LAPLACE TRANSFORM, DYNAMICS AND SPECTRAL GEOMETRY

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Abstract. We consider vector fields X on a closed manifold M with rest points of Morse type. For such vector fields we define the property of exponential growth. A cohomology class $\xi \in H^1(M;\mathbb{R})$ which is Lyapunov for X defines counting functions for isolated instantons and closed trajectories. If X has exponential growth property we show, under a mild hypothesis generically satisfied, how these counting functions can be recovered from the spectral geometry associated to (M, g, ω) where g is a Riemannian metric and ω is a closed one form representing ξ. This is done with the help of Dirichlet series and their Laplace transform.

Contents

1. Introduction

1.1. Vector fields with Morse zeros and Lyapunov cohomology class. Let X be a smooth vector field on a smooth manifold M. A

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point $x \in M$ is called a *zero* or *rest point* of X if $X(x) = 0$. Denote by $\mathcal{X} := \{x \in M | X(x) = 0\}$ the set of rest points.

Recall that:

(i) A parameterized trajectory is a map $\theta : \mathbb{R} \to M$ so that $\theta'(t) = X(\theta(t))$. A trajectory is an equivalence class of parameterized trajectories with $\theta_1 \equiv \theta_2$ iff $\theta_1(t + a) = \theta_2(t)$ for some real number a. Any representative θ of a trajectory is called a parametrization.

(ii) An *instanton* from the rest point x to the rest point y is a trajectory with the property that for one and then any parameterization θ, $\lim_{t \to -\infty} \theta(t) = x$, $\lim_{t \to +\infty} \theta(t) = y$, and which is isolated among these trajectories.

(iii) A parameterized *closed trajectory* is a pair $\tilde{\theta} = (\theta, T)$, with θ a parametrized trajectory so that $\theta(t + T) = \theta(t)$. A parameterized closed trajectory gives rise to a smooth map $\theta : S^1 := \mathbb{R}/T\mathbb{Z} \to M$. A *closed trajectory* is an equivalence class of parameterized closed trajectories with $(\theta_1, T_1) \equiv (\theta_2, T_2)$ iff $\theta_1 \equiv \theta_2$ and $T_1 = T_2$.

Recall that a zero $x \in \mathcal{X}$ is called *Morse zero* if there exist coordinates $(t_1, \dots, t_n)$ around x so that $X = 2\sum_{i=1}^{q} t_i \frac{\partial}{\partial t_i} - 2\sum_{i=q+1}^{n} t_i \frac{\partial}{\partial t_i}$. The integer q is independent of the chosen coordinates $(t_1, \dots, t_n)$. It is referred to as the *Morse index* of x and denoted by $\operatorname{ind}(x)$.[1] Therefore $\mathcal{X} = \bigsqcup_q \mathcal{X}_q$ with $\mathcal{X}_q$ the set of rest points of index q.

For any Morse zero x the *stable* resp. *unstable set* is

$$W_x^{\pm} := \{y | \lim_{t \to \pm\infty} \Psi_t(y) = x\}$$

where $\Psi_t : M \to M$ denotes the flow of X at time t. The stable and unstable sets are images of injective smooth immersions $i_x^{\pm} : W_x^{\pm} \to M$.[2] The manifold W_x^- resp. W_x^+ is diffeomorphic to $\mathbb{R}^{\operatorname{ind}(x)}$ resp. $\mathbb{R}^{n-\operatorname{ind}(x)}$.

Convention. Unless explicitly mentioned all the vector fields in this paper are assumed to have Morse, hence isolated, rest points.

Definition 1. A vector field X is said to have the *exponential growth property at a zero x* if for some (and then any) Riemannian metric g there exists a positive constant C so that $\operatorname{Vol}(D_r(x)) \leq e^{Cr}$, for all $r \geq 0$. Here $D_r(x) \subseteq W_x^-$ denotes the disk of radius r centered at $x \in W_x^-$ with respect to the induced Riemannian metric $(i_x^-)^* g$ on W_x^-. A vector field is said to have the *exponential growth property* if it has the exponential growth property at all of its zeros.

[1] A Morse zero is non-degenerate and its Hopf index is $(-1)^{n-q}$.

[2] By abuse of notation we denote the source manifold also by $W_x^{\pm}$.

We expect that every vector field which has a Lyapunov cohomology class, see Definition 2 below, and satisfies the Morse–Smale property, see Definition 3 below, has the exponential growth property, cf. the conjecture in section 3.2. For the sake of Theorem 4 we introduce in section 6.1, cf. Definition 9, the *strong exponential growth property.* Both concepts are superfluous if the conjecture is true.

Definition 2. A cohomology class $\xi \in H^1(M;\mathbb{R})$ is called *Lyapunov class* for a vector field X if there exits a Riemannian metric g and a closed one form ω representing ξ, so that $X = -\operatorname{grad}_g \omega$.[3][4][5]

In this paper we will show that a vector field X and a Lyapunov class ξ for X provide *counting functions for the instantons* from x to y when $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$[6] and *counting functions for closed trajectories.* Moreover these counting functions can be interpreted as Dirichlet series.

If the vector field has exponential growth property these series have a finite abscissa of convergence, hence have a Laplace transform. Their Laplace transform can be read off from the spectral geometry of a pair (g,ω) where g is a Riemannian metric and ω is a closed one form representing ξ.

We will describe these counting functions and prove our results under the hypotheses that properties MS and NCT defined below are satisfied. Generically these properties are always satisfied, cf. Proposition 2 below.

Also in this paper, for any vector field X and cohomology class $\xi \in H^1(M;\mathbb{R})$ we define an invariant $\rho(\xi,X) \in \mathbb{R}\cup\{\pm\infty\}$ and show that if ξ is Lyapunov for X then exponential growth property is equivalent to $\rho(\xi,X) < \infty$.

Definition 3. The vector field X is said to satisfy the *Morse–Smale property*, MS for short, if for any $x, y \in \mathcal{X}$ the maps i_x^- and i_y^+ are transversal.

In this case the set $\mathcal{M}(x,y) = W_x^- \cap W_y^+$ with $x, y \in \mathcal{X}$ is the image by an injective immersion of a smooth manifold of dimension

[3]An alternative definition is the following: There exists a closed one form ω representing ξ so that $\omega(X) < 0$ on $M \setminus \mathcal{X}$ and such that in a neighborhood of any rest point the vector field X is of the form $-\operatorname{grad}_g \omega$ for some Riemannian metric g. It is proved in section 3 that the two definitions are actually equivalent.

[4]This implies that ω is a Morse form, i.e. locally it is the differential of a smooth function whose critical points are non-degenerate.

[5]Not all vector fields admit Lyapunov cohomology classes.

[6]This is the only case when, generically, the instantons can be isolated.

$\mathrm{ind}(x)-\mathrm{ind}(y)$ on which $\mathbb{R}$ acts freely with quotient a smooth manifold $\mathcal{T}(x,y)$ of dimension $\mathrm{ind}(x)-\mathrm{ind}(y)-1$. The manifold $\mathcal{T}(x,y)$ is called the manifold of *unparameterized trajectories* from x to y. If $\mathrm{ind}(x)-\mathrm{ind}(y)=1$ it will be zero dimensional. In this case the unparameterized trajectories are isolated and referred to as *instantons* from x to y.

Choose $\mathcal{O}=\{\mathcal{O}_x\}_{x\in\mathcal{X}}$ a collection of orientations of the unstable manifolds of the critical points, with $\mathcal{O}_x$ an orientation of W_x^-. Any instanton $[\theta]$ from $x\in\mathcal{X}_q$ to $y\in\mathcal{X}_{q-1}$ has a sign $\epsilon([\theta])=\epsilon^{\mathcal{O}}([\theta])=\pm1$ defined as follows: The orientations $\mathcal{O}_x$ and $\mathcal{O}_y$ induce an orientation on $[\theta]$. Take $\epsilon([\theta])=+1$ if this orientation is compatible with the orientation from x to y and $\epsilon([\theta])=-1$ otherwise.

Let $\tilde\theta$ be a parameterized closed trajectory and let Ψ_t denote the flow of X at time t. The closed trajectory $[\tilde\theta]$ is called non-degenerate if for some (and then any) $t_0\in\mathbb{R}$ and parameterization $\tilde\theta=(\theta,T)$ the differential $D_{\theta(t_0)}\Psi_T: T_{\theta(t_0)}M\to T_{\theta(t_0)}M$ is invertible with the eigenvalue 1 of multiplicity one.

Definition 4. The vector field X is said to satisfies the *non-degenerate closed trajectories property*, NCT for short, if all (unparameterized) closed trajectories of X are non-degenerate.

Any non-degenerate closed trajectory $[\tilde\theta]$ has a *period* $p([\tilde\theta])\in\mathbb{N}$ and a *sign* $\epsilon([\tilde\theta]):=\pm1$ defined as follows:

(i) $p([\tilde\theta])$ is the largest positive integer p such that $\theta: S^1\to M$ factors through a self map of S^1 of degree p.

(ii) $\epsilon([\tilde\theta]):=\operatorname{sign}\det D_{\theta(t_0)}\Psi_T$ for some (and hence any) $t_0\in\mathbb{R}$ and parameterization $\tilde\theta$.

A cohomology class $\xi\in H^1(M;\mathbb{R})$ induces the homomorphism $\xi: H_1(M;\mathbb{Z})\to\mathbb{R}$ and then the injective group homomorphism

$$\xi:\Gamma_\xi\to\mathbb{R},\ \text{with}\ \Gamma_\xi:=H_1(M;\mathbb{Z})/\ker\xi.$$

For any two points $x,y\in M$ denote by $\mathcal{P}_{x,y}$ the space of continuous paths from x to y. We say that $\alpha\in\mathcal{P}_{x,y}$ is equivalent to $\beta\in\mathcal{P}_{x,y}$, iff the closed path $\beta^{-1}\star\alpha$[7] represents an element in $\ker\xi$. We denote by $\hat{\mathcal{P}}_{x,y}=\hat{\mathcal{P}}^\xi_{x,y}$ the set of equivalence classes of elements in $\mathcal{P}_{x,y}$. Note that Γ_ξ acts freely and transitively, both from the left and from the right, on $\hat{\mathcal{P}}^\xi_{x,y}$. The action $\star$ is defined by juxtaposing at x resp. y a closed curve representing an element $\gamma\in\Gamma_\xi$ to a path representing the element $\hat\alpha\in\hat{\mathcal{P}}^\xi_{x,y}$.

[7]Here $\star$ denotes the juxtaposition of paths. Precisely if $\alpha,\beta:[0,1]\to M$ and $\beta(0)=\alpha(1)$, then $\beta\star\alpha:[0,1]\to M$ is given by $\alpha(2t)$ for $0\le t\le 1/2$ and $\beta(1-2t)$ for $1/2\le t\le 1$.

Any closed one form ω representing ξ defines a map, $\overline{\omega} : \mathcal{P}_{x,y} \to \mathbb{R}$, by

$$\overline{\omega}(\alpha) := \int_{[0,1]} \alpha^* \omega$$

which in turn induces the map $\overline{\omega} : \hat{\mathcal{P}}^{\xi}_{x,y} \to \mathbb{R}$. We have:

$$\begin{aligned} \overline{\omega}(\gamma \star \hat{\alpha}) &= \xi(\gamma) + \overline{\omega}(\hat{\alpha}) \\ \overline{\omega}(\hat{\alpha} \star \gamma) &= \overline{\omega}(\hat{\alpha}) + \xi(\gamma) \end{aligned}$$

Note that for $\omega' = \omega + dh$ we have $\overline{\omega'} = \overline{\omega} + h(y) - h(x)$.

Proposition 1. *Suppose $\xi \in H^1(M;\mathbb{R})$ is a Lyapunov class for the vector field X.*

(i) *If X satisfies MS, $x \in \mathcal{X}_q$ and $y \in \mathcal{X}_{q-1}$ then the set of instantons from x to y in each class $\hat{\alpha} \in \hat{\mathcal{P}}^{\xi}_{x,y}$ is finite.*
(ii) *If X satisfies both MS and NCT then for any $\gamma \in \Gamma_\xi$ the set of closed trajectories representing the class γ is finite.*

The proof is a straight consequence of the compacity of space of trajectories of bounded energy, cf. [5] and [8].

Suppose X is a vector field which satisfies MS and NCT and suppose ξ is a Lyapunov class for X. In view of Proposition 1 we can define the *counting function of closed trajectories* by

$$\mathbb{Z}^{\xi}_X : \Gamma_\xi \to \mathbb{Q}, \qquad \mathbb{Z}^{\xi}_X(\gamma) := \sum_{[\tilde{\theta}] \in \gamma} \frac{(-1)^{\epsilon([\tilde{\theta}])}}{p([\tilde{\theta}])} \in \mathbb{Q}.$$

If a collection of orientations $\mathcal{O} = \{\mathcal{O}_x\}_{x \in \mathcal{X}}$ is given one defines the *counting function of the instantons* from x to y by

$$\mathbb{I}^{X,\mathcal{O},\xi}_{x,y} : \hat{\mathcal{P}}^{\xi}_{x,y} \to \mathbb{Z}, \qquad \mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat{\alpha}) := \sum_{[\theta] \in \hat{\alpha}} \epsilon([\theta]). \tag{1}$$

Note that the change of the orientations $\mathcal{O}$ might change the function $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}$ but only up to multiplication by ± 1. A key observation in this work is the fact that the counting functions $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}$ and $\mathbb{Z}^{\xi}_X$ can be interpreted as Dirichlet series.

As long as Hypotheses MS and NCT are concerned we have the following genericity result. For a proof consult [5] and the references in [7, page 211].

Proposition 2. *Suppose X has $\xi \in H^1(M;\mathbb{R})$ as a Lyapunov cohomology class.*

(i) *One can find a vector fields X' arbitrarily close to X in the C^1–topology which satisfy MS and have ξ as Lyapunov cohomology class. Moreover one can choose X' equal to X in some neighborhood of $\mathcal{X}$ and away from any given neighborhood of $\mathcal{X}$.*
(ii) *If in addition X above satisfies MS one can find vector fields X' arbitrary closed to X in the C^1–topology which satisfy MS and NCT, and have ξ as Lyapunov cohomology class. Moreover one can choose X' equal to X in some neighborhood of $\mathcal{X}$.*
(iii) *Consider the space of vector fields which have the same set of rest points as X, and agree with X in some neighborhood of $\mathcal{X}$. Equip this set with the C^1–topology. The subset of vector fields which satisfy MS and NCT is Baire residual set.*

1.2. **Dirichlet series and their Laplace transform.** Recall that a Dirichlet series f is given by a pair of finite or infinite sequences:

$$\begin{pmatrix} \lambda_1 & < & \lambda_2 & < & \cdots & < & \lambda_k & < & \lambda_{k+1} & \cdots \\ a_1 & & a_2 & & \cdots & & a_k & & a_{k+1} & \cdots \end{pmatrix}$$

The first sequence is a sequence of real numbers with the property that $\lambda_k \to \infty$ if the sequences are infinite. The second sequence is a sequence of non-zero complex numbers. The associated series

$$L(f)(z) := \sum_i e^{-z\lambda_i} a_i$$

has an *abscissa of convergence* $\rho(f) \leq \infty$, characterized by the following properties, cf. [17] and [18]:

(i) If $\Re z > \rho(f)$ then $f(z)$ is convergent and defines a holomorphic function.
(ii) If $\Re z < \rho(f)$ then $f(z)$ is divergent.

A Dirichlet series can be regarded as a complex valued measure with support on the discrete set $\{\lambda_1, \lambda_2, \dots\} \subseteq \mathbb{R}$ where the measure of λ_i is equal to a_i. Then the above series is the Laplace transform of this measure, cf. [18]. The following proposition is a reformulation of results which lead to the Novikov theory and to the work of Hutchings–Lee and Pajitnov etc, cf. [5] and [8] for more precise references.

Proposition 3.

(i) *(Novikov) Suppose X is a vector field on a closed manifold M which satisfies MS and has ξ as a Lyapunov cohomology class. Suppose ω is a closed one form representing ξ. Then for any*

$x \in \mathcal{X}_q$ and $y \in \mathcal{X}_{q-1}$ the collection of pairs of numbers

$$\mathbb{I}^{X,\mathcal{O},\omega}_{x,y} := \left\{ \left(-\overline{\omega}(\hat\alpha), \mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha)\right) \,\middle|\, \mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha) \neq 0, \hat\alpha \in \hat{\mathcal{P}}^{\xi}_{x,y} \right\}$$

defines a Dirichlet series. The sequence of λ's consists of the numbers $-\overline{\omega}(\hat\alpha)$ when $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha)$ is non-zero, and the sequence a's consists of the numbers $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha) \in \mathbb{Z}$.

(ii) *(D. Fried, M. Hutchings) If in addition X satisfies NCT* [8] *then the collection of pairs of numbers*

$$\mathbb{Z}^{\xi}_X := \left\{ \left(-\xi(\gamma), \mathbb{Z}^{\xi}_X(\gamma)\right) \,\middle|\, \mathbb{Z}^{\xi}_X(\gamma) \neq 0, \gamma \in \Gamma_\xi \right\}$$

defines a Dirichlet series. The sequence of λ's consists of the real numbers $-\xi(\gamma)$ when $\mathbb{Z}^{\xi}_X(\gamma)$ is non-zero and the sequence of a's consists of the numbers $\mathbb{Z}^{\xi}_X(\gamma) \in \mathbb{Q}$.

We will show that if X has exponential growth property then the abscissa of convergence will be finite, hence the above Dirichlet series will have Laplace transform and the main results of this paper, Theorems 3 and 4 below, will provide explicit formulae for them in terms of the spectral geometry of (M, g, ω). To explain such formulae we need additional considerations and results.

1.3. **The Witten–Laplacian.** Let M be a closed manifold and (g, ω) a pair consisting of a Riemannian metric g and a closed one form ω. We suppose that ω is a *Morse form.* This means that locally $\omega = dh$, h smooth function with all critical points non-degenerate. A critical point or a zero of ω is a critical point of h and since non-degenerate, has an index, the index of the Hessian $d^2_x h$, denoted by $\mathrm{ind}(x)$. Denote by $\mathcal{X}$ the set of critical points of ω and by $\mathcal{X}_q$ be the subset of critical points of index q.

For $t \in \mathbb{R}$ consider the complex $(\Omega^*(M), d^*_\omega(t))$ with differential $d^q_\omega(t) : \Omega^q(M) \to \Omega^{q+1}(M)$ given by

$$d^q_\omega(t)(\alpha) := d\alpha + t\omega \wedge \alpha.$$

Using the Riemannian metric g one constructs the formal adjoint of $d^q_\omega(t)$, $d^q_\omega(t)^\sharp : \Omega^{q+1}(M) \to \Omega^q(M)$, and one defines the Witten–Laplacian $\Delta^q_\omega(t) : \Omega^q(M) \to \Omega^q(M)$ associated to the closed 1–form ω by:

$$\Delta^q_\omega(t) := d^q_\omega(t)^\sharp \circ d^q_t + d^{q-1}_\omega(t) \circ d^{q-1}_\omega(t)^\sharp.$$

[8] It seems possible to prove the above proposition without the hypothesis MS and NCT, of course with properly modified definition of the counting functions $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}$ and $\mathbb{Z}^{\xi}_X$.We will return to this matter in a future paper.

