Estimates for the Approximation numbers and n-widths of the weighted Hardy-type operators

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Abstract

Let we have a weighted Hardy-type integral operator $T : L^p(a,b) \to L^p(a,b)$, $-\infty \leq a < b \leq \infty$, which is defined by

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)dt.$$

In papers Edmunds, Evans & Harris (J.London Math. Soc.,1988,(2) 37; Studia Math., 1994, 109 (1)) and Edmunds, Harris & Lang (Studia Math., 1998, 130 (2)) upper and lower estimates and asymptotic results were obtained for the approximation numbers $a_n(T)$ of T. In case $p = 2$ for "nice" u and v these results were improved in Edmunds, Kerman & Lang (J. Anal. Math., 2001, 85) and lately extended for $1 < p < \infty$ in Lang (J. Approx. Theory, 2003, 121 (1)). In this paper we will improved these results and obtain second asymptotic term and also extend these results for Kolmogorov, Genfald and Bernstein numbers.

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1 Introduction.

For the weighed Hardy operator T defined by

$$(Tf)(x) = v(x) \int_0^x u(t)f(t)dt, \tag{1}$$

and being a map from $L^p(a,b)$ into $L^p(a,b)$, for $1 \le p \le \infty$, properties of the approximation numbers were studied in (EEH1), (EEH2), citeLL and (EHL1). From papers (NS2), (NS1) and (EHL2) under some conditions on u and v was shoved that the approximation numbers $a_n(T)$ of T in the case $1 < p < \infty$ satisfy

$$\lim_{n\to\infty} n a_n(T) = \alpha_n \int_a^b |u(t)v(t)|dt,$$

where $\alpha_p = (1/\lambda_p)^{1/p}$ (λ_p corresponds to the first eigenvalue of the p-Laplacian problem on interval $(0,1)$ and $\lambda_p = \left(\frac{2\pi}{\sin(\pi/p)}\right) \frac{1}{p' p^{p-1}}$) (see (EL)) . From this follow

$$\frac{1}{C} \le \lim_{n\to\infty} \left(a_n(T) - \frac{\alpha_p}{n} \|uv\|_{1,(a,b)} \right) \le C, \qquad \text{for some } C > 0$$

Under slightly stricter conditions on weights u, v these results were improved in (EKL) (case $p = 2$) and latter in (L) (case $1 < p < \infty$). It was show that

$$\limsup_{n\to\infty} n^{1/2} \left| \alpha_p \int_a^b |u(t)v(t)|dt - n a_n(T) \right| \tag{2}$$
$$\le c(\|u'\|_{p'/(p'+1)} + \|v'\|_{p/(p+1)}) \left(\|u\|_{p'} + \|v\|_p\right) + 3\alpha_p \|uv\|_1$$

In this paper techniques from (L) are improved and by using information about properties of $A(I)$ (introduced in (EHL1)) we obtain information about first asymptotic for n-widths and also information about second asymptotic for the approximation numbers and n-widths numbers for the weighted Hardy operator. Mainly we proved:

$$\rho_n(T) = \frac{1}{n} \int_I u(x)v(x)dx + O(n^{-2})$$

where $\rho_n(T)$ stands for any of the followings: the Approximation numbers of T, Kolmogorv, Gel'fand or Bernstain n-widths of T.

Let we mentioned here that in the case $u = v = 1$ (i.e. non-weighted case) problem of description of approximation numbers and n-widths for the non-weighted Hardy operator and corresponding Sobolev embedding was already studied and described in (M), (BMN), (TB), (EL) and (L1).

Also we would like to put in the reader attention a recent elegant paper (B) in which similar results were obtained by using different techniques.

2 Asymptotic estimate for the Approximation and n-widths numbers.

Let we start by recalling the definitions of the Approximative numbers and n-widths.

