|}{n+1} = \frac{1}{\lambda_{n+1}^{1/p}}$$

- *the operator P_0 (see (10)) is the optimal operator for a_n,*
- *the Range of the operator P_0, (see (10)) is the optimal subspace for d_{n-1},*
- *the space X_{n+1} (see (21)) is the optimal subspace for b_n.*

(2) *For every $n \in \mathbf{N}$:*

$$a_{n+1}(E_1) = \sigma_n(BW^{1,p}(I), L^p(I)) = C(p)\frac{|I|}{n} = \frac{1}{\lambda_n^{1/p}}.$$

- *the operator P_1 (see (11)) is the optimal operator for a_{n+1},*
- *the Range of the operator P_1, (see (11)) is the optimal subspace for d_n,*
- *the space X_{n+1} (see (22)) is the optimal subspace for b_n.*

(3) *For every $n \in \mathbf{N}$:*

$$a_{n+1}(E_3) = \sigma_n(BW^{1,p}(I)/\operatorname{span}\{1\}, L^p(I)/\operatorname{span}\{1\}) = C(p)\frac{|I|}{n+1} = \frac{1}{\lambda_{n+1}^{1/p}}.$$

- *the operator P_1 (see (12)) is the optimal operator for a_n,*
- *the Range of the operator P_1, (see (11)) is the optimal subspace for d_{n-1},*
- *the space X_{n+1} (see (23)) is the optimal subspace for b_n.*

(4) *For every $n \in \mathbf{N}$:*

$$a_{n+1}(E_a) = a_{n+1}(T_a) = \sigma_n(BW_a^{1,p}(I), L^p(I)) =$$

$$= \sigma_n(T_a BL^p(I), L^p(I)) = C(p)\frac{|I|}{n+1/2}.$$

- *the operator P_T (see (13)) is the optimal operator for $a_{n+1}(T_a)$ and the operator P_a (see (14)) is the optimal operator for $a_{n+1}(E_a)$,*
- *the Range of the operator P_T (see (13)) is the optimal subspace for $d_n(T_a BL^p(I), L^p(I))$ and the Range of the operator P_a (see (14)) is the optimal subspace for $d_n(BW_a^p(I), L^p(I))$,*
- *the space X_{n+1} (see (19)) is the optimal subspace for $b_n(BW_a^p(I), L^p(I))$ (see (4)).*

(5) *For n odd and $c = (a+b)/2$:*

$$a_{n+1}(E_{mid}) = a_{n+1}(T_c) = a_n(E_{mid}) = a_n(T_c) =$$

$$= \sigma_n(BW^{1,p}_{mid}(I), L^p(I)) = \sigma_n(T_c BL^p(I), L^p(I)) =$$

$$= \sigma_{n-1}(BW^{1,p}_{mid}(I), L^p(I)) = \sigma_{n-1}(T_c BL^p(I), L^p(I)) = C(p)\frac{|I|}{n}$$

- *the operator P_{T_c} (see (15)) is the optimal operator for $a_n(T_c)$ and $a_{n+1}(T_c)$ and the operator P_c (see (16)) is the optimal operator for $a_n(E_{mid})$ and $a_{n+1}(E_{mid})$*
- *the Range of the operator P_{T_c} (see (15)) is the optimal subspace for $d_n(T_c BL^p(I), L^p(I))$ and $d_{n-1}(T_c BL^p(I), L^p(I))$ and the Range of the operator P_c (see (16)) is the optimal subspace for $d_n(BW^p_{mid}(I), L^p(I))$ and $d_{n-1}(BW^p_{mid}(I), L^p(I))$,*
- *the space X_{n+1} (see (20)) is the optimal subspace for $b_n(BW^p_{mid}(I), L^p(I))$ and for $b_{n-1}(BW^p_{mid}(I), L^p(I))$ (see (5)).*

Proof: Part **(1)** follows from Lemma 1.9 (i), Lemma 2.7 and Observation 1.3.

Part **(2)** can be obtained from combination of Lemma 1.9 (ii), Lemma 2.8 and Observation 1.3.

Part **(3)** is consequence of Lemma 1.9 (iii), Lemma 2.9 and Observation 1.3.

Part **(4)** follows from Lemma 2.1, Lemma 2.5 and Observation 1.3.

Part **(5)** turn to be consequence of Lemma 2.2, Lemma 2.6 and Observation 1.3. □

Let us state our results on the unit circle $\mathbb{T}$.