Thus, $\Delta^q_\omega(t)$ is a second order differential operator, with $\Delta^q_\omega(0) = \Delta^q$, the Laplace–Beltrami operator. The operators $\Delta^q_\omega(t)$ are elliptic, selfadjoint and nonnegative, hence their spectra $\operatorname{Spect}\Delta^q_\omega(t)$ lie in the interval $[0,\infty)$. It is not hard to see that

$$\Delta^q_\omega(t) = \Delta^q + t(L + L^\sharp) + t^2||\omega||^2 \operatorname{Id},$$

where L denotes the Lie derivative along the vector field $-\operatorname{grad}_g\omega$, $L^\sharp$ the formal adjoint of L and $||\omega||^2$ is the fiber wise norm of ω.

The following result extends a result due to E. Witten (cf. [19]) in the case that ω is exact and its proof was sketched in [5].

Theorem 1. *Let M be a closed manifold and (g,ω) be a pair as above. Then there exist constants $C_1, C_2, C_3, T > 0$ so that for $t > T$ we have:*

(i) $\operatorname{Spect}\Delta^q_\omega(t) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset$.
(ii) $\sharp\big(\operatorname{Spect}\Delta^q_\omega(t) \cap [0, C_1 e^{-C_2 t}]\big) = \sharp\mathcal{X}_q$.
(iii) $1 \in (C_1 e^{-C_2 t}, C_3 t)$.

Here $\sharp A$ denotes cardinality of the set A.

Theorem 1 can be complemented with the following proposition, proved also by Mityagin and Novikov, cf. [14], whose proof is included in Appendix A.

Proposition 4. *For all but finitely many t the dimension of* $\ker\Delta^q_\omega(t)$ *is constant in t.*

Denote by $\Omega^*_{\mathrm{sm}}(M)(t)$ the $\mathbb{R}$–linear span of the eigen forms which correspond to eigenvalues smaller than 1 referred bellow as the small eigenvalues. Denote by $\Omega^*_{\mathrm{la}}(M)(t)$ the orthogonal complement of $\Omega^*_{\mathrm{sm}}(M)(t)$ which, by elliptic theory, is a closed subspace of $\Omega^*(M)$ with respect to C^∞–topology, in fact with respect to any Sobolev topology. The space $\Omega^*_{\mathrm{la}}(M)(t)$ is the closure of the span of the eigen forms which correspond to eigenvalues larger than one. As an immediate consequence of Theorem 1 we have for $t > T$:

$$\big(\Omega^*(M), d_\omega(t)\big) = \big(\Omega^*_{\mathrm{sm}}(M)(t), d_\omega(t)\big) \oplus \big(\Omega^*_{\mathrm{la}}(M)(t), d_\omega(t)\big) \qquad (2)$$

With respect to this decomposition the Witten–Laplacian is diagonalized

$$\Delta^q_\omega(t) = \Delta^q_{\omega,\mathrm{sm}}(t) \oplus \Delta^q_{\omega,\mathrm{la}}(t). \qquad (3)$$

and by Theorem 1(ii), we have for $t > T$

$$\dim\Omega^q_{\mathrm{sm}}(M)(t) = \sharp\mathcal{X}_q.$$

The cochain complex $(\Omega^*_{\mathrm{la}}(M)(t), d_\omega(t))$ is acyclic and in view of Theorem 1(ii) of finite codimension in the elliptic complex $(\Omega^*(M), d_\omega(t))$.

Therefore we can define the function

$$\log T_{\mathrm{an,la}}(t) = \log T^{\omega,g}_{\mathrm{an,la}}(t) := \frac{1}{2}\sum_q (-1)^{q+1} q \log\det \Delta^q_{\omega,\mathrm{la}}(t) \qquad (4)$$

where $\det \Delta^q_{\omega,\mathrm{la}}(t)$ is the zeta-regularized product of all eigenvalues of $\Delta^q_{\omega,\mathrm{la}}(t)$ larger than one.[9] This quantity will be referred to as the *large analytic torsion.*

1.4. **Canonical base of the small complex.** Let M be a closed manifold and (g, g', ω) be a triple consisting of two Riemannian metrics g and g' and a Morse form ω. The vector field $X = -\operatorname{grad}_{g'} \omega$ has $[\omega]$ as a Lyapunov cohomology class.

Suppose that X satisfies MS and has exponential growth. Choose $\mathcal{O} = \{\mathcal{O}_x\}_{x\in\mathcal{X}}$ a collection of orientations of the unstable manifolds with $\mathcal{O}_x$ orientation of W^-_x. Let $h_x : W^-_x \to \mathbb{R}$ be the unique smooth map defined by $dh_x = (i^-_x)^*\omega$ and $h_x(x) = 0$. Clearly $h_x \le 0$.

In view of the exponential growth property, cf. section 3, there exists T so that for $t > T$ the integral

$$\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)(a)(x) := \int_{W^-_x} e^{th_x}(i^-_x)^* a, \quad a \in \Omega^q(M), \qquad (5)$$

is absolutely convergent, cf. section 4, and defines a linear map:

$$\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^q(M) \to \mathrm{Maps}(\mathcal{X}_q, \mathbb{R}).$$

Theorem 2. *Suppose (g, g', ω) is a triple as above with X of exponential growth and satisfying MS. Equip $\Omega^*(M)$ with the scalar product induced by g and $\mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$ with the unique scalar product which makes $E_x \in \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$, the characteristic functions of $x \in \mathcal{X}_q$, an orthonormal base.*

Then there exists T so that for any q and $t \ge T$ the linear map $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ defined by (5), when restricted to $\Omega^q_{\mathrm{sm}}(M)(t)$, is an isomorphism and an $O(1/t)$ isometry. In particular $\Omega^q_{\mathrm{sm}}(M)(t)$ has a canonical base $\{E^{\mathcal{O}}_x(t) | x \in \mathcal{X}_q\}$ with $E^{\mathcal{O}}_x(t) = (\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t))^{-1}(E_x)$.

As a consequence we have

$$d^{q-1}_\omega(E^{\mathcal{O}}_y(t)) =: \sum_{x\in\mathcal{X}_q} I^{X,\mathcal{O},\omega,g}_{x,y}(t) \cdot E^{\mathcal{O}}_x(t), \qquad (6)$$

where $I^{X,\mathcal{O},\omega,g}_{x,y} : [T, \infty) \to \mathbb{R}$ are smooth, actually analytic functions, cf. Theorem 3 below.

[9] Which, by the ellipticity, are all eigenvalues of $\Delta^q_\omega(t)$ but finitely many.

In addition to the functions $I^{X,\mathcal{O},\omega,g}_{x,y}(t)$ defined for $t \geq T$, cf. (6), we consider also the function

$$\log \mathbb{V}(t) = \log \mathbb{V}_{\omega,g,X}(t) := \sum_q (-1)^q \log \operatorname{Vol}\{E_x(t) | x \in \mathcal{X}_q\}. \qquad (7)$$

Observe that the change in the orientations $\mathcal{O}$ does not change the right side of (7), so $\mathcal{O}$ does not appear in the notation $\mathbb{V}(t)$.

1.5. **A geometric invariant associated to (X, ω, g) and a smooth function associated with the triple (g, g', ω).** Recall that Mathai–Quillen [11] (cf. also [1]) have introduced a differential form $\Psi_g \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M)$ for any Riemannian manifold (M, g) of dimension n. Here $\mathcal{O}_M$ denotes the orientation bundle of M pulled back to TM. For any closed one form ω on M we consider the form $\omega \wedge X^*\Psi_g \in \Omega^n(M \setminus \mathcal{X}; \mathcal{O}_M)$. Here $X = -\operatorname{grad}_{g'} \omega$ is regarded as a map $X : M \setminus \mathcal{X} \to TM \setminus M$ and M is identified with the image of the zero section of the tangent bundle.

The integral

$$\int_{M\setminus\mathcal{X}} \omega \wedge X^*\Psi_g$$

is in general divergent. However it does have a regularization defined by the formula

$$R(X, \omega, g) := \int_M \omega_0 \wedge X^*\Psi_g - \int_M f E_g + \sum_{x\in\mathcal{X}} (-1)^{\operatorname{ind}(x)} f(x) \qquad (8)$$

where

(i) f is a smooth function whose differential df is equal to ω in a small neighborhood of $\mathcal{X}$ and therefore $\omega_0 := \omega - df$ vanishes in a small neighborhood of $\mathcal{X}$ and
(ii) $E_g \in \Omega^n(M; \mathcal{O}_M)$ is the Euler form associated with g.

It will be shown in section 5 below that the definition is independent of the choice of f, see also [6]. Finally we introduce the function

$$\log \hat{T}^{X,\omega,g}_{\mathrm{an}}(t) := \log T^{\omega,g}_{\mathrm{an,la}}(t) - \log \mathbb{V}_{\omega,g,X}(t) + tR(X, \omega, g) \qquad (9)$$

where $X = -\operatorname{grad}_{g'} \omega$. $\hat{T}_{\mathrm{an}}(t) = \hat{T}^{X,\omega,g}_{\mathrm{an}}(t)$ is referred to as as the *corrected large analytic torsion.*

1.6. **The main results.** The main results of this paper are Theorems 3 and 4 below.

Theorem 3. *Suppose X is a vector field which is MS and has exponential growth and suppose ξ is a Lyapunov cohomology class for X. Let (g, g', ω) be a system as in Theorem 2 so that $X = -\operatorname{grad}_{g'} \omega$ and ω a*

Morse form representing ξ*. Let* $I^{X,\mathcal{O},\omega,g}_{x,y} : [T,\infty) \to \mathbb{R}$ *be the functions defined by* (6)*. Then the Dirichlet series* $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}$ *have finite abscissa of convergence and their Laplace transform are exactly the functions* $I^{X,\mathcal{O},\omega,g}_{x,y}(t)$*. In particular* $I^{X,\mathcal{O},\omega,g}_{x,y}(t)$ *is the restriction of a holomorphic function on* $\{z \in \mathbb{C} | \Re z > T\}$*.*

Theorem 4. *Suppose* X *is a vector field with* ξ *a Lyapunov cohomology class which satisfies MS and NCT. Let* (g, g', ω) *be a system as in Theorem 2 so that* $X = -\operatorname{grad}_{g'} \omega$ *and* ω *a Morse form representing* ξ*. Let* $\log \hat{T}^{X,\omega,g}_{\mathrm{an}}(t)$ *be the function defined by* (9)*.*

If in addition X *has exponential growth and* $H^*(M, t[\omega]) = 0$ *for* t *sufficiently large*[10] *or* X *has strong exponential growth then the Dirichlet series* $\mathbb{Z}_X$ *has finite abscissa of convergence and its Laplace transform is exactly the functions* $\log \hat{T}^{X,\omega,g}_{\mathrm{an}}(t)$*. In particular* $\log \hat{T}^{X,\omega,g}_{\mathrm{an}}(t)$ *is the restriction of a holomorphic function on* $\{z \in \mathbb{C} | \Re z > T\}$*.*

If the conjecture in section 3.2 is true, then the additional hypothesis (exponential growth resp. strong exponential growth) are superfluous.

Remark 1. The Dirichlet series $\mathbb{Z}_X$ depends only on X and $\xi = [\omega]$, while $\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}$ depends only on X and ξ up to multiplication with a constant (with a real number r for the sequence of λ's and with $\epsilon = \pm 1$ for the sequence of a's).

Corollary 1 (J. Marcsik cf. [10] or [6])**.** *Suppose* X *is a vector field with no rest points,* $\xi \in H^1(M;\mathbb{R})$ *a Lyapunov class for* X*,* ω *a closed one form representing* ξ *and let* g *a Riemannian metric on* M*. Suppose all closed trajectories of* X *are non-degenerate and denote by*

$$\log T_{\mathrm{an}}(t) := 1/2 \sum (-1)^{q+1} q \log \det(\Delta^q_\omega(t)).$$

Then

$$\log T_{\mathrm{an}}(t) + t \int_M \omega \wedge X^* \Psi_g$$

is the Laplace transform of the Dirichlet series $\mathbb{Z}_X$ *which counts the set of closed trajectories of* X *with the help of* ξ*.*

Strictly speaking J. Marcsik proved the above result, see [10], in the case $X = -\operatorname{grad}_g \omega$. The same arguments could also yield the result in the generality stated above.

Remark 2. In case that M is the mapping torus of a diffeomorphism $\phi : N \to N$, $M = N_\phi$ whose periodic points are all non-degenerate,

[10] Or, more general, $H^*_{\mathrm{sing}}(M; \Lambda_{\xi,\rho})$ is free for ρ large enough, cf. section 6 for definitions.

the Laplace transform of the Dirichlet series $\mathbb{Z}_X$ is the Lefschetz zeta function $\mathrm{Lef}(Z)$ of ϕ, with the variable Z replaced by e^{-z}.

Theorems 3, 4 and Corollary 1 can be routinely extended to the case of a compact manifolds with boundary.

In section 2 we discuss one of the main topological tools in this paper, the completion of the unstable sets and of the space of unparameterized trajectories, cf. Theorem 5. This theorem was also proved in [5]. In this paper we provide a significant short cut in the proof and a slightly more general formulation.

In section 3 we define the invariant ρ and discuss the relationship with the exponential growth property. Additional results of independent interest pointing toward the truth of the conjecture in section 3.2 are also proved. The results of this section are not needed for the proofs of Theorems 2–4.

The proof of Theorem 1 as stated is contained in [5] and so is the proof of Theorem 2 but in a slightly different formulation and (apparently) less generality. For this reason and for the sake of completeness we will review and complete the arguments (with proper references to [5] when necessary) in section 4. Section 4 contains the proof of Theorem 2 and 3. Section 5 treats the numerical invariant $R(X, \omega, g)$. The proof of Theorem 4 is presented in section 6 and relies on some previous work of Hutchings–Lee, Pajitnov [9], [8], [15] and the work of Bismut–Zhang and Burghelea–Friedlander–Kappeler [1].

2. Topology of the space of trajectories and unstable sets

In this section we discuss the completion of the unstable manifolds and of the manifolds of trajectories to manifolds with corners, which is a key topological tool in this work. The main result, Theorem 5 is of independent interest.

Definition 5. Suppose $\xi \in H^1(M; \mathbb{R})$. We say a covering $\pi : \tilde{M} \to M$ satisfies *property P with respect to* ξ if $\tilde{M}$ is connected and $\pi^*\xi = 0$.

Let X be vector field on a closed manifold M which has $\xi \in H^1(M; \mathbb{R})$ as a Lyapunov cohomology class, see Definition 2. Suppose that X is MS. Let $\pi : \tilde{M} \to M$ be a covering satisfying property P with respect to ξ. Since ξ is Lyapunov there exists a closed one form ω representing ξ and a Riemannian metric g so that $X = -\operatorname{grad}_g \omega$. Since the covering has property P we find $h : \tilde{M} \to \mathbb{R}$ with $\pi^*\omega = dh$.

Denote by $\tilde{X}$ the vector field $\tilde{X} := \pi^* X$. We write $\tilde{\mathcal{X}} = \pi^{-1}(\mathcal{X})$ and $\tilde{\mathcal{X}}_q = \pi^{-1}(\mathcal{X}_q)$. Clearly $\mathrm{Cr}(h) = \pi^{-1}(\mathrm{Cr}(\omega))$ are the zeros of $\tilde{X}$.

Given $\tilde{x} \in \tilde{\mathcal{X}}$ let $i^+_{\tilde{x}} : W^+_{\tilde{x}} \to \tilde{M}$ and $i^-_{\tilde{x}} : W^-_{\tilde{x}} \to \tilde{M}$, denote the one to one immersions whose images define the stable and unstable sets of $\tilde{x}$ with respect to the vector field $\tilde{X}$. The maps $i^{\pm}_{\tilde{x}}$ are actually smooth embeddings because $\tilde{X}$ is gradient like for the function h, and the manifold topology on $W^{\pm}_{\tilde{x}}$ coincides with the topology induced from $\tilde{M}$. Clearly, for any $\tilde{x}$ with $\pi(\tilde{x}) = x$ one can canonically identify $W^{\pm}_{\tilde{x}}$ to $W^{\pm}_x$ and then we have $\pi \circ i^{\pm}_{\tilde{x}} = i^{\pm}_x$.

As the maps $i^-_{\tilde{x}}$ and $i^+_{\tilde{y}}$ are transversal, $\mathcal{M}(\tilde{x}, \tilde{y}) := W^-_{\tilde{x}} \cap W^+_{\tilde{y}}$ is a submanifold of $\tilde{M}$ of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y})$. The manifold $\mathcal{M}(\tilde{x}, \tilde{y})$ is equipped with the action $\mu : \mathbb{R} \times \mathcal{M}(\tilde{x}, \tilde{y}) \to \mathcal{M}(\tilde{x}, \tilde{y})$, defined by the flow generated by $\tilde{X}$. If $\tilde{x} \neq \tilde{y}$ the action μ is free and we denote the quotient $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ by $\mathcal{T}(\tilde{x}, \tilde{y})$. The quotient $\mathcal{T}(\tilde{x}, \tilde{y})$ is a smooth manifold of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y}) - 1$, possibly empty, which, in view of the fact that $\tilde{X} \cdot h = \omega(X) < 0$ is diffeomorphic to the submanifold $h^{-1}(c) \cap \mathcal{M}(\tilde{x}, \tilde{y})$, where c is any regular value of h with $h(\tilde{x}) > c > h(\tilde{y})$.

Note that if $\mathrm{ind}(\tilde{x}) \leq \mathrm{ind}(\tilde{y})$, and $\tilde{x} \neq \tilde{y}$, in view the transversality required by the Hypothesis MS, the manifolds $\mathcal{M}(\tilde{x}, \tilde{y})$ and $\mathcal{T}(\tilde{x}, \tilde{y})$ are empty. We make the following convention: $\mathcal{T}(\tilde{x}, \tilde{x}) := \emptyset$. This is very convenient for now $\mathcal{T}(\tilde{x}, \tilde{y}) \neq \emptyset$ implies $\mathrm{ind}(\tilde{x}) > \mathrm{ind}(\tilde{y})$ and in particular $\tilde{x} \neq \tilde{y}$.