Definition 2.1 *Let $T : L^p(I) \to L^p(I)$ be a bounded operator and $n \in \mathbf{N}$.*

(i) *The n-th approximation number $a_n(T)$ of T is defined by*

$$a_n(T) := \inf \|T - F|L^p(T) \to L^p(I)\|,$$

where the infimum is taken over all bounded linear maps $F : L^p(I) \to L^p(I)$ with rank less than n.

(ii) *The n-th Kolmogorov widths $d_n(T)$ of T is defined by*

$$d_n(T) = d_n(T(L^p(I)), L^p(I)) = \inf_{X_n} \sup_{\|x\|_{L^p(I)} \le 1} \inf_{y \in X_n} \|Tx - y\|_{L^p(I)}$$

where the infimum is taken over all n-dimensional subspaces X_n of $L^p(I)$.

(iii) *The n-th widths in the sense of Gel'fand $d^n(T)$ of T is defined by*

$$d^n(T) = d^n(T(L^p(I)), L^p(I)) = \inf_{L^n} \sup_{\|x\|_{L^p(I)} \le 1, x \in L^n} \|Tx\|_{L^p(I)}$$

where the infimum is taken over all n-dimensional subspaces X_n of $L^p(I)$.

(iv) *The Bernstein n-th widths $b_n(T)$ of T is defined by*

$$b_n(T) = b_n(T(L^p(I)), L^p(I)) = \sup_{X_{n+1}} \inf_{Tx \in X_{n+1}, Tx \neq 0} \|Tx\|_{L^p(I)} / \|x\|_{L^p(I)}$$

where X_{n+1} is any subspace of span $\{Tx; x \in X\}$ of dimension $\ge n+1$.

(v) *The linear n-th widths $\delta_n(T)$ of T is defined by*

$$\delta_n(T) = \delta_n(T(L^p(I)), L^p(I)) = \inf_{P_n} \sup_{\|x\|_{L^p(I)} \le 1} \|Tx - P_n x\|_{L^p(I)}$$

where P_n is any continuous linear operator of $L^p(I)$ into $L^p(I)$ of rank at most n.

The following lemma will give us information about relation between the approximation numbers and n-widths.

Lemma 2.2 *Let $T : L^p(I) \in L^p(I)$ be a bounded operator and $n \in \mathbf{N}$, then*

$$a_{n+1}(T) = \delta_n(T) \geq d_n(T), d^n(T) \geq b_n(T).$$

Proof. The first equality is obvious for the rest see (P) $\square$

Throughout the paper we shall assume that $-\infty \leq a < b \leq \infty$ and that

$$u \in L^{p'}(a,b), \quad v \in L^p(a,b) \quad \text{ and } u, v > 0 \text{ on } (a,b). \tag{3}$$

Under these restrictions on u and v it is well known (see, for example, (EEH1), Theorem 1) that the norm $\|T\|$ of the operator $T : L^p(a,b) \to L^p(a,b)$ in (1) satisfies

$$\|T\| \sim \sup_{x \in (a,b)} \|u\chi_{(a,x)}\|_{p',(a,b)} \|v\chi_{(x,b)}\|_{p,(a,b)}. \tag{4}$$

Here χ_S denotes the characteristic function of the set S and

$$\|f\|_{p,I} = \left(\int_I |f(t)|^p dt \right)^{1/p}, \quad 1 < p < \infty, \quad I \subset (a,b).$$

Moreover, by $F_1 \sim F_2$ we mean that $C^{-1}F_1 \leq F_2 \leq CF_1$ for some positive constant $C \geq 1$ independent of any variables in $F_1, F_2 \geq 0$.

From (3) also follows that the operator T is a compact operator from L^p into L^p (see (EGP) or (OK)).