Theorem 3.2 *Let $\mathbb{T}$ be the unit circle, $1 < p < \infty$ and let us denote by σ_n any of the following n-widths d_n, d^n, δ_n, b_n, then we have:*

(1) *For every $n \in \mathbf{N}$:*

$$\sigma_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) \geq C(p)\frac{2\pi}{n+1}$$

For every odd n:

$$a_{n+1}(E_2) = \sigma_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = C(p)\frac{2\pi}{n+1}$$

For every even n:

$$a_{n+1}(E_2) = \delta_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = d_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) =$$

$$= d^n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = C(p)\frac{2\pi}{n}$$

- *for n even the operator $P_{\mathbb{T}}$ (see (17)) is the optimal operator for $a_n(E_2)$ and when n odd then the operator $P_{\mathbb{T}}$ (see (18)) is the optimal operator for $a_n(E_2)$.*
- *for n even the range of the operator $P_{\mathbb{T}}$ (see (18)) is the optimal subspace for $d_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T}))$ and $d_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T}))$, for n odd the range of the operator $P_{\mathbb{T}}$ (see (17))*
- *for n even the space X_{n+1} (see 24) is the optimal subspace for $b_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})$.*

(2) *For every $n \in \mathbf{N}$:*

$$\sigma_{n-1}(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) \geq C(p)\frac{2\pi}{n+1}$$

For every even n:

$$a_{n+1}(E_4) = \sigma_n(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) = C(p)\frac{2\pi}{n+2}$$

For every odd n:

$$a_{n+1}(E_4) = \delta_n(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) =$$

$$= d_n(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) =$$

$$= d^n(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) = C(p)\frac{2\pi}{n+1}$$

Proof: *Proof of the part* **(1)** *of the Theorem:* The first inequality follows from Lemma 2.10. The first equation is consequence of Lemma 2.10, Lemma 2.3 and Observation 1.3.

For the second equation: From Lemma 1.11 we have that the right-hand side is equal to $\lambda_{n/2}(p, G)$ which is defined by: $\lambda_n(p, G) := \sup\{\|G*h\|_{p,\mathbb{T}} : h \in D_{n,p}\}$ where $D_{n,p}$ is the class of functions $h(x)$ such that $\|h\|_{p,\mathbb{T}} \leq 1$ and

$$h(x + \frac{\pi}{n}) = -h(x), \qquad \text{for } x \in \mathbb{T}$$

$$h(x) \geq 0, \qquad \text{for } x \in \left[-\pi, \frac{\pi}{n} - \pi\right),$$

and $G(x) = \chi_{\mathbf{R}_-}$. Since $G * f$ corresponds to the Hardy operator then from Lemma 1.4 follows that

$$\lambda_n(p, G) = C(p)\frac{\pi}{n}$$

and then the second equation is proved.

Proof of the part **(2)** *of the Theorem:* The first inequality follows from Lemma 2.11. The first equation is consequence of Lemma 2.11, Lemma 2.4 and Observation 1.3. The second equation can be obtained from the second equation of the part **(1)** via techniques and modifications used in Lemma 2.11 and Lemma 2.4. □

Remark 3.3 *We can see that for the full description of n-widths for periodic functions (i.e. for E_2 and E_4 embeddings) only information about exact values of b_n for even n in the case E_2 and odd n in the case E_4 are missing. We only have:*

$$\frac{C(p)}{n+1} \leq b_n(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) \leq \frac{C(p)}{n} \qquad \textit{for } n \textit{ even}$$

and

$$\frac{C(p)}{n+2} \leq b_n(BW^{1,p}(\mathbb{T})/\operatorname{span}\{1\}, L^p(\mathbb{T})/\operatorname{span}\{1\}) \leq \frac{C(p)}{n+1} \qquad \textit{for } n \textit{ odd.}$$

Only for $p = 2$ can be shown that b_n are equal to the upper bound.

The last note in our paper corresponds to relation of our results for the Sobolev embedding E_2 with Lemma 1.10 (i.e. with the Main theorem from [BMN]).

From Lemma 1.6 we can see that $\lambda(p)$ from Lemma 1.10 is equal to $C(p)$ from Lemma 1.4. Then Lemma 1.10 is giving us that

$$b_{2n-1}(BW^{1,p}(\mathbb{T}), L^p(\mathbb{T})) = \frac{2}{\pi}\frac{C(p)}{n},$$

which is in contradiction with our Theorem 3.2 and showing us that the Main theorem from [BMN] is not correct.

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