An *unparameterized broken trajectory* from $\tilde{x} \in \tilde{\mathcal{X}}$ to $\tilde{y} \in \tilde{\mathcal{X}}$, is an element of the set $\mathcal{B}(\tilde{x}, \tilde{y}) := \bigcup_{k \geq 0} \mathcal{B}(\tilde{x}, \tilde{y})_k$, where

$$\mathcal{B}(\tilde{x}, \tilde{y})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}) \tag{10}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$ and $\tilde{y}_{k+1} = \tilde{y}$.

For $\tilde{x} \in \tilde{\mathcal{X}}$ introduce the *completed unstable set* $\hat{W}^-_{\tilde{x}} := \bigcup_{k \geq 0} (\hat{W}^-_{\tilde{x}})_k$, where

$$(\hat{W}^-_{\tilde{x}})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k} \tag{11}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$.

To study $\hat{W}^-_{\tilde{x}}$ we introduce the set $\mathcal{B}(\tilde{x}; \lambda)$ of *unparameterized broken trajectories from $\tilde{x} \in \tilde{\mathcal{X}}$ to the level $\lambda \in \mathbb{R}$* as $\mathcal{B}(\tilde{x}; \lambda) := \bigcup_{k \geq 0} \mathcal{B}(\tilde{x}; \lambda)_k$ where

$$\mathcal{B}(\tilde{x}; \lambda)_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times (W^-_{\tilde{y}_k} \cap h^{-1}(\lambda))$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$. Clearly, if $\lambda > h(\tilde{x})$ then $\mathcal{B}(\tilde{x}; \lambda) = \emptyset$.

Since any broken trajectory of $\tilde{X}$ intersects each level of h in at most one point one can view the set $\mathcal{B}(\tilde{x}, \tilde{y})$ resp. $\mathcal{B}(\tilde{x}; \lambda)$ as a subset of $C^0\big([h(\tilde y),h(\tilde x)],\tilde M\big)$ resp. $C^0\big([\lambda,h(\tilde x)],\tilde M\big)$. One parameterizes the points of a broken trajectory by the value of the function h on these points. This leads to the following characterization (and implicitly to a canonical parameterization) of an unparameterized broken trajectory.

Remark 3. Let $\tilde x,\tilde y\in\tilde{\mathcal{X}}$ and set $a:=h(\tilde y)$, $b:=h(\tilde x)$. The parameterization above defines a one to one correspondence between $\mathcal{B}(\tilde x,\tilde y)$ and the set of continuous mappings $\gamma:[a,b]\to\tilde M$, which satisfy the following two properties:

(i) $h(\gamma(s))=a+b-s$, $\gamma(a)=\tilde x$ and $\gamma(b)=\tilde y$.

(ii) There exists a finite collection of real numbers $a=s_0<s_1<\cdots<s_{r-1}<s_r=b$, so that $\gamma(s_i)\in\tilde{\mathcal{X}}$ and γ restricted to (s_i,s_{i+1}) has derivative at any point in the interval (s_i,s_{i+1}), and the derivative satisfies

$$\gamma'(s)=\frac{\tilde X}{-\tilde X\cdot h}\big(\gamma(s)\big).$$

Similarly the elements of $\mathcal{B}(\tilde x;\lambda)$ correspond to continuous mappings $\gamma:[\lambda,b]\to\tilde M$, which satisfies (i) and (ii) with a replaced by λ and the condition $\gamma(b)=\tilde y$ ignored.

We have the following proposition, which can be found in [5].

Proposition 5. *For any $\tilde x,\tilde y\in\tilde{\mathcal{X}}$ and $\lambda\in\mathbb{R}$, the spaces*

(i) *$\mathcal{B}(\tilde x,\tilde y)$ with the topology induced from $C^0\big([h(\tilde y),h(\tilde x)],\tilde M\big)$, and*

(ii) *$\mathcal{B}(\tilde x;\lambda)$ with the topology induced from $C^0\big([\lambda,h(\tilde x)],\tilde M\big)$*

are compact.

Let $\hat i^-_{\tilde x}:\hat W^-_{\tilde x}\to\tilde M$ denote the map whose restriction to $\mathcal{T}(\tilde y_0,\tilde y_1)\times\cdots\times\mathcal{T}(\tilde y_{k-1},\tilde y_k)\times W^-_{\tilde y_k}$ is the composition of the projection on $W^-_{\tilde y_k}$ with $i^-_{\tilde y_k}$. Moreover let $\hat h_{\tilde x}:=h^{\tilde x}\circ\hat i^-_{\tilde x}:\hat W^-_{\tilde x}\to\mathbb{R}$, where $h^{\tilde x}=h-h(\tilde x)$.

Recall that an n–dimensional *manifold with corners* P, is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi:U\to\varphi(U)\subseteq\mathbb{R}^n_+$, where $\mathbb{R}^n_+=\{(x_1,\dots,x_n)\mid x_i\geq 0\}$. The collection of points of P which correspond by some (and hence every) chart to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of P called the *k–corner of P* and it will be denoted by P_k. It has a structure of a smooth $(n-k)$–dimensional manifold. The union $\partial P=P_1\cup P_2\cup\cdots\cup P_n$ is a closed subset which is a topological manifold and $(P,\partial P)$ is a topological manifold with boundary ∂P.

The following theorem was proven in [5] for the case that $\tilde M$ is the minimal covering which has property P.

Theorem 5. *Let M be a closed manifold, X a vector field which is MS and suppose ξ is a Lyapunov class for X. Let $\pi : \tilde{M} \to M$ be a covering which satisfies property P with respect to ξ and let $h : \tilde{M} \to \mathbb{R}$ be a smooth map as above. Then:*

(i) *For any two rest points $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$ the smooth manifold $\mathcal{T}(\tilde{x}, \tilde{y})$ has $\mathcal{B}(\tilde{x}, \tilde{y})$ as a canonical compactification. Moreover there is a canonic way to equip $\mathcal{B}(\tilde{x}, \tilde{y})$ with the structure of a compact smooth manifold with corners, whose k–corner is $\mathcal{B}(\tilde{x}, \tilde{y})_k$ from (10).*

(ii) *For any rest point $\tilde{x} \in \tilde{\mathcal{X}}$, the smooth manifold $W^-_{\tilde{x}}$ has $\hat{W}^-_{\tilde{x}}$ as a canonical completion. Moreover there is a canonic way to equip $\hat{W}^-_{\tilde{x}}$ with the structure of a smooth manifold with corners, whose k–corner coincides with $(\hat{W}^-_{\tilde{x}})_k$ from (11).*

(iii) *$\hat{i}^-_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \tilde{M}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

(iv) *$\hat{h}_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \mathbb{R}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

Proof. In view of Lemma 4 in section 3, the set of Lyapunov classes for X is open in $H^1(M; \mathbb{R})$. So we can find a closed one form ω and a Riemannian metric g such that $X = -\operatorname{grad}_g \omega$ and such that ω has degree of rationality one. Consider the minimal covering on which $\xi = [\omega]$ becomes exact. Since ξ has degree of rationality one[11] the critical values of h form a discrete set. In [5, paragraphs 4.1–4.3] one can find all details of the proof of Theorem 5 for this special ξ and this special covering.

Note that as long as properties (i) through (iii) are concerned they clearly remain true when we pass to the universal covering of M which obviously has property P. One easily concludes that they also remain true for every covering which has property P. So we have checked (i) through (iii) in the general situation.

Next observe that $\hat{h}^-_{\tilde{x}} = h^{\tilde{x}} \circ \hat{i}_{\tilde{x}-}$ is certainly smooth as a composition of two smooth mappings. The properness of $\hat{h}_{\tilde{x}}$ follows from Proposition 5(ii). □

It will be convenient to formulate Theorem 5 without any reference to the covering $\pi : \tilde{M} \to M$ or to lifts $\tilde{x}$ of rest points x.

Let $\xi \in H^1(M; \mathbb{R})$ be a one dimensional cohomology class so that $\pi^*\xi = 0$. As in section 1 denote by $\mathcal{P}_{x,y}$ the set of continuous paths from x to y and by $\hat{\mathcal{P}}^{\tilde{M}}_{x,y}$ the equivalence classes of paths in $\mathcal{P}_{x,y}$ with respect to the following equivalence relation.

[11] Recall that the closed one form has degree of rationality k if the image of $[\omega](\Gamma) \subset \mathbb{R}$ is a free abelian group of rank k.

Definition 6. Two paths $\alpha, \beta \in \mathcal{P}_{x,y}$ are equivalent if for some (and then for any) lift $\tilde{x}$ of x the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β originating from $\tilde{x}$ end up in the same point $\tilde{y}$.

The reader might note that the present situation is slightly more general than the one considered in introduction which correspond to the case the covering π is the Γ_ξ–principal covering with Γ_ξ induced from ξ as described in section 1. For this covering we have $\hat{\mathcal{P}}^{\tilde{M}}_{x,y} = \hat{\mathcal{P}}^{\xi}_{x,y}$.

Note that any two lifts $\tilde{x}, \tilde{y} \in \tilde{M}$ determine an element $\hat{\alpha} \in \hat{\mathcal{P}}^{\tilde{M}}_{x,y}$ and the set of unparameterized trajectories from $\tilde{x}$ to $\tilde{y}$ identifies to the set $\mathcal{T}(x, y, \hat{\alpha})$ of unparameterized trajectories of X from x to y in the class $\hat{\alpha}$.

Theorem 5 can be reformulated in the following way:

Theorem 6 (Reformulation of Theorem 5)**.** *Let M be a smooth manifold, X a smooth vector field which is MS and suppose ξ is a Lyapunov class for X. Let $\tilde{M}$ be a covering of M which has property P with respect to ξ. Then:*

(i) *For any two rest points $x, y \in \mathcal{X}$ and every $\hat{\alpha} \in \hat{\mathcal{P}}^{\tilde{M}}_{x,y}$ the set $\mathcal{T}(x, y, \hat{\alpha})$ has the structure of a smooth manifold of dimension* $\operatorname{ind}(x) - \operatorname{ind}(y) - 1$ *which admits a canonical compactification to a compact smooth manifold with corners $\mathcal{B}(x, y, \hat{\alpha})$. Its k–corner is*

$$\mathcal{B}(x, y, \hat{\alpha})_k = \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_k, y_{k+1}, \hat{\alpha}_k)$$

where the union is over all (tuples of) critical points $y_i \in \mathcal{X}$ and $\hat{\alpha}_i \in \hat{\mathcal{P}}^{\tilde{M}}_{y_i,y_{i+1}}$ with $y_0 = x$, $y_{k+1} = y$ and $\hat{\alpha}_0 \star \cdots \star \hat{\alpha}_k = \hat{\alpha}$.

(ii) *For any rest point $x \in \mathcal{X}$ the smooth manifold W^-_x has a canonical completion to a smooth manifold with corners $\hat{W}^-_x$. Its k–corner is*

$$(W^-_x)_k = \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_{k-1}, y_k, \hat{\alpha}_{k-1}) \times W^-_{y_k}$$

where the union is over all (tuples of) critical points $y_i \in \mathcal{X}$ and $\hat{\alpha}_i \in \hat{\mathcal{P}}^{\tilde{M}}_{y_i,y_{i+1}}$ with $y_0 = x$.

(iii) *The mapping $\hat{i}^-_x : \hat{W}^-_x \to M$ which on $(W^-_x)_k$ is given by the composition of the projection onto $W^-_{y_k}$ with $i^-_{y_k} : W^-_{y_k} \to M$ is smooth, for all $x \in \mathcal{X}$.*

(iv) *Let ω be a closed one form representing ξ. Then the mappings $\hat{h}_x : \hat{W}^-_x \to \mathbb{R}$ which on $(W^-_x)_k$ is given by the composition of the projection onto $W^-_{y_k}$ with $h^\omega_{y_k} : W^-_{y_k} \to \mathbb{R}$ plus $\overline{\omega}(\hat{\alpha}_0 \star \cdots \star \hat{\alpha}_{k-1})$ is smooth and proper, for all $x \in \mathcal{X}$.*

The above results can be easily extended to the case of compact manifolds with boundary.

2.1. **Appendix to section 2.** Given a compact smooth manifold M with boundary ∂M we will consider only *admissible metrics*, i.e. Riemannian metrics g which are *product like* near the boundary. In this case denote by g_0 the induced metric on the boundary. This means that there exists a collar neighborhood $\varphi : \partial M \times [0, \epsilon) \to M$ with φ equal to the identity when restricted to $\partial M \times \{0\}$ and $\varphi^* g = g_0 + ds^2$.

Convention. Unless explicitly mentioned in this paper all the vector fields on a compact manifold with boundary are assumed to be tangent to the boundary and have rest points of Morse type.

Definition 7. The vector field X has $\xi \in H^1(M; \mathbb{R})$ as Lyapunov cohomology class if the following conditions hold:

(i) There exists a closed one form representing ξ and an admissible metric so that $X = -\operatorname{grad}_g \omega$. In particular $X_{\partial M} = -\operatorname{grad}_{g_0} \omega_{\partial M}$, where $\omega_{\partial M}$ denotes the pullback of ω to ∂M.

(ii) If we set

$$\begin{aligned} \mathcal{X}''_- &:= \big\{x \in \mathcal{X} \cap \partial M \,\big|\, \operatorname{ind}_{\partial M}(x) = \operatorname{ind}(x)\big\} \\ \mathcal{X}''_+ &:= \big\{x \in \mathcal{X} \cap \partial M \,\big|\, \operatorname{ind}_{\partial M}(x) = \operatorname{ind}(x) - 1\big\} \end{aligned}$$

then $\mathcal{X}''_-$ resp. $\mathcal{X}''_+$ lie in different components ∂M_- resp. ∂M_+ of M.

This definition implies that $\mathcal{X} = \mathcal{X}' \sqcup \mathcal{X}''$, where $\mathcal{X}'$ is the set of rest points inside M and $\mathcal{X}''$ of the rest points on ∂M which is the same as the set of rest points of $X_{\partial M}$. For $x \in \mathcal{X}''$ denote by $i_x^- : W_x^- \to M$ the unstable manifold with respect to X and by $j_x^- : W_{\partial M,x}^- \to \partial M$ the unstable manifold with respect to $X_{\partial M}$.

Remark 4.

(i) If $x \in \mathcal{X}''_-$ then the unstable manifold of x with respect to X and $X_{\partial M}$ are the same. More precisely $i_x^- : W_x^- \to M$ identifies to $j_x^- : W_{\partial M,x}^- \to \partial M$ followed by the inclusion of $\partial M \subset M$.

(ii) If $x \in \mathcal{X}''_+$ then
  (a) $(W_x^-, W_{\partial M,x}^-)$ is a smooth manifold with boundary diffeomorphic to $(\mathbb{R}^k_+, \mathbb{R}^{k-1})$ with $k = \operatorname{ind}(x)$; and
  (b) $i_x^- : W_x^- \to M$ is transversal to the boundary of M and $i_x^- : (i_x^-)^{-1}(\partial M) \to \partial M$ can be identified to $j_x^- : W_{\partial M,x}^- \to \partial M$.

Theorems 5 and 6 remain true as stated with the following specifications. Set $P_y^- := W_y^- \setminus W_{\partial M,y}^-$ for $y \in \mathcal{X}''_+$, and $P_y^- := W_y^-$ for $y \in \mathcal{X}' \sqcup \mathcal{X}''_-$. For $x \in \mathcal{X}''_+$ the k–corner of $\hat{W}_x^-$ then is

$$(\hat{W}_x^-)_k = (\hat{W}_{\partial M,x}^-)_{k-1} \cup \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_{k-1}, y_k, \hat{\alpha}_{k-1}) \times P_{y_k}^-$$

where the big union is over all (tuples of) $y_i \in \mathcal{X}$ and $\hat{\alpha}_i \in \hat{\mathcal{P}}_{y_i,y_{i+1}}$ with $y_0 = x$.

3. Exponential growth property and the invariant ρ

In this section we introduce for a pair (X, ξ) consisting of a vector field X and a cohomology class $\xi \in H^1(M;\mathbb{R})$ an invariant $\rho(\xi, X) \in \mathbb{R} \cup \{\pm\infty\}$. For the purpose of this paper we are interested in the case this invariant is smaller than ∞. One expects that this is always the case if ξ is Lyapunov for X at least in the case X satisfies MS. If X has ξ as a Lyapunov cohomology class we prove that the exponential growth and $\rho < \infty$ are equivalent. The discussion of this section is not needed for the proofs of Theorem 2, 3 and 4.

Throughout this section M will be a closed manifold and X a vector field with Morse zeros.

3.1. **The invariant ρ.** For a critical point $x \in \mathcal{X}$, i.e. a zero of X, we let $i_x^- : W_x^- \to M$ denote the smooth immersion of the unstable manifold into M. If M is equipped with a Riemannian metric we get an induced Riemannian metric $g_x := (i_x^-)^* g$ on W_x^- thus a volume density $\mu(g_x)$ on W_x^- and hence the spaces $L^p(W_x^-)$, $p \geq 1$. Though the L^p–norm depends on the metric g the space $L^p(W_x^-)$ and its topology does not. Indeed for another Riemannian metric g' on M we find a constant $C > 0$ so that $1/C \leq \frac{g'(X,Y)}{g(X,Y)} \leq C$ for all tangent vectors X and Y which implies $1/C' \leq \frac{\mu(g'_x)}{\mu(g_x)} \leq C'$ for some constant $C' > 0$.

Given a closed 1–form ω on M we let h_x^ω denote the unique smooth function on W_x^- which satisfies $dh_x^\omega = (i_x^-)^*\omega$ and $h_x^\omega(x) = 0$. We are interested in the space of 1–forms for which $e^{h_x^\omega} \in L^1(W_x^-)$. This condition actually only depends on the cohomology class of ω. Indeed we have $h_x^{\omega+df} = h_x^\omega + (i_x^-)^* f - f(x)$ and so $|h_x^{\omega+df} - h_x^\omega| \leq C''$ and $e^{-C''} \leq e^{h_x^{\omega+df}}/e^{h_x^\omega} \leq e^{C''}$ for some constant $C'' > 0$. So we define

$$R_x(X) := \big\{[\omega] \in H^1(M) \bigm| e^{h_x^\omega} \in L^1(W_x^-)\big\}$$

and set $R(X) := \bigcap_{x \in \mathrm{Cr}(X)} R_x(X)$. Let us also define

$$\rho_x(\xi, X) := \inf\{t \in \mathbb{R} \mid t\xi \in R_x(X)\} \in \mathbb{R} \cup \{\pm\infty\}$$

as well as:

$$\rho(\xi, X) := \inf\{t \in \mathbb{R} \mid t\xi \in R(X)\} \in \mathbb{R} \cup \{\pm\infty\}.$$

Observe that $\rho(\xi, X) = \max_{x \in \mathcal{X}} \rho_x(\xi, X)$.

Lemma 1. *The sets $R_x(X)$ and $R(X)$ are convex. Particularly*

$$\rho\big(\lambda\xi_1 + (1-\lambda)\xi_2, X\big) \leq \max\{\rho(\xi_1, X), \rho(\xi_2, X)\}$$

for all $0 \leq \lambda \leq 1$.