In the next we introduce a function A which will play a key role in the paper. Given $I = (c,d) \subset (a,b)$, set

$$A(I,u,v) := \sup_{\|f\|_{p,I}=1} \inf_{\alpha \in \Re} \|Tf - \alpha v\|_{p,I}.$$

(we will write shortly $A(I)$ in situation in which is obvious which functions are u, v.) Since T is a compact operator then from (EHL2), Theorem 3.8. we have that

$$A(I,u,v) = \inf_{x\in I} \|T_{x,I}|L^p(I) \to L^p(I)\|,$$

where

$$T_{x,I}f(.) := v(.)\chi_I(.) \int_x^{.} v(t)\chi_I(t)dt.$$

Lemma 2.3 *Let $I = (c,d) \subset (a,b)$ and $1 \le p \le \infty$, then $\|T_{x,I}|L^p(I) \to L^p(I)\|$ is continuous in x.*

Proof. See Lemma 3.4 in (EHL2) and put $\Gamma = (a,b)$ and $K = I$. $\square$

Lemma 2.4 *Suppose that u and v satisfy (3), $a \le c < d \le b$ and $1 < p < \infty$. Then:*

1. The function $A(.,d)$ is non-increasing and continuous on (a,d).

2. The function $A(c,.)$ is non-decreasing and continuous on (c,b).

3. $\lim_{y\to a_+} A(a,y) = \lim_{y\to b_-} A(y,b) = 0$.

Proof. See Lemma 2.2 in (L) $\square$

From the previous two lemmas we can obtained the next lemma.

Lemma 2.5 *Suppose that $T : L^p(a,b) \to L^p(a,b)$ is compact and $1 < p < \infty$. Let $I = (c,d)$ and $J = (c',d')$ be subintervals of (a,b), with $J \subset I$, $|J| > 0$, $|I - J| > 0$, $\int_a^b v^p(x)dx < \infty$ and $u, v > 0$ on I. Then*

$$A(I) > A(J) > 0. \tag{5}$$

Proof. See Lemma 2.3 in (L). $\square$

Now we will introduce $N(\varepsilon)$ which play important rule in the next.

Remark 2.6 *It follows from the continuity of A that for sufficiently small $\varepsilon > 0$ there is an a_1, $a < a_1 < b$, for which $A(a_1,b) = \varepsilon$. Indeed, since T is compact, there exists a positive integer $N(\varepsilon)$ and points $b = a_0 > a_1 > \ldots > a_{N(\varepsilon)} = a$ with $A(a_i,a_{i-1}) = \varepsilon, i = 1,2,\ldots,N(\varepsilon)-1$ and $A(a,a_{N(\varepsilon)-1}) \le \varepsilon$.*

By the same arguments as in the proof of Lemma 2.6 from [EKL] we have as a consequence of the previous two lemmas the following lemma:

Lemma 2.7 *If $T : L^p(a,b) \to L^p(a,b)$ be compact and $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ then the number $N(\varepsilon)$ is a non-increasing function of ε which takes on every sufficiently large an integer value.*

The quantity $N(\varepsilon)$ is useful in the derivation of upper and lower estimates for the approximation numbers of T as we can see from the following lemma which is an easy consequence of Lemma 3.19 from (EHL2) (put $K = (a,b)$).

Lemma 2.8 *For all $\varepsilon \in (0, \|T\|)$,*

$$a_{N(\varepsilon)+2}(T) \le \varepsilon \le a_{N(\varepsilon)-1}(T).$$

In next we extend this lemma also for n-widths.

Lemma 2.9 *Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a,b)$. If $N := N(\epsilon)$ then*

$$a_{N+1}(T) \le \varepsilon.$$

Proof. At first let we recall that T is compact and then from Lemma 2.7 follows that for any $\varepsilon > 0$ we have $N(\varepsilon) < \infty$. Let $\{I_i\}_{i=1}^N$ be the partition of I which defines $N := N(\varepsilon)$ in Remark 2.6 and set $Pf = \sum_{i=1}^N P_i f$ where

$$P_i f(x) := \chi_{I_i} v(x) \left[\int_a^{a_i} fu + \left(\int_{a_i}^{c_i} fu\chi_{I_i} \right) \right],$$

and a_i is the left end point of I_i, c_i is the point from I_i such that $A(I_i) = \|T_{c_i,I_i} | L^p(I) \to L^p(I)\|$ (for existence such point see Lemma 3.14 in (EHL2)).

Then $\operatorname{rank}(P) \le N$ and we have

$$\begin{aligned}
\|(T-P)f\|_{p,I}^p &= \sum_{i=1}^N \|Tf - Pf\|_{p,I_i}^p \\
&= \sum_{i=1}^N \|\chi_{I_i} v(.) \int_a^{.} v(t)f(t)dt - P_i f(.)\|_{p,I_i}^p \\
&= \sum_{i=1}^N \|\chi_{I_i} v(.) \int_{c_i}^{.} v(t)f(t)dt\|_{p,I_i}^p \\
&\le \sum_{i=1}^N (A(I_i))^p \, \|f\|_{p,I_i}^p \\
&= \left(\max_{i=1,\dots,N} A(I_i) \right)^p \|f\|_{p,I}^p
\end{aligned}$$

and then the lemma follows. $\square$

Lemma 2.10 *Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a, b)$. If $N := N(\epsilon)$ then*

$$b_{N-2}(T) \geq \varepsilon.$$

Proof. From the definition of $N(\varepsilon)$ we have that for $i = 1, ..., N-1$ we have $A(I_i) = \varepsilon$. Let $\lambda \in (0,1)$ then from the definition of $A(I_i)$ we have that for $i = 1, ..., N-1$ there is a function $\phi_i \in L^p(I)$, where $\|\phi_i\|_{p,I} = 1$, with support in I_i such that

$$\inf_{\alpha \in \mathbf{R}} \|T\phi_i - \alpha v\|_{p,I_i} > \lambda A(I_i) \geq \lambda\varepsilon.$$

Let $X_{N-1} = \operatorname{span}\{T\phi; \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i, \lambda_i \in \mathbf{R}\}$ then we can see that rank $X_{N-1} \geq N-1$. Take $0 \neq T\phi \in X_{N-1}$ then $0 \neq \phi = \sum_{i=1}^{N-1} \lambda_i \phi_i$ with $\lambda_i \neq 0$ for some i.

$$\begin{aligned}
\|T\phi\|_{p,I}^p &\geq \sum_{i=1}^{N-1} \|(T\phi)\chi_{I_i}\|_{p,I}^p \\
&= \sum_{i=1}^{N-1} \|\chi_{I_i}(x) v(x) \left(\int_{a_i}^{x} \lambda_i \phi_i(t) \chi_{I_i}(t) dt + \int_{a}^{a_i} \phi(t) u(t) dt \right) \|_{p,I}^p \\
&= \sum_{i=1}^{N-1} \| \left(T\phi_i(x) + v(x) \frac{\eta_i}{\lambda_i} \right) \lambda_i \|_{p,I_i}^p \\
&\qquad\qquad \left(\text{where } \eta_i := \int_a^{a_i} \phi(t) u(t) dt \right) \\
&\geq \sum_{i=1}^{N-1} \inf_{\alpha \in \mathbf{R}} \|T\phi_i(x) - v(x)\alpha\|_{p,I_i}^p |\lambda_i|^p \\
&\geq (\lambda\varepsilon)^p \sum_{i=1}^{N-1} \|\phi_i\|_{p,I_i}^p |\lambda_i|^p \geq (\lambda\varepsilon)^p \|\phi\|_{p,I_i}^p.
\end{aligned}$$

and the lemma follows.$\square$

From these lemmas follows the next theorem:

Theorem 2.11 *Let $\varepsilon > 0$, $1 < p < \infty$ and let $I = (a, b)$. If $N := N(\varepsilon)$ then*

$$a_{N+1}(T) \leq \varepsilon \leq b_{N-2}(T).$$

We can see that this theorem is giving us:

$$a_{N+1}(T) \leq \varepsilon \leq a_{N-1}(T) \qquad \text{and} \qquad \rho_N(T) \leq \varepsilon \leq \rho_{N-2}(T)$$

where $\rho_N(T)$ stands for any of the followings $\delta_N(T), d_N(T), d^N(T)$ or $b_N(T)$.