Proof. Indeed let $[\omega_0]$, $[\omega_1] \in H^1(M)$, $\lambda \in [0,1]$ and set $\omega_\lambda := \lambda\omega_1 + (1-\lambda)\omega_0$. Then $h_x^{\omega_\lambda} = \lambda h_x^{\omega_1} + (1-\lambda)h_x^{\omega_0}$. For $\lambda \in (0,1)$ we set $p := 1/\lambda > 1$ and $q := 1/(1-\lambda)$. Then $1/p + 1/q = 1$ and by Hölder's inequality

$$\begin{aligned} ||e^{h_x^{\omega_\lambda}}||_1 &= ||e^{\lambda h_x^{\omega_1}} e^{(1-\lambda)h_x^{\omega_0}}||_1 \\ &\leq ||e^{\lambda h_x^{\omega_1}}||_p ||e^{(1-\lambda)h_x^{\omega_0}}||_q \\ &= ||e^{h_x^{\omega_1}}||_1^{\lambda} ||e^{h_x^{\omega_0}}||_1^{1-\lambda} \end{aligned}$$

So, if $[\omega_0]$ and $[\omega_1] \in R_x(X)$ then $[\omega_\lambda] \in R_x(X)$, and thus $R_x(X)$ is convex. As an intersection of convex sets $R(X)$ is convex too. ☐

Next we introduce:

$$B_x(X) := \big\{[\omega] \in H^1(M) \bigm| e^{h_x^\omega} \in L^\infty(W_x^-)\big\}$$

and set $B(X) := \bigcap_{x \in \mathrm{Cr}(X)} B_x(X)$. Note if ξ is a Lyapunov cohomology class for X then $\xi \in B(X)$, cf. Lemma 4 below.

Most obviously we have:

Lemma 2. *The sets $B_x(X)$ and $B(X)$ are convex cones. Moreover we have $R_x(X) + B_x(X) \subseteq R_x(X)$ and $R(X) + B(X) \subseteq R(X)$.*

Next define

$$L(X) := \big\{\xi \in H^1(M) \bigm| \xi \text{ is Lyapunov class for } X\big\}$$

Recall from Definition 2 that $\xi \in L(X)$ if there exists a closed one form ω representing ξ and a Riemannian metric g such that $X = -\operatorname{grad}_g \omega$.

Lemma 3. *Let M be a smooth manifold, X a vector field, ω a closed one form and g a Riemannian metric. Suppose $U \subset M$ is an open set and*

(i) *the vector fields X and $-\operatorname{grad}_g \omega$ agree on U and*
(ii) *$\omega(X) < 0$ on a neighborhood of $M \setminus U$.*

Then there exists a Riemannian metric g' so that:

(i) $X = -\operatorname{grad}_{g'} \omega$
(ii) *g and g' agree on U.*

Proof. Let N be an open neighborhood of $M \setminus U$ so that $\omega(X) < 0$ and therefore $X_x \neq 0$, $x \in N$. For $x \in N$ the tangent space T_x decomposes as the direct sum $T_xM = V_x \oplus [X_x]$ where $[X_x]$ denotes the one dimensional vector space generated by X_x and $V_x = \ker(\omega(x) : T_xM \to \mathbb{R})$. Clearly on U the function $-\omega(X)$ is the square of the length of X_x with respect to the metric g and X_x is orthogonal to V_x and on N it is strictly negative. Define a new Riemannian metric g' on M as follows: For $x \in U$ the scalar product in T_xM is the same as the one defined by g. For $x \in N$ the scalar product on T_xM agrees to the one defined by g but make V_x and $[X_x]$ perpendicular and the length of X_x equal to $\sqrt{-\omega(X)(x)}$. It is clear that the new metric is well defined and smooth. □

Corollary 2. *Let X be a vector field on M and let $\xi \in H^1(M)$. Then ξ is Lyapunov for X if and only if there is a closed one form ω representing ξ and a Riemannian metric g such that the following hold:*

(i) $\omega(X) < 0$ *on* $M \setminus \mathcal{X}$.
(ii) $X = -\operatorname{grad}_g \omega$ *on a neighborhood of* $\mathcal{X}$.

Lemma 4. *The set $L(X) \subseteq H^1(M)$ is open and contained in $B(X)$. Moreover $L(X)$ is a convex cone.*

Proof. The subset $L(X) \subseteq H^1(M)$ is open, for we can change the cohomology class $[\omega]$ by adding a form whose support is disjoint from $\mathcal{X}$ and hence not affecting condition in Corollary 2(ii). If the form we add is sufficiently small the condition in Corollary 2(i) will still be satisfied.

We have $L(X) \subseteq B_x(X)$ for $X = -\operatorname{grad}_g \omega$ implies that h^ω_x attains its maximum at x and is thus bounded from above.

Next note that both conditions (i) and (ii) in Corollary 2 are convex and homogeneous conditions on ω. Thus $L(X)$ is a convex cone. □

Lemma 5. *Suppose $L(X) \cap R(X) \neq \emptyset$. Then every ray of $L(X)$, i.e. a half line starting at the origin which is contained in $L(X)$, intersects $R(X)$.*

Proof. Pick $\xi \in L(X) \cap R(X)$. Since $\xi \in R(X)$, Lemma 4 and Lemma 2 imply:

$$\xi + L(X) \subseteq R(X) + L(X) \subseteq R(X) + B(X) \subseteq R(X) \tag{12}$$

On the other hand $L(X)$ is open and $\xi \in L(X)$, so every ray in $L(X)$ has to intersect $\xi + L(X)$. In view of (12) it has to intersect $R(X)$ too. □

Corollary 3. *Suppose ξ_0 and ξ are Lyapunov for X. Then $\rho(\xi_0, X) < \infty$ implies $\rho(\xi, X) < \infty$.*

3.2. **Exponential growth versus ρ.** Let $x \in \mathcal{X}$ be a zero of X, W_x^- the unstable manifold, let g be a Riemannian metric on M and let $r := \mathrm{dist}(x, \cdot) : W_x^- \to [0, \infty)$ denote the distance to x with respect to the induced metric $g_x = (i_x^-)^* g$ on W_x^-. Clearly $r(x) = 0$. Moreover let $B_s(x) := \{y \in W_x^- | r(y) \leq s\}$ denote the ball of radius s centered at x.

Recall from Definition 1 that X has the exponential growth property at a zero x if there exists a constant $C \geq 0$ such that $\mathrm{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$. Clearly this does not depend on the Riemannian metric g on M even though the constant C will depend on g.

Lemma 6. *Suppose we have $C \geq 0$ such that $\mathrm{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$. Then $e^{-(C+\epsilon)r} \in L^1(W_x^-)$ for every $\epsilon > 0$.*

Proof. We have

$$\int_{W_x^-} e^{-(C+\epsilon)r} = \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r} \tag{13}$$

On $B_{n+1}(x) \setminus B_n(x)$ we have $e^{-(C+\epsilon)r} \leq e^{-(C+\epsilon)n}$ and thus

$$\begin{aligned} \int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r} &\leq \mathrm{Vol}(B_{n+1}(x)) e^{-(C+\epsilon)n} \\ &\leq e^{C(n+1)} e^{-(C+\epsilon)n} = e^C e^{-\epsilon n} \end{aligned}$$

So (13) implies

$$\int_{W_x^-} e^{-(C+\epsilon)r} \leq e^C \sum_{n=0}^{\infty} e^{-\epsilon n} = e^C (1 - e^{-\epsilon})^{-1} < \infty$$

and thus $e^{-(C+\epsilon)r} \in L^1(W_x^-)$. □

Lemma 7. *Suppose we have $C \geq 0$ such that $e^{-Cr} \in L^1(W_x^-)$. Then there exists a constant C_0 such that $\mathrm{Vol}(B_s(x)) \leq C_0 e^{Cs}$ for all $s \geq 0$.*

Proof. We start with the following estimate for $N \in \mathbb{N}$:

$$\begin{aligned}
&\operatorname{Vol}(B_{N+1}(x))e^{-C(N+1)} = \\
&= \sum_{n=0}^{N} \operatorname{Vol}(B_{n+1}(x))e^{-C(n+1)} - \operatorname{Vol}(B_n(x))e^{-Cn} \\
&\leq \sum_{n=0}^{\infty} \big(\operatorname{Vol}(B_{n+1}(x)) - \operatorname{Vol}(B_n(x))\big)e^{-C(n+1)} \\
&= \sum_{n=0}^{\infty} \operatorname{Vol}\big(B_{n+1}(x) \setminus B_n(x)\big)e^{-C(n+1)} \\
&\leq \sum_{n=0}^{\infty} \int_{B_{n+1}(x)\setminus B_n(x)} e^{-Cr} = \int_{W_x^-} e^{-Cr}
\end{aligned}$$

Given $s \geq 0$ we choose an integer N with $N \leq s \leq N+1$. Then $\operatorname{Vol}(B_s(x))e^{-Cs} \leq \operatorname{Vol}(B_{N+1}(x))e^{-CN} = e^C \operatorname{Vol}(B_{N+1}(x))e^{-C(N+1)}$. So the computation above shows

$$\operatorname{Vol}(B_s(x))e^{-Cs} \leq e^C \int_{W_x^-} e^{-Cr} =: C_0 < \infty$$

and thus $\operatorname{Vol}(B_s(x)) \leq C_0 e^{Cs}$ for all $s \geq 0$. □

As immediate consequence of the two preceding lemmas we have

Proposition 6. *Let x be a zero of X. Then the following are equivalent:*

(i) *X has the exponential growth property at x with respect to one (and hence every) Riemannian metric on M.*
(ii) *For one (and hence every) Riemannian metric on M there exists a constant $C \geq 0$ such that $e^{-Cr} \in L^1(W_x^-)$.*

Let g be a Riemannian metric on M, ω a closed one form and consider $X = -\operatorname{grad}_g \omega$. Assume X has Morse zeros and let x be one of them. Recall that we have a smooth function $h_x^\omega : W_x^- \to (-\infty, 0]$ defined by $(i_x^-)^*\omega = dh_x^\omega$ and $h_x^\omega(x) = 0$. The next two lemmas tell, that $-h_x^\omega : W_x^- \to [0,\infty)$ is comparable with $r : W_x^- \to [0,\infty)$.

Lemma 8. *In this situation there exists a constant $C_{\omega,g} \geq 0$ such that $r \leq 1 - C_{\omega,g} h_x^\omega$ on W_x^-.*

Proof. The proof is exactly the same as the one in [5, Lemma 3(2)]. Note that the Smale condition is not used there. □

Lemma 9. *In this situation there exists a constant $C'_{\omega,g} \geq 0$ such that $-h_x^\omega \leq C'_{\omega,g} r$.*

Proof. Let $\gamma : [0,1] \to W_x^-$ be any path starting at $\gamma(0) = x$. For simplicity set $h := h_x^\omega$. Since $h(x) = 0$ we find

$$|h(\gamma(1))| = \Big|\int_0^1 (dh)(\gamma'(t))dt\Big| \leq ||\omega||_\infty \int_0^1 |\gamma'(t)|dt = ||\omega||_\infty \operatorname{length}(\gamma)$$

with $||\omega||_\infty$ the supremums norm of ω. We conclude

$$||\omega||_\infty r(\gamma(1)) = ||\omega||_\infty \operatorname{dist}(x, \gamma(1)) \geq |h(\gamma(1))| \geq -h(\gamma(1))$$

and thus $-h \leq C'_{\omega,g} r$ with $C'_{\omega,g} := ||\omega||_\infty$. □

Let us collect what we have found so far.

Proposition 7. *Suppose ξ is Lyapunov for X and let x be a zero of X. Then the following are equivalent.*

(i) *X has the exponential growth property at x with respect to one (and hence every) Riemannian metric on M.*
(ii) *For one (and hence every) Riemannian metric on M there exists a constant $C \geq 0$ such that $e^{-Cr} \in L^1(W_x^-)$.*
(iii) *$\rho_x(\xi, X) < \infty$.*

Proof. The equivalence of (i) and (ii) was established in Proposition 6 without the assumption that ξ is Lyapunov for X. The implication (ii) $\Rightarrow$ (iii) follows from Lemma 8; the implication (iii) $\Rightarrow$ (ii) from Lemma 9. □

Note that this again implies Corollary 3. We expect that these equivalent statements hold true, at least in the generic situation. More precisely:

Conjecture (Exponential growth)**.** *Let g be a Riemannian metric on a closed manifold M, ω a closed one form (and assume $X = -\operatorname{grad}_g \omega$ is Morse–Smale.) Let x be a zero and let $i_x^- : W_x^- \to M$ denote its unstable manifold. Let $\operatorname{Vol}(B_s(x))$ denote the volume of the ball $B_s(x) \subseteq W_x^-$ of radius s centered at $x \in W_x^-$ with respect to the induced Riemannian metric $(i_x^-)^* g$ on W_x^-. Then there exists a constant $C \geq 0$ such that $\operatorname{Vol}(B_s(x)) \leq e^{Cs}$ for all $s \geq 0$.*

3.3. **A criterion for exponential growth.** The rest of the section is dedicated to a criterion which guarantees that the exponential growth property, and hence $\rho < \infty$, holds in simple situations.

Suppose $x \in \mathcal{X}_q$. Let $B \subseteq W_x^-$ denote a small ball centered at x. The submanifold $i_x^-\big(W_x^- \setminus B\big) \subseteq M$ gives rise to a submanifold $\operatorname{Gr}(W_x^- \setminus B) \subseteq \operatorname{Gr}_q(TM)$ in the Grassmannified tangent bundle, the space of q–dimensional subspaces in TM. For a critical point $y \in \mathcal{X}$ we define

$$K_x(y) := \mathrm{Gr}_q(T_yW_y^-) \cap \overline{\mathrm{Gr}(W_x^- \setminus B)}$$

where $\mathrm{Gr}_q(T_yW_y^-) \subseteq \mathrm{Gr}_q(T_yM) \subseteq \mathrm{Gr}_q(TM)$. Note that $K_x(y)$ does not depend on the choice of B.

Remark 5.

(i) Even though we removed a neighborhood of x from the unstable manifold W_x^- the set $K_x(x)$ need not be empty. However if we did not remove B the set $K_x(x)$ would never be vacuous for trivial reasons.

(ii) If $q = \mathrm{ind}(x) > \mathrm{ind}(y)$ we have $K_x(y) = \emptyset$, for $\mathrm{Gr}_q(T_yW_y^-) = \emptyset$ since $q > \dim(T_yW_y^-) = \mathrm{ind}(y)$.

(iii) If $\dim(M) = n = q = \mathrm{ind}(x)$ we always have $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$.

Proposition 8. *Let ξ be Lyapunov for X and suppose $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$. Then $\rho_x(\xi, X) < \infty$.*

We start with a little

Lemma 10. *Let (V, g) be an Euclidean vector space and $V = V^+ \oplus V^-$ an orthogonal decomposition. For $\kappa \geq 0$ consider the endomorphism $A_\kappa := \kappa\,\mathrm{id} \oplus -\mathrm{id} \in \mathrm{End}(V)$ and the function*

$$\delta^{A_\kappa} : \mathrm{Gr}_q(V) \to \mathbb{R}, \quad \delta^{A_\kappa}(W) := \mathrm{tr}_{g|W}(p_W^\perp \circ A_\kappa \circ i_W),$$

where $i_W : W \to V$ denotes the inclusion and $p_W^\perp : V \to W$ the orthogonal projection. Suppose we have a compact subset $K \subseteq \mathrm{Gr}_q(V)$ for which $\mathrm{Gr}_q(V^+) \cap K = \emptyset$. Then there exists $\kappa > 0$ and $\epsilon > 0$ with $\delta^{A_\kappa} \leq -\epsilon$ on K.

Proof. Consider the case $\kappa = 0$. Let $W \in \mathrm{Gr}_q(V)$ and choose a $g|_W$–orthonormal base $e_i = (e_i^+, e_i^-) \in V^+ \oplus V^-$, $1 \leq i \leq q$, of W. Then

$$\delta^{A_0}(W) = \sum_{i=1}^q g(e_i, A_0 e_i) = -\sum_{i=1}^q g(e_i^-, e_i^-).$$

So we see that $\delta^{A_0} \leq 0$ and $\delta^{A_0}(W) = 0$ iff $W \in \mathrm{Gr}_q(V^+)$. Thus $\delta^{A_0}|_K < 0$. Since δ^{A_κ} depends continuously on κ and since K is compact we certainly find $\kappa > 0$ and $\epsilon > 0$ so that $\delta^{A_\kappa}|_K \leq -\epsilon$. □

Proof of Proposition 8. Let $S \subseteq W_x^-$ denote a small sphere centered at x. Let $\tilde{X} := (i_x^-)^* X$ denote the restriction of X to W_x^- and let Φ_t denote the flow of $\tilde{X}$ at time t. Then

$$\varphi : S \times [0, \infty) \to W_x^-, \quad \varphi(x, t) = \varphi_t(x) = \Phi_t(x)$$

parameterizes W_x^- with a small neighborhood of x removed.

Let $\kappa > 0$. For every $y \in \mathcal{X}$ choose a chart $u_y : U_y \to \mathbb{R}^n$ centered at y so that

$$X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i}.$$

Let g be a Riemannian metric on M which restricts to $\sum_i du_y^i \otimes du_y^i$ on U_y and set $g_x := (i_x^-)^* g$. Then

$$\nabla X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i}.$$

In view of our assumption $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$ Lemma 10 permits us to choose $\kappa > 0$ and $\epsilon > 0$ so that after possibly shrinking U_y we have

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla \tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0,\infty)) \cap (i_x^-)^{-1}\Big(\bigcup_{y \in \mathcal{X}} U_y\Big). \tag{14}$$

Next choose a closed 1–form ω so that $[\omega] = \xi$ and $\omega(X) < 0$ on $M \setminus \mathcal{X}$. Choose $\tau > 0$ so that

$$\tau\omega(X) + \mathrm{ind}(x)||\nabla X||_g \leq -\epsilon < 0 \quad \text{on} \quad M \setminus \bigcup_{y \in \mathcal{X}} U_y. \tag{15}$$

Using $\tau\tilde{X} \cdot h_x^\omega \leq 0$ and

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla\tilde{X}) \leq \mathrm{ind}(x)||\nabla\tilde{X}||_{g_x} \leq \mathrm{ind}(x)||\nabla X||_g$$

(14) and (15) yield

$$\tau\tilde{X} \cdot h_x^\omega + \mathrm{div}_{g_x}(\tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0,\infty)). \tag{16}$$

Choose an orientation of W_x^- and let μ denote the volume form on W_x^- induced by g_x. Consider the function

$$\psi : [0,\infty) \to \mathbb{R}, \quad \psi(t) := \int_{\varphi(S \times [0,t])} e^{\tau h_x^\omega} \mu \geq 0.$$

For its first derivative we find

$$\psi'(t) = \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu > 0$$

and for the second derivative, using (16),

$$\begin{aligned}\psi''(t) &= \int_{\varphi_t(S)} \big(\tau\tilde X\cdot h^\omega_x+\operatorname{div}_{g_x}(\tilde X)\big)e^{\tau h^\omega_x} i_{\tilde X}\mu\\ &\leq -\epsilon\int_{\varphi_t(S)} e^{\tau h^\omega_x} i_{\tilde X}\mu = -\epsilon\psi'(t).\end{aligned}$$

So $(\ln\circ\psi')'(t)\leq -\epsilon$ hence $\psi'(t)\leq\psi'(0)e^{-\epsilon t}$ and integrating again we find

$$\psi(t)\leq\psi(0)+\psi'(0)(1-e^{-\epsilon t})/\epsilon\leq\psi'(0)/\epsilon.$$

So we have $e^{\tau h^\omega_x}\in L^1\big(\varphi(S\times[0,\infty))\big)$ and hence $e^{\tau h^\omega_x}\in L^1(W^-_x)$ too. We conclude $\rho_x(\xi,X)\leq\tau<\infty$. □

Remark 6.