For general u and v it is impossible to find a simple relation between ε and $N(\varepsilon)$, but by using the properties of $A(I)$ the behavior of $\varepsilon N(\varepsilon)$ when $\varepsilon \to 0_+$ can be determined.

Lemma 2.12 *Given $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ then we have*

$$\lim_{\varepsilon \to 0_+} \varepsilon N(\varepsilon) = \alpha_p \int_a^b |u(t)v(t)|dt.$$

This result follows from an adaptation of the argument of (EHL2); see, in particular, Theorem 6.4 of that paper. Together with Theorem 2.11 this shows, again using the techniques of (EHL2), that the following theorem holds.

Theorem 2.13 *Given $v \in L^p(a,b)$, $u \in L^{p'}(a,b)$ the operator T defined in (1) satisfies*

$$\lim_{n \to \infty} n\rho_n(T) = \alpha_p \int_a^b |u(t)v(t)|dt,$$

where $\alpha_p = A((0,1),1,1)$ and $\rho_n(T)$ stands for any of the followings $b_n(T)$, $\delta_n(T)$, $d_n(T)$, $d^n(T)$ or $a_n(T)$.

From paper (L) we have better estimate for the relation between $\varepsilon > 0$ and $N(\varepsilon)$ (see Theorem 4.1 in (L))

Lemma 2.14 *Let $-\infty \leq a < b \leq \infty$ and $I = (a,b)$, let $u \in L^{p'}(I), v \in L^p(I)$ and suppose that $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b]), v' \in L^{p/(p+1)}(a,b) \cap C([a,b])$. Then*

$$\limsup_{\varepsilon \to 0_+} |\alpha_p \int_a^p |u(t)v(t)dt - \varepsilon N(\varepsilon)| N^{1/2}(\varepsilon) \leq$$

$$c(p,p') \left(\|u'\|_{p'/(p'+1),I} + \|v'\|_{p/(p+1),I} \right) \left(\|u\|_{p',I} + \|v\|_{p,(a,b)} \right) + 3\alpha_p \|uv\|_{1,I},$$

where $\alpha_p = A((0,1),1,1)$ and $c(p,p')$ is a constant depending only on p and p'.

From this Lemma and from Theorem 2.11 we can easily obtain the following theorem:

Theorem 2.15 *Let $-\infty \leq a < b \leq \infty$ and $I = (a,b)$, let $u \in L^{p'}(I), v \in L^p(I)$ and suppose that $u' \in L^{p'/(p'+1)}(a,b) \cap C([a,b]), v' \in L^{p/(p+1)}(a,b) \cap$*

$C([a,b])$. *Then*

$$\limsup_{n\to\infty} |\alpha_p \int\limits_a^p |u(t)v(t)dt - \rho_n(T)n|n^{1/2} \leq$$

$$c(p,p')\left(\|u'\|_{p'/(p'+1),I} + \|v'\|_{p/(p+1),I}\right)\left(\|u\|_{p',I} + \|v\|_{p,(a,b)}\right) + 3\alpha_p\|uv\|_{1,I},$$

where $\alpha_p = A((0,1),1,1)$, $c(p,p')$ *is a constant depending only on* p *and* p' *and* $\rho_n(T)$ *stands for any of the followings* $\delta_n(T)$, $d_n(T)$, $d^n(T)$, $b_n(T)$ *or* $a_n(T)$.

3 The second asymptotic term

In this section we will use properties of $A(I)$ to obtain better estimate about the Approximation numbers and n-widths numbers.

At first let we make some observation about $A(I)$.