(i) In view of Remark 5(iii) Proposition 8 implies $\rho_x(\xi,X)<\infty$ whenever ξ is Lyapunov for X and $\operatorname{ind}(x)=\dim(M)$. However there is a much easier argument for this special case. Indeed, in this case W^-_x is an open subset of M and therefore its volume has to be finite. Since ξ is Lyapunov for X we immediately even get $\rho_x(\xi,X)\leq 0$.

(ii) In the case $\operatorname{ind}(x)=1$ we certainly have $\rho_x(\xi,X)\leq 0$.

(iii) Throughout the whole section we did not make use of a Morse–Smale condition.

Using Proposition 7, Proposition 8, Remark 5(ii) and Remark 6(ii) we get

Corollary 4. *Suppose ξ is Lyapunov for X and x a zero of X. If $1<\operatorname{ind}(x)<\dim(M)$ assume moreover that $K_x(y)=\emptyset$ for all zeros y of X with $\dim(M)>\operatorname{ind}(y)\geq\operatorname{ind}(x)$. Then X has the exponential growth property at x and $\rho_x(\xi,X)<\infty$.*

4. Proof of Theorems 2 and 3

Let X be a vector field with $\xi\in H^1(M;\mathbb R)$ a Lyapunov cohomology class. Recall that in Section 1 we have defined the instanton counting function (or the Novikov incidence) $\mathbb I^{X,\mathcal O,\xi}_{x,y}:\hat{\mathcal P}^\xi_{x,y}\to\mathbb Z$, cf. (1).

The following proposition is a reformulation of a basic observation made by S.P. Novikov [13] in order to define his celebrated complex.

Proposition 9.

(i) *For any $x\in\mathcal X_q$, $y\in\mathcal X_{q-1}$ and every real number R the set*

$$\big\{\hat\alpha\in\hat{\mathcal P}^\xi_{x,y}\mid \mathbb I^{X,\mathcal O,\xi}_{x,y}(\hat\alpha)\neq 0, \overline\omega(\hat\alpha)\geq R\big\}$$

is finite. Here ω is any closed one form representing ξ.

(ii) *For any* $x \in \mathcal{X}_q$, $z \in \mathcal{X}_{q-2}$ *and* $\hat\gamma \in \hat{\mathcal{P}}^\xi_{x,z}$ *one has*

$$\sum \mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha) \cdot \mathbb{I}^{X,\mathcal{O},\xi}_{y,z}(\hat\beta) = 0. \tag{17}$$

where the sum is over all $y \in \mathcal{X}_{q-1}$, $\hat\alpha \in \hat{\mathcal{P}}^\xi_{x,y}$ *and all* $\hat\beta \in \hat{\mathcal{P}}^\xi_{y,z}$ *with* $\hat\alpha \star \hat\beta = \hat\gamma$.

Formula (17) *implicitly states that the left side of the equality contains only finitely many non-zero terms.*

Proposition 9 above is equivalent to Theorem 2 parts 1 and 2 in [5]. The proof, originally due to Novikov can be also found in [5].

The following proposition will be the main tool in the proof of Theorem 2.

Proposition 10. *Suppose* $t \in \mathbb{R}$, ω *a closed one form representing* ξ *and* $t > \rho(\xi, X)$. *Then:*

(i) *For every* $a \in \Omega^q(M)$ *and every* $x \in \mathcal{X}_q$ *the integral*

$$\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)(a)(x) := \int_{W^-_x} e^{th_x}(i^-_x)^* a$$

converges absolutely.[12] *In particular it defines a linear map* $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^q(M) \to \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$.

(ii) *The map* $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^q(M) \to \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$ *is surjective and*

$$\mathrm{Int}^{q+1}_{X,\omega,\mathcal{O}}(t)(d_\omega(t)a)(x) = \sum_{y\in\mathcal{X}_q,\ \hat\alpha\in\hat{\mathcal{P}}^\xi_{x,y}} e^{t\overline{\omega}(\hat\alpha)}\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha)\,\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)(a)(y) \tag{18}$$

where the right side of (18) *is a potentially infinite sum which is convergent.*

The proof of Proposition 10 is given in details in [5] section 5, (cf. Proposition 4) and uses in an essential way Theorem 5 and Stokes' theorem. Particular care is necessary in view of the fact that W^-_x are not compact. The integration has to be performed on a non compact manifold and Stokes' theorem applied to non-compact manifolds with corners.

The proof of Theorem 2 boils down to the verification of the following claims:

[12] Recall that for an oriented n–dimensional manifold N and $a \in \Omega^n(N)$ one has $|a| := |a'|\,\mathrm{Vol} \in \Omega^n(M)$, where $\mathrm{Vol} \in \Omega^n(N)$ is any volume form and $a' \in C^\infty(N, \mathbb{R})$ is the unique function satisfying $a = a' \cdot \mathrm{Vol}$. The integral $\int_N a$ is called absolutely convergent, if $\int_N |a|$ converges.

Claim 1. For any $t > \sup\{\rho(\xi, X), T\}$, $x \in \mathcal{X}_{q+1}$ and $y \in \mathcal{X}_q$ the possibly infinite sum

$$\sum_{\hat\alpha \in \hat{\mathcal{P}}^\xi_{x,y}} \mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha) e^{t\overline{\omega}(\hat\alpha)}$$

is convergent and the formula

$$\delta^q_{X,\omega,\mathcal{O}}(t)(E_y) = \sum_{x\in\mathcal{X}_{q+1},\ \hat\alpha \in \hat{\mathcal{P}}^\xi_{x,y}} \big(\mathbb{I}^{X,\mathcal{O},\xi}_{x,y}(\hat\alpha) e^{t\overline{\omega}(\hat\alpha)}\big) E_x \tag{19}$$

defines a linear map $\delta^q_{X,\omega,\mathcal{O}}(t) : C^q(X) = \mathrm{Maps}(\mathcal{X}_q, \mathbb{R}) \to C^{q+1}(X) = \mathrm{Maps}(\mathcal{X}_{q+1}, \mathbb{R})$ which makes $(C^*(X), \delta^*_{X,\omega,\mathcal{O}}(t))$ a smooth (actually analytic) family of cochain complexes of finite dimensional Euclidean spaces. Recall that $\{E_x\}_{x\in\mathcal{X}}$ denotes the characteristic functions of $x \in \mathcal{X}$ and $\{E_x\}_{x\in\mathcal{X}}$ provide the canonical base of $C^*(X)$ which, implicitly equips each component $C^q(X)$ with a scalar product, the unique scalar product which makes this base orthonormal. Recall also that $\overline{\omega} : \hat{\mathcal{P}}^\xi_{x,y} \to \mathbb{R}$ was defined in section 1 before Proposition 1 and makes sense even when ω is not a representative of ξ but still, its pull back on $\tilde M$ is exact.

Claim 2. The linear maps $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ are surjective and define a morphism of cochain complexes.

Claim 3. There exists T larger than $\rho(\omega, X)$ so that for $t > T$ the linear map $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ when restricted to $\Omega^q_{\mathrm{sm}}(M)(t)$ is an isomorphism and actually an $O(1/t)$–isometry.

Everything but the $O(1/t)$–isometry statement in Claim 3 is a straight forward consequence of Theorem 1 and Proposition 10 above. To check this part of Claim 3 we have to go back to the proof of Theorems 3 and 4 in [5], section 6. We observe that if t is large enough the restriction of $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ to the subspace $H_1(t) \subset \Omega^q(M)$ defined in [5], section 4 page 172 (cf. Proof of Theorem 3) is surjective and then by Lemma 7 in [5] so is the restriction of $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ to $\Omega^q_{\mathrm{sm}}(M)(t)$. This because $H_1(t)$ and $\Omega^q_{\mathrm{sm}}(M)(t)$ are, by Lemma 7 in [5] section 6, as close as we want for t large enough. Since the spaces $\Omega^q_{\mathrm{sm}}(M)(t)$ and $C^q(X)$ have the same finite dimension, by the surjectivity in Claim 2, $\mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)$ is an isomorphism and, as shown in [5] section 4 page 172 an $O(1/t)$–isometry. We take as the base $E^{\mathcal{O}}_x(t)$ the differential forms $E^{\mathcal{O}}_x(t) = \mathrm{Int}^q_{X,\omega,\mathcal{O}}(t)^{-1}(E_x)$. This finishes the proof of Theorem 2. Theorem 3 is a consequence of Theorem 2 and of Claim 3.

We conclude this section with the following remarks. Let X be a vector field which has ξ as Lyapunov cohomology class. Suppose X

satisfies MS and $\rho(\xi, X) < \infty$. Let ω be a closed one form representing ξ.

For $t > \rho(\xi, X)$ the finite dimensional vector spaces

$$C^q(X) := \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$$

and the linear maps

$$\delta^q_{X,\omega,\mathcal{O}}(t) : \mathrm{Maps}(\mathcal{X}_q, \mathbb{R}) \to \mathrm{Maps}(\mathcal{X}_{q+1}, \mathbb{R})$$

defined by

$$\delta^q_{X,\omega,\mathcal{O}}(t)(E_x) := \sum_{y \in \mathcal{X}_{q+1},\ \hat{\alpha} \in \mathcal{P}^{\xi}_{y,x}} \mathbb{I}^{X,\mathcal{O},\xi}_{y,x}(\hat{\alpha}) e^{t\overline{\omega}(\hat{\alpha})} E_y$$

give rise to a cochain complex of finite dimensional vector spaces

$$\mathbb{C}^*(X, \omega, \mathcal{O})(t) := \{C^q(X), \delta^q_{X,\omega,\mathcal{O}}(t)\},$$

and to a morphism of such complexes:

$$\mathrm{Int}^*_{X,\mathcal{O},\omega}(t) : (\Omega^*(M), d_\omega(t)) \to \mathbb{C}^*(X, \omega, \mathcal{O})(t)$$

One can show (implicit in Theorem 3) that $\mathrm{Int}^*_{X,\mathcal{O},\omega}(t)$ induces an isomorphism in cohomology.[13]

Let ω_1 and ω_2 be two closed one forms representing the same cohomology class ξ and let $f : M \to \mathbb{R}$ be a smooth function so that $\omega_1 - \omega_2 = df$. The collections of linear maps

$$m^q_f(t) : \Omega^q(M) \to \Omega^q(M), \qquad m^q_f(t)(a) := e^{tf} a,$$

where $a \in \Omega^q(M)$, and

$$s^q_f(t) : \mathbb{C}^q(X, \omega_1, \mathcal{O}) \to \mathbb{C}^q(X, \omega_2, \mathcal{O}), \quad s^q_f(t)(E_x) := e^{tf(x)} E_x,$$

where $E_x \in \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$ denotes the characteristic function of $x \in \mathcal{X}_q$, define morphisms of cochain complexes making the diagram

$$\begin{array}{ccc} \big(\Omega^*(M), d_{\omega_1}(t)\big) & \xrightarrow{m^*_f(t)} & \big(\Omega^*(M), d_{\omega_2}(t)\big) \\ {\scriptstyle \mathrm{Int}^*_{X,\mathcal{O},\omega_1}(t)} \downarrow & & \downarrow {\scriptstyle \mathrm{Int}^*_{X,\mathcal{O},\omega_2}(t)} \\ \mathbb{C}^*(X, \mathcal{O}, \omega_1)(t) & \xrightarrow{s^*_f(t)} & \mathbb{C}^*(X, \mathcal{O}, \omega_2)(t) \end{array}$$

commutative for any $t > \rho(\xi, X)$.

Indeed because $h^1_x - h^2_x = (f - f(x)) \cdot i^-_x$ is bounded,[14] $\int_{W^-_x} e^{th^2_x} (i^-_x)^* a$ is absolutely convergent iff $\int_{W^-_x} e^{th^1_x} (i^-_x)^* a$ is.

[13] This will not be used in this paper but in the case that X is the gradient of a smooth function (i.e. coming from a generalized triangulation) in which case the statement is a consequence of deRham's theorem with local coefficients.

[14] h^1_x is associated to ω_1 and h^2_x is associated to ω_2.

5. The regularization $R(X,\omega,g)$

In this section we discuss the numerical invariant $R(X,\omega,g)$ associated to a vector field X, a closed one form ω and a Riemannian metric g. The invariant is defined by a possibly divergent integral but regularizable and is implicit in the work of [1]. More on this invariant is contained in [6].

In section 1.5 we have considered the Mathai–Quillen form $\Psi_g \in \Omega^{n-1}(TM\setminus M;\mathcal{O}_M)$ of an n–dimensional Riemannian manifold (M,g). The Mathai–Quillen form (see [11]) is actually associated to a pair $\tilde\nabla = (\nabla,\mu)$ consisting of a connection and a parallel Euclidean structure on a vector bundle $E\to M$. If E is of rank k it is a $k-1$ form $\Psi_{\tilde\nabla}\in\Omega^{k-1}(E\setminus M;\mathcal{O}_E)$ with values in the pull back of the orientation bundle $\mathcal{O}_E$ of E to the total space of E. Here M is identified with the zero section in the bundle E. If g is a Riemannian metric let $\tilde\nabla^g := (\nabla^g, g)$ denote the Levi–Civita pair associated to g and write $\Psi_g := \Psi_{\tilde\nabla^g}$.

The Mathai–Quillen form has the following properties:

(i) For the Euler form $E_{\tilde\nabla}\in\Omega^k(M;\mathcal{O}_E)$ associated to $\tilde\nabla$ we have $d\Psi_{\tilde\nabla} = \pi^* E_{\tilde\nabla}$.

(ii) For two $\tilde\nabla^1$ and $\tilde\nabla^2$ we have $\Psi_{\tilde\nabla^2}-\Psi_{\tilde\nabla^1} = \pi^*\operatorname{cs}(\tilde\nabla^1,\tilde\nabla^2)$ modulo exact forms. Here $\operatorname{cs}(\tilde\nabla^1,\tilde\nabla^2)\in\Omega^{k-1}(M;\mathcal{O}_E)/d\Omega^{k-2}(M;\mathcal{O}_E)$ is the Chern–Simon invariant.

(iii) For every $x\in M$ the form $-\Psi_{\tilde\nabla}$ restricts to the standard generator of $H^{k-1}(E_x\setminus 0;\mathcal{O}_E)$, where E_x denotes the fiber over $x\in M$. Note that the restriction of $-\Psi_{\tilde\nabla}$ is closed by (i).

(iv) Suppose $E = TM$, $\tilde\nabla^g$ is the Levi–Civita pair, and suppose that on the open set U we have coordinates $x^1,\dots,x^n$ in which the Riemannian metric $g|_U$ is given by $g_{ij}=\delta_{ij}$. Then, with respect to the induced coordinates $x^1,\dots,x^n,\xi^1,\dots,\xi^n$ on TU, the form Ψ_g is given by

$$\Psi_g = \frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_i(-1)^i\frac{\xi^i}{\left(\sum_j(\xi^j)^2\right)^{n/2}}d\xi^1\wedge\cdots\wedge\widehat{d\xi^i}\wedge\cdots\wedge d\xi^n,$$

cf. [11].

Let X be a vector field on M, i.e. a section of the tangent bundle TM. We suppose that it has only isolated zeros, that is its zero set $\mathcal{X}$ is a discrete subset of M. The vector field defines an integer valued map $\mathrm{IND}:\mathcal{X}\to\mathbb{Z}$, where $\mathrm{IND}(x)$ denotes the Hopf index of the vector field X at the zero $x\in\mathcal{X}$. This integer $\mathrm{IND}(x)$ is the degree of the map $(U,U\setminus x)\to(T_xM,T_xM\setminus 0)$ obtained by composing $X:U\to TU$

with the projection $p : TU \to T_xM$ induced by a local trivialization of the tangent bundle on a small disk $U \subseteq M$ centered at x.

Choose coordinates around x so that we can speak of the disk U_ϵ with radius $\epsilon > 0$ centered x. It is well known that we have:

$$\mathrm{IND}_x = -\lim_{\epsilon \to 0} \int_{\partial U_\epsilon} X^* \Psi_g \tag{20}$$

Indeed, by (ii) we may assume that g is flat on U_ϵ. Thus $E_g = 0$ and Ψ_g is closed on U_ϵ by (i). Using (iii) we see that $-\Psi_g$ gives the standard generator of $H^{n-1}(TU_\epsilon \backslash U_\epsilon; \mathcal{O}_{U_\epsilon})$ and thus certainly $\mathrm{IND}(x) = -\int_{\partial U_\epsilon} X^* \Psi_g$.

The vector field X has its rest points (zeros) non-degenerate and in particular isolated, if the map X is transversal to the zero section in TM. In this case $\mathcal{X}$ is an oriented zero dimensional manifold, whose orientation is specified by $\mathrm{IND}(x)$. Moreover we have

$$\mathrm{IND}(x) = \operatorname{sign} \det H \in \{\pm 1\},$$

where $H : T_xM \to T_xM$ denotes the Hessian. Particularly, if there exist coordinates $x^1, \dots, x^n$ centered at x so that

$$X = -\sum_{1 \leq i \leq k} x^i \tfrac{\partial}{\partial x^i} + \sum_{i > k} x^i \tfrac{\partial}{\partial x^i} \tag{21}$$

we get $\mathrm{IND}(x) = (-1)^k$.

Let X^1 and X^2 be two vector fields and $\mathbb{X} := \{X_s\}_{s \in [-1,1]}$ a smooth homotopy from X^1 to X^2, i.e. $X_s = X^1$ for $s \leq -1 + \epsilon$ and $X_s = X^2$ for $s \geq 1 - \epsilon$. The homotopy is called non-degenerate if the map $\mathbb{X} : [-1, 1] \times M \to TM$ defined by $\mathbb{X}(s, x) := X_s(x)$ is transversal to the zero section of TM. In this case necessarily X^1 and X^2 are vector fields with non-degenerate zeros and so are all but finitely many X_s. Moreover all X_s have isolated zeros with indexes in $\{0, 1, -1\}$ and the zero set $\tilde{\mathcal{X}}$ of $\mathbb{X}$ is an oriented one dimensional smooth submanifold of $[-1, 1] \times M$. Note that we have

$$\partial \tilde{\mathcal{X}} = \sum_{y \in \mathcal{X}^2} \mathrm{IND}(y) y - \sum_{x \in \mathcal{X}^1} \mathrm{IND}(x) x.$$

If $\mathbb{X}'$ is a second homotopy joining X^1 with X^2 then $\tilde{\mathcal{X}}' - \tilde{\mathcal{X}}$ is the boundary of a smooth 2–cycle. Indeed, if we choose a homotopy of homotopies joining $\mathbb{X}$ with $\mathbb{X}'$ which is transversal to the zero section, then its zero set will do the job.

Given a closed one form ω on M denote by

$$I_{\mathbb{X},\omega} := \int_{\tilde{\mathcal{X}}} p_2^* \omega,$$

where $p_2 : \tilde{\mathcal{X}} \to M$ denotes the restriction of the projection $[-1,1] \times M \to M$. It follows from the previous paragraph that $I_{\mathbb{X},\omega}$ does not depend on the homotopy $\mathbb{X}$ — only on X^1, X^2 and ω, and therefor will be denoted from now on by $I(X^1, X^2, \omega)$.

Remark 7. If there exists a simply connected open set $V \subset M$ so that $\mathcal{X}_s \subset V$ for all $s \in [-1,1]$ then one can calculate $I_{\mathbb{X},\omega}$ as follows: Choose a smooth function $f : V \to \mathbb{R}$ so that $\omega|_V = df$. Then

$$I_{\mathbb{X},\omega} = \sum_{y \in \mathcal{X}^2} \mathrm{IND}(y) f(y) - \sum_{x \in \mathcal{X}^1} \mathrm{IND}(x) f(x).$$

The proof of this equality is a straight forward application of Stokes' theorem.

With these considerations we will describe now the *regularization* referred to in Section 1.5, cf. (8). First note that for a non-vanishing vector field X, a closed one form ω and a Riemannian metric g the quantity

$$R(X,\omega,g) := \int_M \omega \wedge X^*\Psi_g \tag{22}$$

has the following two properties.

$$R(X,\omega + df, g) - R(X,\omega,g) = -\int_M f E_g$$

for every smooth function f. If g^1 and g^2 are two Riemannian metrics then

$$R(X,\omega,g^2) - R(X,\omega,g^1) = \int_M \omega \wedge \mathrm{cs}(g^1,g^2)$$

where $\mathrm{cs}(g^1,g^2) = \mathrm{cs}(\tilde{\nabla}^{g^1}, \tilde{\nabla}^{g^2})$. This follows from properties (i) and (ii) of the Mathai–Quillen form.

If X has zeros, then the form $\omega \wedge X^*\Psi_g$ is well defined on $M \setminus \mathcal{X}$ but the integral $\int_{M\setminus\mathcal{X}} \omega \wedge X^*\Psi_g$ might be divergent unless ω is zero on a neighborhood of $\mathcal{X}$.

We will define below a regularization of the integral $\int_{M\setminus\mathcal{X}} \omega \wedge X^*\Psi_g$ which in case $\mathcal{X} = \emptyset$ is equal to the integral (22). For this purpose we choose a smooth function $f : M \to \mathbb{R}$ so that the closed 1–form $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$, and put

$$R(X,\omega,g;f) := \int_{M\setminus\mathcal{X}} \omega' \wedge X^*\Psi_g - \int_M f E_g + \sum_{x\in\mathcal{X}} \mathrm{IND}(x) f(x) \tag{23}$$

Proposition 11. *The quantity $R(X,\omega,g;f)$ is independent of f.*

Therefore $R(X,\omega,g;f)$ can be denoted by $R(X,\omega,g)$ and will be called the *regularization* of $\int_{M\setminus\mathcal{X}}\omega\wedge X^*\Psi_g$.

Proof. Suppose f^1 and f^2 are two functions such that $\omega^i := \omega - df^i$ vanishes in a neighborhood U of $\mathcal{X}$, $i=1,2$. For every $x\in\mathcal{X}$ we choose a chart and let $D_\epsilon(x)$ denote the ϵ–disk around x. Put $D_\epsilon := \bigcup_{x\in\mathcal{X}} D_\epsilon(x)$.

For ϵ sufficiently small $D_\epsilon\subseteq U$ and f^2-f^1 is constant on each $D_\epsilon(x)$. From (23), Stokes' theorem and (20) we conclude that

$$
\begin{aligned}
R(X,\omega,g;f^2)-R(X,\omega,g;f^1)-\sum_{x\in\mathcal{X}}\mathrm{IND}(x)\big(f^2(x)-f^1(x)\big) = \\
&= -\int_{M\setminus\mathcal{X}} d\big((f^2-f^1)\wedge X^*\Psi_g\big)\\
&= -\lim_{\epsilon\to 0}\int_{M\setminus D_\epsilon} d\big((f^2-f^1)\wedge X^*\Psi_g\big)\\
&= \sum_{x\in\mathcal{X}}\big(f^2(x)-f^1(x)\big)\lim_{\epsilon\to0}\int_{\partial D_\epsilon(x)} X^*\Psi_g\\
&= -\sum_{x\in\mathcal{X}}\mathrm{IND}(x)\big(f^2(x)-f^1(x)\big)
\end{aligned}
$$

and thus $R(X,\omega,g;f^1)=R(X,\omega,g;f^2)$. □

Proposition 12. *Suppose that $\mathbb{X}$ is a non-degenerate homotopy from the vector field X^1 to X^2 and ω is a closed one form. Then*

$$R(X^2,\omega,g)-R(X^1,\omega,g)=I(X^1,X^2,\omega). \tag{24}$$

Proof. We may assume that there exists a simply connected $V\subseteq M$ with $\mathcal{X}_s\subseteq V$ for all $s\in[-1,1]$. Indeed, since both sides of (24) do not depend on the homotopy $\mathbb{X}$ we may first slightly change the homotopy and assume that no component of $\tilde{\mathcal{X}}$ lies in a single $\{s\}\times M$. Then we find $-1=t_0,\dots,t_k=1$ so that for every $0\le i<k$ we find a simply connected $V_i\subseteq M$ such that $\mathcal{X}_s\subseteq V_i$ for all $s\in[t_i,t_{i+1}]$.

Assuming V as above we choose a function f so that $\omega' := \omega - df$ vanishes on a neighborhood of every $\mathcal{X}_s$, i.e. $p_2^*\omega'$ vanishes on a neighborhood of $\tilde{\mathcal{X}}$. Here $p_2:[-1,1]\times M\to M$ denotes the canonical projection. Moreover let $\tilde p_2:[-1,1]\times TM\to TM$ denote the canonic projection and note that $p_2^*\omega'\wedge\mathbb{X}^*\tilde p_2^*\Psi_g$ is a globally defined form on

$[-1,1]\times TM$. Using Stokes' theorem and Remark 7 we then get:

$$\begin{aligned} R(X^2,\omega,g)-R(X^1,\omega,g)-I_{\mathbb{X},\omega} &= \int_{[-1,1]\times M} d\big(p_2^*\omega'\wedge\mathbb{X}^*\tilde p_2^*\Psi_g\big) \\ &= \int_{[-1,1]\times M} p_2^*(\omega'\wedge E_g) \\ &= 0 \end{aligned}$$

For the second equality we used $d\mathbb{X}^*\tilde p_2^*\Psi_g = p_2^*E_g$. The integrand of the last integral vanishes because of dimensional reasons. □

With little effort, using Stokes' theorem and the properties of the Mathai–Quillen form, one can proof

$$R(X,\omega+df,g)-R(X,\omega,g) = -\int_M fE_g + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)f(x)$$

for every smooth function f, and

$$R(X,\omega,g^2)-R(X,\omega,g^2) = \int_M \omega\wedge \mathrm{cs}(g^1,g^2)$$

for any two Riemannian metrics g^1 and g^2. Its also not difficult to generalize the regularization to vector fields with isolated singularities, cf. [6].

6. Proof of Theorem 4

The proof of Theorem 4 presented here combines results of Hutchings, Pajitnov and others (cf. [8], [15]) with results of Bismut–Zhang, cf. [1], [6] and [3]. A recollection of these results, additional notations and preliminaries are necessary. They will be collected in four preliminary subsections. These subsections will be followed by the fifth where Theorem 4 is proven.

Recall from [3] that a generalized triangulation $\tau=(f,g)$ on a closed manifold M is a pair consisting of a Morse function f and a Riemannian metric g so that $X=-\operatorname{grad}_g f$ satisfies MS.

6.1. **Homotopy between vector fields.**

Let $\xi\in H^1(M;\mathbb{R})$, and $\pi:\tilde M\to M$ be a covering so that $\pi^*\xi=0$.

Recall that a smooth family of sections $\mathbb{X}:=\{X_s\}_{s\in[-1,1]}$, of the tangent bundle will be called a homotopy from the vector field X^1 to the vector field X^2 if there exists $\epsilon>0$ so that $X_s=X^1$ for $s<-1+\epsilon$ and $X_s=X^2$ for $s>1-\epsilon$.

To a homotopy $\mathbb{X} := \{X_s\}_{s\in[-1,1]}$ one associates the vector field Y on the compact manifold with boundary[15] $N := M \times [-1,1]$ defined by

$$Y(x,s) := X(x,s) + 1/2(s^2-1)\frac{\partial}{\partial s}. \tag{25}$$

With this notation we have the following.

Proposition 13. *If $\mathbb{X}$ is a homotopy between two vector fields X^1 and X^2 which both have ξ as a Lyapunov cohomology class. Then the vector field Y has $p^*\xi$ as a Lyapunov cohomology class, cf. Definition 7, where $p : N = M \times [-1,1] \to M$ is the first factor projection.*

Proof. Since X^1 and X^2 are both vector fields with ξ as Lyapunov cohomology class we can choose closed 1–forms ω_i representing ξ and Riemannian metrics g_i on M such that $X^i = -\operatorname{grad}_{g_i}\omega_i$, $i = 1,2$. Choose an admissible Riemannian metric g on N inducing g_i on the boundaries; for example take

$$g := (1-\lambda)p^*g_1 + \lambda p^*g_2 + ds^2,$$

where $\lambda : [-1,1] \to \mathbb{R}$ is a non-negative smooth function satisfying

$$\lambda(s) = \begin{cases} 0 & \text{for } s \leq -1+\epsilon \text{ and} \\ 1 & \text{for } s \geq 1-\epsilon. \end{cases}$$

Next choose a closed 1–form ω on N which restricts to $p^*\omega_1$ on $M \times [-1,-1+\epsilon]$ and which restricts to $p^*\omega_2$ on $M \times [1-\epsilon,1]$. This is possible since ω_1 and ω_2 define the same cohomology class ξ and can be achieved in the following way. Choose a function h on M with $\omega_2 - \omega_1 = dh$ and set $\omega := p^*\omega_1 + d(\lambda p^*h)$. Choose a function $u : [-1,1] \to \mathbb{R}$, such that:

(i) $u(s) = -\frac{1}{2}(s^2-1)$ for all $s \leq -1+\epsilon$ and all $s \geq 1-\epsilon$.

(ii) $u(s) \geq \left\{\frac{-\omega(Y)(x,s)}{\frac{1}{2}(s^2-1)}\right\}$ for all $s \in [-1+\epsilon, 1-\epsilon]$ and all $x \in M$.

This is possible since $\left\{\frac{-\omega(Y)(x,s)}{\frac{1}{2}(s^2-1)}\right\} \leq 0$ for $s = -1+\epsilon$ and $s = 1-\epsilon$.

Then $\tilde\omega := \omega + u(s)ds$ represents the cohomology class $p^*\xi$ and one can verify that Y is a vector field which coincides with $-\operatorname{grad}_{\tilde g}\tilde\omega$ in a neighborhood of ∂N and for which $\tilde\omega(Y) < 0$ on $N \setminus \mathcal{Y}$. □

Let $\mathbb{X}$ be a homotopy between two vector fields X^1 and X^2 which satisfies MS. Let Y be the vector field defined in (25). With the notations from the appendix to section 2 (the case of a compact manifold

[15]See the appendix to section 2 for the definition of vector fields on a compact manifold with boundary.

with boundary) we have

$$\mathcal{Y} = \mathcal{Y}'' = \mathcal{Y}''_- \sqcup \mathcal{Y}''_+$$

with

$$\mathcal{Y}''_- = \mathcal{X}^1 \times \{-1\} \quad \text{and} \quad \mathcal{Y}''_+ = \mathcal{X}^2 \times \{1\}.$$

Definition 8. The homotopy $\mathbb{X}$ is called MS if the vector field Y is MS, i.e. X^1 and X^2 are MS and for any $y \in \mathcal{Y}''_-$ and $z \in \mathcal{Y}''_+$ the maps i^+_y and i^-_z are transversal. The homotopy $\mathbb{X}$ has exponential growth if Y has exponential growth.

Proposition 14. *Let X^1 and X^2 be two vector fields which satisfy MS and $\mathbb{X}$ a homotopy from X^1 to X^2. Then there exists a MS homotopy $\mathbb{X}'$ from X^1 to X^2, arbitrarily close to $\mathbb{X}$ in the C^1–topology.*

Proof. First we modify the vector field Y into Y' by a small change in the C^1–topology, and only in the neighborhood of $M \times \{0\}$, in order to have the Morse–Smale condition satisfied for any $y \in \mathcal{Y}''_-$ and $z \in \mathcal{Y}''_+$. This can be done using Proposition 2. Unfortunately Y' might not have the I–component equal to $1/2(s^2-1)\partial/\partial s$, it is nevertheless C^1–close, so by multiplication with a function which is C^1–close to 1 and equal to 1 on the complement of a small compact neighborhood of the locus where Y and Y' are not the same, one obtains a vector field Y'' whose I–component is exactly $1/2(s^2-1)\partial/\partial s$. The $M-$component of Y'' defines the desired homotopy. By multiplying a vector field with a smooth positive function the stable and unstable sets do not change, and their transversality continues to hold. □

In view of Theorem 6 for compact manifolds with boundary we have the following

Remark 8. For any $y = (x, 1) \in \mathcal{Y}''_+$ the 1–corner of $\hat{W}^-_y$ is given by

$$\partial_1(\hat{W}^-_y) = V_0 \sqcup V_1 \sqcup V_2$$

where

$$\begin{aligned} V_0 &\simeq W^-_x \\ V_1 &\simeq \bigcup_{\substack{v \in \mathcal{Y}''_+, \hat{\alpha} \in \hat{\mathcal{P}}_{y,v} \\ \operatorname{ind}(v) = \operatorname{ind}(y)-1}} \mathcal{T}(y, v, \hat{\alpha}) \times (W^-_v \setminus \partial N) \\ V_2 &\simeq \bigcup_{\substack{u \in \mathcal{Y}''_-, \hat{\alpha} \in \hat{\mathcal{P}}_{y,u} \\ \operatorname{ind}(u) = \operatorname{ind}(y)-1}} \mathcal{T}(y, u, \hat{\alpha}) \times W^-_u \end{aligned}$$

It is understood that W_x^- represents the unstable manifold in $M = M \times \{1\}$ if $x \in \mathcal{X}^2$.

In view of (25) we introduce the invariant $\rho(\xi, \mathbb{X}) \in \mathbb{R} \cup \{\pm\infty\}$ for any homotopy $\mathbb{X}$ by defining

$$\rho(\xi, \mathbb{X}) := \rho(p^*\xi, Y)$$

Clearly $\rho(\xi, \mathbb{X}) \geq \rho(\xi, X^i)$ for $i = 1, 2$.

Suppose $\mathbb{X}$ is a MS homotopy from the MS vector field X^1 to the MS vector field X^2. For each X^i choose the orientations $\mathcal{O}^i$, $i = 1, 2$. Observe that the set $\hat{\mathcal{P}}_{x',x}$ identifies to $\hat{\mathcal{P}}_{(x',1),(x',-1)}$. The orientations $\mathcal{O}^1$ and $\mathcal{O}^2$ define the orientations $\mathcal{O}$ for the unstable manifolds of the rest points of Y. For $x^1 \in \mathcal{X}^1$, $x^2 \in \mathcal{X}^2$ and $\hat{\alpha} \in \hat{\mathcal{P}}_{x^2,x^1}$ define the incidences

$$\mathbb{I}^{\mathbb{X},\mathcal{O}^2,\mathcal{O}^1}_{x^2,x^1}(\hat{\alpha}) := \mathbb{I}^{Y,\mathcal{O}}\big((x^2,1),(x^1,-1)\big)(\hat{\alpha}). \tag{26}$$

Suppose in addition that $\rho(\xi, \mathbb{X}) < \infty$. For any $t > \rho(\xi, \mathbb{X})$ and ω a closed one form representing ξ define the linear maps

$$u^q_\omega(t) := u^q_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t) : \mathrm{Maps}(\mathcal{X}^1_q, \mathbb{R}) \to \mathrm{Maps}(\mathcal{X}^2_q, \mathbb{R})$$

and the linear maps

$$h^q_\omega(t) := h^q_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t) : \Omega^q(M) \to \mathrm{Maps}(\mathcal{X}^2_{q-1}, \mathbb{R})$$

by

$$u^q_\omega(t)(E_{x^1}) := \sum_{\substack{x^2 \in \mathcal{X}^2 \\ \hat{\alpha} \in \hat{\mathcal{P}}_{x^2,x^1}}} \mathbb{I}^{\mathbb{X},\mathcal{O}^2,\mathcal{O}^1}_{x^2,x^1}(\hat{\alpha}) e^{t\omega(\hat{\alpha})} E_{x^2}, \quad x^1 \in \mathcal{X}^1_q \tag{27}$$

and

$$(h^q_\omega(t)(a))(x^2) = \int_{W_y^-} e^{tF_y}(i_y^-)^* p^* a, \quad x^2 \in \mathcal{X}^2_{q-1} \text{ and } y = (x^2, 1).$$

The right side of (27) is a convergent infinite sum since it is a sub sum of the right hand side of (19) when applied to the vector field Y.