Lemma 3.1 *Let* $I = (c,d) \subseteq (a,b)$ *and* $d = (c+d)/2$. *Suppose that* u *and* v *are constant functions over* I. *Then*

$$A(I,u,v) = |I||u||v|A((0,1),1,1)$$

and

$$\sup_{f\in L^p(I)} \inf_{c\in\mathbf{R}} \frac{\|v(x)\int_a^x u(t)f(t)dt - c\|_{p,I}}{\|f\|_{p,I}} =$$

$$\sup_{f\in L^p(I)} \frac{\|v(x)\int_d^x u(t)f(t)dt - c\|_{p,I}}{\|f\|_{p,I}} =$$

$$= |u||v|\frac{\|\sin_p(\pi_p(x-a)/(b-a))\|_{p,I}}{\|\cos_p(\pi_p(x-a)/(b-a))\|_{p,I}} =$$

$$= \alpha_p|u||v| = \left(\frac{1}{\lambda_p}\right)^{1/p}|u||v|,$$

where $\lambda_p = \left(\frac{2\pi}{\sin(\pi/2)}\right)\frac{1}{p'p^{p-1}}$ *is the first non-zero eigenvalue of the p-Laplacian problem on interval* $(0,1)$.

Proof. See Lemma 4.1 in (EHL2) and Lemma 2.7 in (EL). □

In this lemma $\sin_p(.)$ and $\cos_p(.)$ means special goniometric functions which corresponds to first non-constant eigenfunctions of the one-dimensional p-Laplacian (see (EL) or (DM) for more).

From the definition of $A(I)$ we have:

Lemma 3.2 *Let $I = (c,d) \subset (a,b)$. Let $u_1 \geq u_2 > 0$ and $v_1 \geq v_2 > 0$ than we have:*

$$A(I, u_1, v_1) \geq A(I, u_2, v_2) \geq 0.$$

Now we are ready to prove the following lemma about behavior of $\varepsilon N(\varepsilon)$.

Lemma 3.3 *Let $1 < p < \infty$, $I = (a,b)$, $u \in L^{p'}(I)$, $v \in L^p(I)$ and $(v'/v), (u'/u) \in L^1(I) \cap C[a,b]$ then*

$$\lim_{\varepsilon \to 0_+} \left| N(\varepsilon) \left[\varepsilon N(\varepsilon) - \int_I u(x)v(x)dx \right] \right|$$

$$\leq \int_I u(x)v(x)dx \left[\int_I \frac{v'(x)}{v(x)} dx + \int_I \frac{u'(x)}{u(x)} dx + 1 \right.$$

$$\left. + \left(\int_I \frac{u'(x)}{u(x)} dx \right) \left(\int_I \frac{v'(x)}{v(x)} dx \right) \right]$$

Proof. Let we take $\|T\| > \varepsilon > 0$ and $N := N(\varepsilon)$. Then we have the following partition: $I = \cup_{i=1}^N$, $A(I_i) = \varepsilon$ for $i = \{1, ..., N-1\}$ and $A(I_N) < \varepsilon$. Define the following step functions:

$$u^{+,\varepsilon}(x) = \sum_{i=1}^N u_i^{+,\varepsilon} \chi_{I_i}(x), \qquad v^{+,\varepsilon}(x) = \sum_{i=1}^N v_i^{+,\varepsilon} \chi_{I_i}(x)$$

$$u^{-,\varepsilon}(x) = \sum_{i=1}^N u_i^{-,\varepsilon} \chi_{I_i}(x), \qquad v^{-,\varepsilon}(x) = \sum_{i=1}^N v_i^{-,\varepsilon} \chi_{I_i}(x)$$

where

$$u_i^{+,\varepsilon} = \sup_{x \in I_i} |u(x)|, \qquad u_i^{-,\varepsilon} = \inf_{x \in I_i} |u(x)|$$

$$v_i^{+,\varepsilon} = \sup_{x \in I_i} |v(x)|, \qquad v_i^{-,\varepsilon} = \inf_{x \in I_i} |v(x)|.$$

Then we have from the previous two lemmas:

$$u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| \leq A(I_i) \leq u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i|, \tag{6}$$

and we can see that

$$\int_I u^{-,\varepsilon}(x) v^{-,\varepsilon}(x) dx \leq \int_I u(x)v(x)dx \leq \int_I u^{+,\varepsilon}(x) v^{+,\varepsilon}(x) dx.$$