Proposition 15. *Suppose X^1, X^2 are two MS vector fields having ξ as a Lyapunov cohomology class and suppose $\mathbb{X}$ is a MS homotopy. Suppose that $\rho := \rho(\xi, \mathbb{X}) = \rho(p^*\xi, Y) < \infty$ and ω is a closed one form with $p^*\omega$ exact; here $p : \tilde{M} \to M$ is the Γ–principal covering associated with Γ. Then for $t > \rho$ we have:*

(i) *The collection of linear maps $\{u^q_\omega(t)\}$ defines a morphism of cochain complexes:*

$$u^*_\omega(t) := u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t) : \mathbb{C}^*(X^1, \mathcal{O}^1, \omega)(t) \to \mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t)$$

(ii) *The collection of linear maps $h^q_\omega(t)$ defines an algebraic homotopy between* $\mathrm{Int}^*_{X^2,\mathcal{O}^2,\omega}(t)$ *and* $u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t)\circ\mathrm{Int}^*_{X^2,\mathcal{O}^2,\omega}(t)$. *Precisely, we have:*

$$h^{*+1}_\omega(t)\circ d^*_\omega(t)+\delta^{*-1}_\omega(t)\circ h^*_\omega(t)=u^*_\omega(t)\circ\mathrm{Int}^*_{X^1,\mathcal{O}^1,\omega}(t)-\mathrm{Int}^*_{X^2,\mathcal{O}^2,\omega}(t)$$

Proof. Statement (i) follows from the equality

$$\sum_{\substack{x'\in\mathcal{X}^1_{q+1},\ \hat\alpha\in\hat{\mathcal{P}}_{z,x'}\\ \hat\beta\in\hat{\mathcal{P}}_{x',x},\ \hat\alpha\star\hat\beta=\hat\gamma}}\mathbb{I}^{\mathbb{X},\mathcal{O}^2,\mathcal{O}^1}_{z,x'}(\hat\alpha)\mathbb{I}^{X^1,\mathcal{O}^1}_{x',x}(\hat\beta)-\sum_{\substack{z'\in\mathcal{X}^2_{q},\ \hat\alpha\in\hat{\mathcal{P}}_{z,z'}\\ \hat\beta\in\hat{\mathcal{P}}_{z',x},\ \hat\alpha\star\hat\beta=\hat\gamma}}\mathbb{I}^{X^2,\mathcal{O}^2}_{z,z'}(\hat\alpha)\mathbb{I}^{\mathbb{X},\mathcal{O}^2,\mathcal{O}^1}_{z',x}(\hat\beta)=0$$

for any $x\in\mathcal{X}^1_q$, $z\in\mathcal{X}^2_{q+1}$ and $\hat\gamma\in\hat{\mathcal{P}}_{z,x}$ which is a reinterpretation of equation (17) when applied to the vector field Y, the rest points $(x,-1)$ and $(z,1)$ and $\hat\gamma\in\hat{\mathcal{P}}_{z,x}=\hat{\mathcal{P}}_{(z,1),(x,-1)}$. The sign stems from the fact that the sign associated to a trajectory from z to z' changes when it is considered as trajectory in $M\times[-1,1]$ instead of $M\times\{1\}$.

To verify statement (ii) we first observe that:

(a) If $y,u\in\mathcal{Y}$ the restriction of F_y to $\mathcal{T}(y,u)(\hat\alpha)\times W^-_u$, $\hat\alpha\in\hat{\mathcal{P}}_{y,u}$, when this set is viewed as a subset of $\hat W^-_y$ is given by

$$F_u\circ\mathrm{pr}_{W^-_u}+\overline{\omega}(\hat\alpha).$$

(b) If $y=(x,-1)$, $x\in\mathcal{X}^1$, via the identification of W^-_x to W^-_y, we have $F_y=h_x$.

In view of the uniform convergence of all integrals which appear in the formulae below, guaranteed by the hypothesis $t>\rho$, the Stokes theorem for manifolds with corners gives for any $a\in\Omega^q(M)$ and $y\in(\mathcal{Y}''_+)_q$

$$\int_{\hat W^-_y}d(e^{tF_y}c)=\int_{V_0}e^{tF_y}c+\int_{V_1}e^{tF_y}c+\int_{V_2}e^{tF_y}c,\tag{28}$$

where $c:=(i^-_y)^*p^*a\in\Omega^q(\hat W^-_y)$.

In view of the Remark 8 we have

$$\int_{V_0}e^{tF_y}c=\mathrm{Int}^q_{X^2,\mathcal{O}^2,\omega}(t)(a),\tag{29}$$

and

$$\begin{aligned}\int_{V_2}e^{tF_y}c&=\sum_{\substack{u\in\mathcal{Y}''_+,\ \hat\alpha\in\hat{\mathcal{P}}_{y,u}\\ \mathrm{ind}(u)=\mathrm{ind}(y)-1}}\mathbb{I}^{Y,\mathcal{O}}_{y,u}(\hat\alpha)e^{t\omega(\hat\alpha)}\int_{\hat W^-_u}e^{tF_u}(\hat i^-_u)^*p^*a\\&=-(\delta^{q-1}_\omega(t)\circ h^q_\omega(t))(a)\end{aligned}\tag{30}$$

and

$$\begin{aligned}\int_{V_1} e^{tF_y} c &= -\sum_{\substack{v\in\mathcal{Y}''_-,\ \hat\alpha\in\hat{\mathcal{P}}_{y,v}\\ \operatorname{ind}(v)=\operatorname{ind}(y)-1}} \mathbb{I}^{Y,\mathcal{O}}_{y,v}(\hat\alpha)e^{t\omega(\hat\alpha)}\int_{\hat W_v^-} e^{tF_v}(i_v^-)^* p^* a\\ &= -(u^q_\omega(t)\circ \operatorname{Int}^q_{X^1,\mathcal{O}^1,\omega}(t))(a). \qquad (31)\end{aligned}$$

Moreover

$$\begin{aligned}(h^{q+1}_\omega(t)\circ d_\omega(t))(b)(y) &= \int_{\hat W_y^-} e^{tF_y}(i_y^-)^* p^*(db+t\omega\wedge b)\\ &= \int_{\hat W_y^-} d(e^{tF_y}(i_y^-)^* p^* b) \qquad (32)\end{aligned}$$

and the statement follows combining the equalities (28)–(32). □

The following proposition will be important in the proof of Theorem 4.

Proposition 16.

(i) *Let (f,g) be a pair consisting of a Morse function and a Riemannian metric. Then the vector field $-\operatorname{grad}_g f$ has any $\xi\in H^1(M;\mathbb{R})$ as a Lyapunov cohomology class.*

(ii) *Let X be a vector field which has MS property and has ξ as Lyapunov cohomology class. Let $\tau=(f,g)$ be a generalized triangulation. Then there exists a homotopy $\mathbb{X}$ from $X^1 := X$ to X^2 which is MS and is C^0–close to the family $X_s = \frac{1-s}{2}X - \frac{1+s}{2}\operatorname{grad}_g f$. One can choose $\mathbb{X}$ to be C^1–close to a family $l(s)X-(1-l(s))\operatorname{grad}_g f$ where $l:[-1,1]\to[0,1]$ is a smooth function with $l'(s)\le 0$ and $l'(s)=0$ in a neighborhood of $\{\pm1\}$.*

Proof of (i). Let ω be a closed one form representing ξ with support disjoint from a neighborhood of $\operatorname{Cr}(f)$. Clearly for C a large constant the form $\omega' := \omega + Cdf$ represents ξ and satisfies $\omega'(-\operatorname{grad}_g f)<0$. □

Proof of (ii). First consider the family $X_s := (\frac{1-s}{2})X - (\frac{1+s}{2})\operatorname{grad}_g f$. Change the parametrization to make this family locally constant near $\{\pm1\}$, hence get a homotopy and apply Proposition 14 to change this homotopy into one which satisfies MS. □

Definition 9. A vector field X which satisfies MS and has ξ as a Lyapunov cohomology class is said to have *strong exponential growth* if for one (and then any) generalized triangulation $\tau=(f,g)$ there exists a homotopy $\mathbb{X}$ from X to $-\operatorname{grad}_g f$ which has exponential growth.

To summarize the discussion in this subsection consider:

(i) a vector field $X^1 = -\operatorname{grad}_{g'} \omega$ with ω a Morse form representing ξ and g' a Riemannian metric so that X^1 satisfies MS,
(ii) A vector field $X^2 = -\operatorname{grad}_{g''} f$, $\tau = (f, g'')$ a generalized triangulation,
(iii) A homotopy $\mathbb{X}$ from X^1 to X^2 which is MS,

Since $\rho(\xi, X^2) = -\infty$, for any $t \in \mathbb{R}$ we have a well defined morphism of cochain complexes

$$\operatorname{Int}^*_{X^2,\mathcal{O}^2,\omega}(t) : (\Omega^*(M), d_\omega(t)) \to \mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t).$$

If X^1 has $\rho(\xi, X^1) < \infty$, equivalently X^1 has exponential growth, then for t large enough we have a well defined morphism of cochain complexes

$$\operatorname{Int}^*_{X^1,\mathcal{O}^1,\omega}(t) : (\Omega^*(M), d_\omega(t)) \to \mathbb{C}^*(X^1, \mathcal{O}^1, \omega)(t).$$

If $\mathbb{X}$ has $\rho(\xi, \mathbb{X}) < \infty$ then for t large enough we have the morphism of cochain complexes

$$u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t) : \mathbb{C}^*(X^1, \mathcal{O}^1, \omega)(t) \to \mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t)$$

and the algebraic homotopy $h^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t)$ making the diagram below homotopy commutative.

$$\begin{array}{ccc}
(\Omega^*(M), d_\omega(t)) & \xrightarrow{\operatorname{Id}} & (\Omega^*(M), d_\omega(t)) \\
{\scriptstyle \operatorname{Int}^*_{X^1,\mathcal{O}^1,\omega}(t)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Int}^*_{X^2,\mathcal{O}^2,\omega}(t)} \\
\mathbb{C}^*(X^1, \mathcal{O}_1, \omega)(t) & \xrightarrow{u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t)} & \mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t).
\end{array}$$

6.2. **A few observations about torsion.** Consider cochain complexes (C^*, d^*) of free A–modules of finite rank whose cohomology $H^* := H^*(C^*, d^*)$ is also a graded A–free module[16] of finite rank. Here A is a commutative ring with unit.

For two equivalence classes of bases, $[\underline{c}]$ of C^* and $[\underline{h}]$ of H^*, Milnor, cf. [12], has defined the torsion $\tau((C^*, d^*), [\underline{c}], [\underline{h}]) \in A^+/\{\pm 1\}$ where A^+ denotes the multiplicative group of invertible elements of A.

Recall that the bases $\underline{m}' \equiv \{m'_1, \dots, m'_k\}$ and $\underline{m}$" $\equiv \{m''_1, \dots, m''_k\}$ of the free A–module M are equivalent iff the isomorphism $T : M \to M$ defined by $T(m'_i) = m''_i$ has determinant ± 1.

If the complex (C^*, d^*) is acyclic there is no need of the base $\underline{h}$ and one has $\tau((C^*, d^*), [\underline{c}]) \in A^+/\{\pm 1\}$. If $\alpha : A \to B$ is a unit preserving ring homomorphism, by tensoring $(C^*, d^*), [\underline{c}], [\underline{h}]$ with B, regarded as

[16]Actually one can weaken the hypothesis free to projective but this is of no interest in the present discussion.

an $A-$module via α, one obtains $((C')^*, (d')^*), [\underline{c}'], [\underline{h}']$ a cochain complex of free $B-$modules whose cohomology is a free $B-$module and the bases $[\underline{c}'], [\underline{h}']$. Clearly

$$\tau(((C')^*, (d')^*), [\underline{c}'], [\underline{h}']) = \alpha(\tau((C^*, d^*), [\underline{c}], [\underline{h}])).$$

If A is the field $\mathbb{R}$ or $\mathbb{C}$, hence (C^*, d^*) is a cochain complex of finite dimensional vector spaces, and $\langle\cdot,\cdot\rangle$ are scalar products in C^* one can define the T–torsion, $T((C^*, d^*), \langle\cdot,\cdot\rangle) \in \mathbb{R}_+$, by the formula

$$\log T((C^*, d^*), \langle\cdot,\cdot\rangle) = 1/2 \sum_i (-1)^{i+1} i \log \det{}' \Delta_i$$

where $\det{}'\Delta_i$ is the product of the non-zero eigen values of $\Delta_i := (d^{i+1})^\sharp \cdot d^i + d^{i-1} \cdot (d^i)^\sharp$. Here $(d^i)^\sharp$ denotes the adjoint of d^i with respect of the scalar product $\langle\cdot,\cdot\rangle$

If in addition scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ in cohomology $H^* := H^*(C^*, d^*)$, is given, one defines the positive real numbers $\mathrm{Vol}(H^i, \langle\cdot,\cdot\rangle, \langle\langle\cdot,\cdot\rangle\rangle)$ to be the volume of the isomorphism $\ker d^i / d^{i-1}(C^{i-1}) \to H^i$.[17] Here the first vector space is equipped with the scalar product induced from $\langle\cdot,\cdot\rangle$ and the second with the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$.

If A is $\mathbb{R}$ or $\mathbb{C}$ then any base $\underline{c}$ resp. $\underline{h}$ induce a scalar product $\langle\cdot,\cdot\rangle_{\underline{c}}$ resp. $\langle\langle\cdot,\cdot\rangle\rangle_{\underline{h}}$, the unique scalar product which makes the base orthonormal. Although equivalent bases do not necessary provide the same scalar products they do however lead to the same T–torsions. This follows from the inspection of the definition. Moreover one has

$$\begin{aligned} &|\tau((C^*, d^*), [\underline{c}], [\underline{h}])| = \\ &= \ T((C^*, d^*), \langle\cdot,\cdot\rangle_{\underline{c}}) + \sum_i (-1)^i \log \mathrm{Vol}(H^i, \langle\cdot,\cdot\rangle_{\underline{c}}, \langle\langle\cdot,\cdot\rangle\rangle_{\underline{h}}) \end{aligned}$$

Let $u^* : (C_1^*, d_1^*) \to (C_2^*, d_2^*)$ be a morphism of cochain complexes of free A–modules of finite rank which induce isomorphism in cohomology. Then the mapping cone $\mathcal{C}u^*$ is an acyclic cochain complex of free A–modules of finite rank.

Two equivalence classes of bases $[\underline{c}_1]$ of C_1^* and $[\underline{c}_2]$ of C_2^* provide an equivalence class of bases $[\underline{c}]$ of $\mathcal{C}u^*$, and permit to define

$$\tau(u^*, [\underline{c}_1], [\underline{c}_2]) := \tau(\mathcal{C}u^*, [\underline{c}]).$$

If A is $\mathbb{R}$ or $\mathbb{C}$ the scalar products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ in C_1^* and C_2^* provide the scalar product $\langle\cdot,\cdot\rangle$ in $\mathcal{C}u^*$ and permit to define

$$T(u^*, \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) := T(\mathcal{C}u^*, \langle\cdot,\cdot\rangle).$$

[17] Recall that the volume of an isomorphism $\theta : (V_1, \langle\cdot,\cdot\rangle_1) \to (V_2, \langle\cdot,\cdot\rangle_2)$ between two Euclidean vector spaces is the positive real number $\log \mathrm{Vol}(\theta) := 1/2 \log \det \theta^\sharp \cdot \theta$.

If the scalar products $\langle\cdot,\cdot\rangle_i := \langle\!\langle\cdot,\cdot\rangle\!\rangle_{\underline{c}_i}$, $i = 1, 2$ are induced from the bases c_i^*, $i = 1, 2$ we also have

$$|\tau((u^*, [\underline{c}_1], [\underline{c}_2])| = T(u^*, \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) \tag{33}$$

It is a simple exercise of linear algebra (cf. [3]) to check that:

Proposition 17.

(i) *Suppose $u^* : (C_1^*, d_1^*) \to (C_2^*, d_2^*)$ is a morphism of cochain complexes which induces the isomorphisms H^*u in cohomology. Suppose that $H^*(C_1^*, d_1^*) = H^*(C_2^*, d_2^*)$ are free A–modules equipped with the bases $\underline{h}_1$ and $\underline{h}_2$ and $\underline{c}_1$ and $\underline{c}_2$ are bases of C_1^* and C_2^*. Suppose in addition that $\prod(\det H^i u)^{(-1)^i} = \pm 1$.*[18] *Then:*

$$\tau(u^*, [\underline{c}_1], [\underline{c}_2]) = \tau(C_2^*, [\underline{c}_2], [\underline{h}_2]) \cdot \tau(C_1^*, [\underline{c}_1], [\underline{h}_1])^{-1}.$$

(ii) *Suppose A is $\mathbb{R}$ or $\mathbb{C}$ and $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are scalar products on C_1^* and C_2^*. Then*

$$\begin{aligned} &\log T(u^*, \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) = \\ &\quad = \log T((C_1^*, d_1^*), \langle\cdot,\cdot\rangle_1) - \log\tau((C_2^*, d_2^*), \langle\cdot,\cdot\rangle_2) + \log\operatorname{Vol} H^*u \end{aligned} \tag{34}$$

where $\log\operatorname{Vol} H^*u = \sum_i (-1)^i \log\operatorname{Vol} H^i u$.[19] *Moreover if u^* is an isomorphism then*

$$\log T(u^*, \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) = \sum_i (-1)^i \log\operatorname{Vol}\theta^i$$

We conclude this subsection by recalling the following result of Bismut–Zhang, see [1] and [3].

Suppose that (M, g) is a closed Riemannian manifold $X = -\operatorname{grad}_{g'} f$ with $\tau = (f, g')$ a generalized triangulation, ρ a representation of $\pi_1(M)$ and μ a Hermitian structure in the flat vector bundle associated with ρ. Consider $\operatorname{Int}^* : (\Omega^*(M, \rho), d_\rho) \to (C^*(\tau, \rho), \delta_{\mathcal{O},\rho})$ and equip each of these complexes with a scalar product, the first complex with the scalar product induced from the Riemannian metric g and the Hermitian structure μ and the second with the scalar product $\langle\cdot,\cdot\rangle_{\mu,\tau}$ induced from the generalized triangulation τ and the Hermitian structure μ, cf. [3]. In this notation $C^q(\tau, \rho)$ can be viewed as the vector space of sections above $\mathcal{X}_q$ of the vector bundle $E_\rho \to M$ equipped with the hermitian structure μ. This is a finite dimensional vector space with a scalar product.

[18] Here $(\det H^i u)$ is calculated with respect to the bases $\underline{h}_1$ and $\underline{h}_2$.

[19] $\operatorname{Vol} H^i u$ is calculated with respect to the scalar product induced from $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$ in cohomology.

Equip the cohomology of these cochain complexes with the induced scalar product. Denote by $H^* \operatorname{Int}$ the isomorphism induced in cohomology and write

$$\log VH(\rho, \mu, g, \tau) = \sum (-1)^q \log \operatorname{Vol}(H^q \operatorname{Int}) \tag{35}$$

Let $\omega(\mu)$ be the closed one form induced by μ as described in [1] and [3]. We have the following result due to Bismut–Zhang, cf. [3].