Let we estimate upper bound for the following quantity:

$$\begin{aligned}
K(\varepsilon) &:= \int_I (u^{+,\varepsilon}(x) v^{+,\varepsilon}(x) - u^{-,\varepsilon}(x) v^{-,\varepsilon}(x)) dx \\
&= \sum_{i=1}^{N} |I_i| (u_i^{+,\varepsilon} v_i^{+,\varepsilon} - u_i^{-,\varepsilon} v_i^{-,\varepsilon}) \\
&= \sum_{i=1}^{N} |I_i| (u_i^{+,\varepsilon} v_i^{+,\varepsilon} - u_i^{+,\varepsilon} v_i^{-,\varepsilon} + u_i^{+,\varepsilon} v_i^{-,\varepsilon} - u_i^{-,\varepsilon} v_i^{-,\varepsilon}) \\
&\qquad \Big(\text{use: } (v_i^{+,\varepsilon} - v_i^{-,\varepsilon}) \leq |I_i| \max_{x \in I_i} |v'(x)| \\
&\qquad\qquad \text{and } \ (u_i^{+,\varepsilon} - u_i^{-,\varepsilon}) \leq |I_i| \max_{x \in I_i} |u'(x)| \ \Big) \\
&\leq \sum_{i=1}^{N} |I_i| \left[u_i^{+,\varepsilon} |I_i| \max_{x \in I_i} |v'(x)| + v_i^{-,\varepsilon} |I_i| \max_{x \in I_i} |u'(x)| \right] \\
&\qquad \Big(\text{use } |I_i| u_i^{-,\varepsilon} v_i^{-,\varepsilon} \leq A(I_i) \leq \varepsilon \Big) \\
&\leq \varepsilon \sum_{i=1}^{N} \left[\frac{u_i^{+,\varepsilon}}{u_i^{-,\varepsilon}} |I_i| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-,\varepsilon}} + \frac{v_i^{-,\varepsilon}}{v_i^{-,\varepsilon}} |I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \\
&\leq \varepsilon \sum_{i=1}^{N} \left[\frac{u_i^{-,\varepsilon} + |I_i| \max_{x \in I_i} |u'(x)|}{u_i^{+,\varepsilon}} |I_i| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \right] \\
&\qquad + \varepsilon \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \\
&\leq \varepsilon \sum_{i=1}^{N} \left[1 + |I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{+,\varepsilon}} \right] |I_i| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \\
&\qquad + \varepsilon \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \\
&\leq \varepsilon \sum_{i=1}^{N} \left[1 + \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \right] \left[|I_i| \frac{\max_{x \in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \right] \\
&\qquad + \varepsilon \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x \in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right]
\end{aligned}$$

$$= \varepsilon \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] + \varepsilon \sum_{i=1}^{N} \left[|I_i| \frac{\max_{x\in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \right]$$
$$+\varepsilon \left(\sum_{i=1}^{N} \left[|I_i| \frac{\max_{x\in I_i} |v'(x)|}{v_i^{-,\varepsilon}} \right] \right) \left(\sum_{i=1}^{N} \left[|I_i| \frac{\max_{x\in I_i} |u'(x)|}{u_i^{-,\varepsilon}} \right] \right).$$

From (6) we have:

$$\sum_{i=1}^{N} u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| \leq \varepsilon N \qquad \text{and} \qquad \sum_{i=1}^{N} u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i| \geq \varepsilon (N-1)$$

and then

$$\sum_{i=1}^{N} u_i^{-,\varepsilon} v_i^{-,\varepsilon} |I_i| - \int_I uv\ dx \leq \varepsilon N - \int_I uv\ dx \leq \sum_{i=1}^{N} u_i^{+,\varepsilon} v_i^{+,\varepsilon} |I_i| + \varepsilon - \int_I uv\ dx$$

which give us

$$-K(\varepsilon) dx \leq \varepsilon N - \int_I uv\ dx \leq K(\varepsilon) + \varepsilon.$$

and

$$-NK(\varepsilon) dx \leq N \left(\varepsilon N - \int_I uv\ dx \right) \leq NK(\varepsilon) + \varepsilon N.$$

Using that $\lim_{\varepsilon\to 0_+} (\varepsilon N(\varepsilon)) = \int_I uv\ dx$ and that

$$\lim_{\varepsilon\to 0_+} \frac{K(\varepsilon)}{\varepsilon} = \int_I \frac{u'}{u} + \int_I \frac{v'}{v} + \int_I \frac{u'}{u} \int_I \frac{v'}{v}$$

we obtain:

$$\limsup_{\varepsilon\to 0_+} \left| N \left(\varepsilon N - \int_I uv\ dx \right) \right| \leq \left(\int_I uv \right) \left(\int_I \frac{u'}{u} + \int_I \frac{v'}{v} + \int_I \frac{u'}{u} \int_I \frac{v'}{v} \right)$$
$$+ \int_I uv.$$

□

With help of Theorem 2.2 we can obtain from the previous lemma the following theorem

Theorem 3.4 *Let* $-\infty \le a < b \le \infty$ *and* $I = (a,b)$, *let* $u \in L^{p'}(I)$, $v \in L^p(I)$ *and* $(v'/v), (u'/u) \in L^1(I) \cap C[a,b]$ *then*

$$\limsup_{n\to\infty} \left| n\left[n\rho_n(T) - \int_I u(x)v(x)dx \right] \right|$$
$$\le \int_I u(x)v(x)dx \left[\int_I \frac{v'(x)}{v(x)}dx + \int_I \frac{u'(x)}{u(x)}dx + 2 \right.$$
$$\left. + \left(\int_I \frac{u'(x)}{u(x)}dx \right) \left(\int_I \frac{v'(x)}{v(x)}dx \right) \right],$$

where $\rho_n(T)$ *stands for any of the followings:* $a_n(T)$, $\delta_n(T)$, $d_n(T)$, $d^n(T)$ *or* $b_n(T)$, *and* T *is the Hardy-type operator.*

Proof. From Lemma 2.7 we have that for any large n there is $\varepsilon > 0$ such $n = N(\varepsilon)$. Then from Theorem 2.11 we have:

$$\begin{aligned} n\left(\varepsilon n - \int_I uv\right) &\ge n\left(a_{n+1}(T)n - \int_I uv\right) \\ &\ge [(n+1)-1]\left(a_{n+1}(T)[(n+1)-1] - \int_I uv\right) \\ &\ge (n+1)\left(a_{n+1}(T)(n+1) - \int_I uv\right) \\ &\qquad + (n+1)(-a_{n+1}(T)) - \left(a_{n+1}(T)n - \int_I uv\right) \end{aligned}$$

and

$$\begin{aligned} n\left(\varepsilon n - \int_I uv\right) &\le n\left(a_{n-1}(T)n - \int_I uv\right) \\ &\le [(n-1)+1]\left(a_{n-1}(T)[(n-1)+1] - \int_I uv\right) \\ &\le (n-1)\left(a_{n-1}(T)(n-1) - \int_I uv\right) \end{aligned}$$

$$+(n-1)a_{n-1}(T) + \left(a_{n-1}(T)n - \int\limits_I uv \right).$$

By taking limit $n \to \infty$ and with help of Theorem 2.13 we have proved our theorem for $\rho_n(T) = a_n(T)$ and by using similar technique we can get the proof of the theorem for $d_n(T)$, $d^n(T)$, $b_n(T)$ and $\delta_n(T)$. $\square$

From Theorem 3.4 we have the following information about the second asymptotic:

$$\rho_n(T) = \frac{1}{n} \int\limits_I u(x)v(x)dx + O(n^{-2}).$$

Remark 3.5 *We have found that our method which is based on studding of the behavior $\varepsilon N(\varepsilon)$ can not be improved behind the second term.*

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