Theorem 7. *With the hypothesis above we have*

$$\log T_{\mathrm{an}}(M, g, \rho, \mu) = \log T(C^*(\tau, \rho), \delta_{\mathcal{O},\rho}, \langle \cdot, \cdot \rangle_{\mu,\rho}) + \\ + \log VH(\rho, \mu, g, \tau) + R(X, \omega(\mu), g)$$

6.3. **A summary of Hutchings–Pajitnov results.** We begin by recalling the results of Hutchings cf. [8].

Let M be a compact smooth manifold, $m \in M$ a base point and $\xi \in H^1(M; \mathbb{R})$. Recall from section 1.1 that ξ defines the free abelian group Γ and induces the injective homomorphism $\xi : \Gamma \to \mathbb{R}$. Denote by $\pi : \tilde{M} \to M$ the principal Γ–covering canonically associated with x and $\pi_1(M, m) \to \Gamma$. To ξ we associate

(i) the Novikov ring Λ_ξ with coefficients in $\mathbb{R}$ which is actually a field,
(ii) the subring $\Lambda_{\xi,\rho} \subset \Lambda_\xi$, for any $\rho \in [0, \infty)$, cf. below,
(iii) the multiplicative groups of invertible elements $\Lambda_\xi^+ \subset \Lambda_\xi$ and $\Lambda_{\xi,\rho}^+ \subset \Lambda_{\xi,\rho}$.

The Novikov ring Λ_ξ consists of functions $f : \Gamma \to \mathbb{R}$ which satisfy the property that for any real number $R \in \mathbb{R}$ the cardinality of the set $\{\gamma \in \Gamma | f(\gamma) \neq 0, \xi(\gamma) \leq R\}$ is finite. The multiplication in Λ_ξ is given by convolution, cf. [5]. We have also shown both in Section 1 and in more details in [5] how to interpret the elements of Λ_ξ as Dirichlet series. In this context $\Lambda_{\xi,\rho}$ is the subring of Λ_ξ consisting of those elements whose corresponding Dirichlet series have the abscissa of convergence smaller than ρ.

Note $\mathbb{Z}[\Gamma] \subset \Lambda_{\xi,\rho} \subset \Lambda_\xi$ and $H^*_{\mathrm{sing}}(M; \Lambda_\xi) := H^*_{\mathrm{sing}}(\tilde{M}; \mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_\xi$ is a finite dimensional vector space over the field Λ_ξ. Let $\det H^*_{\mathrm{sing}}(M; \Lambda_\xi)$ denote the one dimensional vector space over Λ_ξ defined by

$$\det H^*_{\mathrm{sing}}(M; \Lambda_\xi) = \bigotimes_i (\Lambda^i(H^i_X(M; \Lambda_\xi))^{\epsilon(i)}$$

where $V^{\epsilon(i)} = V$ if i is even and $V^{\epsilon(i)}$ is the dual of V if i is odd. Let X be a vector field which satisfies MS and has ξ as a Lyapunov cohomology class and let $\tilde{X}$ be the pullback of X on $\tilde{M}$. Choose $\mathcal{O}$ a collection of orientations for the unstable manifolds of the rest points of X and therefore of the rest points of $\tilde{X}$. Denote by $(NC^q_{X,\xi}, \partial^q_{\mathcal{O}})$ the Novikov cochain complex of free Λ_ξ modules (vector spaces since Λ_ξ is a field) as defined in [5] and by $H^*_X(M;\Lambda_\xi)$ its cohomology. There exists a canonical isomorphism

$$HV^*_X : H^*_X(M;\Lambda_\xi) \to H^*_{\text{sing}}(M;\Lambda_\xi)$$

described below.

The isomorphism HV^*_X : Note that for any two vector fields X^1 and X^2 which are MS and have ξ as Lyapunov cohomology class there exists by Proposition 14 homotopies $\mathbb{X}$ from X^1 to X^2 which satisfy MS. The incidences, $\mathbb{I}^{\mathcal{O}^2,\mathcal{O}^1}_{x^2,x^1}(\hat{\alpha})$ defined in subsection 6.1 provide a morphism

$$u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2} : (NC^*_{X^1,\xi}, \partial^*_{\mathcal{O}^1}) \to (NC^*_{X^2,\xi}, \partial^*_{\mathcal{O}^2})$$

of cochain complexes which induces isomorphism between their cohomology. This cohomology isomorphism is independent of the homotopy $\mathbb{X}$,[20] and will be denoted by

$$H^*u(X^1, X^2) : H^*_{X^1}(M;\Lambda_\xi) \to H^*_{X^2}(M;\Lambda_\xi).$$

For any three vector fields X^i, $i = 1, 2, 3$ which satisfy MS and have ξ as a Lyapunov cohomology class one has

$$H^*u(X^3, X^2) \cdot H^*u(X^2, X^1) = H^*u(X^3, X^1).$$

Let $\tau = (f, g)$ be generalized triangulation, and $\xi \in H^1(M;\mathbb{R})$. Let $X' := -\operatorname{grad}_g f$. By Proposition 16 X has ξ as Lyapunov cohomology class. The Novikov complex $(NC^q_{X',\xi}, \partial^q_{\mathcal{O}})$ identifies to the geometric cochain complex associated to $\tilde{\tau} = (\tilde{f}, \tilde{g})$, the pull back of (f, g) to $\tilde{M}$, tensored (over $\mathbb{Z}[\Gamma]$) by Λ_ξ. Recall that the geometric (or Morse) cochain complex associated to $(\tilde{f}, \tilde{g})$ is a cochain complex of free $\mathbb{Z}[\Gamma]$ modules. This cochain complex is obviously a quotient of $C^*_{\text{sing}}(\tilde{M})$ tensored (over $\mathbb{Z}[\Gamma]$) by Λ_ξ which calculates $H^*_{\text{sing}}(M;\Lambda_\xi)$ and we have an obvious isomorphism from $H^*_{\text{sing}}(M;\Lambda_\xi)$ to the cohomology of $H^*_{X'}(M;\Lambda_\xi)$.

The composition of the this isomorphism with $H^*u(X, X')$ provides the canonical isomorphism $HV^*_X : H^*_X(M;\Lambda_\xi) \to H^*_{\text{sing}}(M;\Lambda_\xi)$.

Denote by $E(M, m)$ the set of Euler structures based at $m \in M$ cf. [2] or [6] for a definition, and let $e \in E(M, m)$. Recall that in the presence of X an Euler structure e is represented by an Euler chain

[20]One can introduce the concept of homotopy of from a homotopy $\mathbb{X}^1$ to the homotopy $\mathbb{X}^2$ (between two vector fields X^1 and X^2 and prove in a standard way that it induces an algebraic homotopy from the morphism induced by $\mathbb{X}^1$ and $\mathbb{X}^2$, hence the same isomorphism in cohomology.

(cf. [6]) which consists of a collection of paths α_x from m to $x \in \mathcal{X}$. Each such path provides a lift $\tilde{x}$ of x (i.e. $\pi(\tilde{x}) = x$) and therefore a base $\{\tilde{E}_x | x \in \mathcal{X}\}$ with $\tilde{E}_x$ the characteristic function of $\tilde{x}$ regarded as an element of $NC^q_{X,\xi,\rho}$, $q = \mathrm{ind}(x)$. Conversely, any lift $s : \mathcal{X} \to \widetilde{X}$, $s(x) = \tilde{x}$ defines an Euler chain and therefore together with X an Euler structure e. The path α_x is the image by π of a smooth path from m to $\tilde{x}$. Different Euler chains representing the same Euler structure might provide nonequivalent bases. All theses bases will be named e–*compatible* and denoted by $\underline{e}$. Any lift s which defines with X the Euler structure e will be also called e–*compatible.*

Choose an element $o_H \in \det H^*_{\mathrm{sing}}(M; \Lambda_\xi) \setminus 0$, and consider bases h^* in $H^*(NC^*_{X,\xi}, \partial^q_{\mathcal{O}})$ which represent via the isomorphism HV^*_X the element o_H. They all will be called o_H–compatible. Again the o_H–compatible bases might not be equivalent, however an inspection of Milnor definition of torsion [12] implies that the element

$$\tau((NC^q_{X,\xi}, \partial^q_{\mathcal{O}}), [\underline{e}], [\underline{h}]) \in \Lambda^+_\xi / \{\pm 1\}$$

as defined in section 6.2 for $\underline{e}$ resp. $\underline{h}$ e–compatible resp. o_H–compatible bases depends only on e and o_H; therefore denoted by $\tau_\xi(X, e, o_H)$.

If $H^*_{\mathrm{sing}}(M; \Lambda_\xi) = 0$ there is no need of o_H and we have $\tau_\xi(X, e) \in \Lambda^+_\xi / \{\pm 1\}$.

Note that if X has exponential growth, in view of Theorem 3, the complex $(NC^q_{X,\xi}, \partial^q_{\mathcal{O}})$ contains, for ρ large enough, a subcomplex of free $\Lambda_{\xi,\rho}$ modules $(NC^q_{X,\xi,\rho}, \partial^q_{\mathcal{O}})$, cf. [5] so that

$$(NC^q_{X,\xi,\rho}, \partial^q_{\mathcal{O}}) \otimes_{\Lambda_{\xi,\rho}} \Lambda_\xi = (NC^q_{X,\xi}, \partial^q_{\mathcal{O}}).$$

Moreover an e–compatible base will provide a base of of $\Lambda_{\xi,\rho}$–modules in this subcomplex. If $H^*_{\mathrm{sing}}(M; \Lambda_\xi) = 0$ then $\tau_\xi(X, e) \in \Lambda^+_{\xi,\rho}/\{\pm 1\}$ for ρ large enough. More general, if $H^*_{\mathrm{sing}}(M; \Lambda_{\xi,\rho})$ is a free $\Lambda_{\xi,\rho}$–module and $o_H \in \det H^*_{\mathrm{sing}}(M; \Lambda_{\xi,\rho}) := \bigotimes_i (\Lambda^i (H^i_X(M; \Lambda_{\xi,\rho}))^{\epsilon(i)}$ we will have $\tau_\xi(X, e, o_H) \in \Lambda^+_{\xi,\rho}/\{\pm 1\}$.

If the homotopy $\mathbb{X}$ has exponential growth then, for ρ big enough, we have $u^q_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2}(NC^q_{X^1,\xi,\rho}, \partial^q_{\mathcal{O}^1}) \subset (NC^q_{X^2,\xi,\rho}, \partial^q_{\mathcal{O}^2})$ and $\tau(u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2}, [\underline{e}_1], [\underline{e}_2])$ which depends only on X^1, X^2 and e, lies in $\Lambda^+_{\xi,\rho}/\{\pm 1\}$.

Note that for $t > \rho$ we denote by $\mathrm{ev}_t : \Lambda_{\xi,\rho} \to \mathbb{R}$ the ring homomorphism which associates to each $f \in \Lambda_{\xi,\rho}$ interpreted as a Dirichlet series f, the value of the Laplace transform $L(f)$ at t, cf. section 1.2. When applied to torsion it calculates the torsion of the corresponding complex tensored by $\mathbb{R}$.

Suppose now that X is MS and satisfies also NCT. As noticed in [8], $\mathbb{Z}_X \in \Lambda_\xi$ and then $e^{\mathbb{Z}_X} \in \Lambda^+_\xi$. The main result of Hutchings can be formulated as follows

Theorem 8. *If X^1 and X^2 are two vector fields which are MS and NCT and have ξ as a Lyapunov cohomology class then*

$$e^{\mathbb{Z}_{X^1}} \cdot \tau_\xi(X^1, e, o_H) = e^{\mathbb{Z}_{X^2}} \cdot \tau_\xi(X^2, e, o_H).$$

The proof of this theorem is given in [8]. The author considers only the acyclic case (in which case o_H is not needed). The acyclicity hypothesis is used only to insure that the Milnor torsion (cf. [12]) $\tau_\xi(X, e)$ can be defined. This can be also defined in the non-acyclic case at the expense of the orientation o_H. The orientation o_H induces via v^*_X an orientation in the cohomology of the Novikov complex associated to X and together with the Euler structure e a class of bases in the Novikov complex. From this moment on the arguments in [8] can be repeated word by word.

Let $\mathbb{X}$ be a homotopy from the vector field X^1 to the vector field X^2 which is MS and suppose that both vector fields have ξ as a Lyapunov cohomology class. The incidences $\mathbb{I}^{\mathcal{O}^1,\mathcal{O}^2}_{\dots}$, cf. (26), induced from $\mathbb{X}$ and the orientation $\mathcal{O} = \mathcal{O}^1 \sqcup \mathcal{O}^2$ provide a morphism $u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2} : (NC^*_{X^1}, \partial^*_{\mathcal{O}_1}) \to (NC^*_{X^2}, \partial^*_{\mathcal{O}_2})$ which induces an isomorphism in cohomology as already indicated.

Choose bases $\underline{e}_i$ in each of the Novikov complexes $(NC^*_{X^i}, \partial^*_{\mathcal{O}^i})$, $i = 1, 2$, which are e–compatible. By the same inspection of the Milnor definition of torsion one concludes that $\tau(u^*_{\mathbb{X}}, [\underline{e}_1], [\underline{e}_2]) \in \Lambda^+_\xi/\{\pm 1\}$ defined in section 6.2 depends only on X^1, X^2 and e. In view of Proposition 17(i) and of Theorem 8 one obtains

Proposition 18. *If X^2 and X^1 are two vector fields which satisfy MS and NCT and have ξ as a Lyapunov cohomology class, e an Euler structure as above. Then*

$$\tau(u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2}, [\underline{e}_1], [\underline{e}_2]) = e^{\mathbb{Z}_{X^2}} \cdot e^{-\mathbb{Z}_{X^1}}.$$

Hence $\tau(u^*_{\mathbb{X}}, [\underline{e}_1], [\underline{e}_2])$ depends only on X^1, X^2 and is independent of e and then can be denoted by $\tau(X^1, X^2)$.

Suppose $X^2 = -\operatorname{grad}_{g''} f$, $\tau = (f, g'')$ a generalized triangulation.

Corollary 5. $\tau(X^1, X^2) = e^{\mathbb{Z}_{X^1}}$.

It is not hard to see that Hutchings theorem is equivalent to this corollary. In this form the result was also established by Pajitnov [15].

Suppose X is a vector field with ξ a Lyapunov cohomology class which satisfies MS and in addition has exponential growth. The exponential growth implies that any e–compatible base of $NC^q_{X,\xi}$ is actually a base of $NC^q_{X,\xi,\rho}$ for ρ large enough. For $t > \rho$ the $\mathbb{R}$–linear maps $\mathrm{Ev}^\omega_t : NC^q_{X^1,\xi,\rho} \to \mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$ defined by

$$\mathrm{Ev}^\omega_t(f) := \sum_{\tilde{x}\in\pi^{-1}(x)} f(\tilde{x})e^{-t\tilde{h}(\tilde{x})}$$

provides a morphism of cochain complexes $\mathrm{Ev}^\omega_t : (NC^*_{X^1,\xi,\rho}, \partial^*_{\mathcal{O}}) \to \mathbb{C}^*(X, \mathcal{O}, \omega)(t)$ with $\mathrm{Ev}^\omega_t(\tilde{E}_x) = e^{-t\tilde{h}(\tilde{x})}E_x$.

Here $\mathbb{C}^*(X, \mathcal{O}, \omega)(t)$ is equipped with the canonical base $\{E_x\}$ with E_x the characteristic function of $x \in \mathcal{X}$. The isomorphism Ev^ω_t factors through an isomorphism from $(NC^*_{X^1,\xi,\rho}, \partial^*_{\mathcal{O}}) \otimes_{\mathrm{ev}_t} \mathbb{R}$ to $\mathbb{C}^*(X, \mathcal{O}, \omega)(t)$. If the Novikov complex $(NC^*_{X,\xi,\rho}, \partial^*_{\mathcal{O}})$ is acyclic so is $\mathbb{C}^*(X, \mathcal{O}, \omega)(t)$.

Let $s : \mathcal{X} \to \widetilde{\mathcal{X}}$ be a compatible lift and $\underline{e}$ the associated e–compatible base. A simple inspection of Milnor definition of torsion leads to

$$\begin{aligned} \mathrm{ev}_t(\tau(X, \xi, e)) &= \mathrm{ev}_t(\tau((NC^*_{X,\xi,\rho}, \partial^*_{\mathcal{O}}), [\underline{e}]) \\ &= \tau(\mathbb{C}^*(X, \mathcal{O}, \omega)(t), [E_x]) \cdot e^{-t\sum_{x\in\mathcal{X}} \mathrm{IND}(x)\tilde{h}(\tilde{x})} \end{aligned} \tag{36}$$

Suppose now that X^i, $i = 1, 2$, are two vector fields which have ξ as a Lyapunov cohomology class and $\mathbb{X}$ is a homotopy between them. Suppose in addition that X^i and $\mathbb{X}$ satisfy MS and have all exponential growth. Then we obtain the morphism of Novikov cochain complexes $u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2} : (NC^*_{X^1,\xi,\rho}, \partial^*_{\mathcal{O}^1}) \to (NC^*_{X^1,\xi,\rho}, \partial^*_{\mathcal{O}^2})$ which induces an isomorphism in cohomology. Clearly when tensored by $\mathbb{R}$ this morphism is conjugate to $u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t)$.

The Euler structure $e \in E(M, p)$ permits to choose $e-$compatible lifts of and then e–compatible bases $\underline{e}_1$ and $\underline{e}_2$. The same inspection of Milnor definition leads then to

$$\begin{aligned} &\mathrm{ev}_t(\tau(u^*_{\mathbb{X}}, \mathcal{O}^1, \mathcal{O}^2), [\underline{e}_1], [\underline{e}_2]) = \\ &\quad = \tau(u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t), [E_{x_1}], [E_{x_2}]) \cdot e^{-t(\sum_{x\in\mathcal{X}^1} \mathrm{IND}(x)\tilde{h}(\tilde{x}) - \sum_{x\in\mathcal{X}^2} \mathrm{IND}(x)\tilde{h}(\tilde{x}))} \end{aligned} \tag{37}$$

where E_{x_1} resp. E_{x_2} are the canonical base provided by the rest points of X^1 resp. X^2.

Note that in view of Proposition 12 (Additional property) for any e–compatible lifts of $\mathcal{X}^1$ and $\mathcal{X}^2$ we have:

$$\sum_{x\in\mathcal{X}^1} \mathrm{IND}(x)\tilde{h}(\tilde{x}) - \sum_{x\in\mathcal{X}^2} \mathrm{IND}(x)\tilde{h}(\tilde{x}) = I(X^1, X^2, \omega). \tag{38}$$

The change of the lifts (providing the same the Euler structure) does not change the left side of (38).

6.4. **The geometry of closed one form.** Suppose M is a connected smooth manifold and $p \in M$ is a base point. The homomorphism $[\omega] : H_1(M;\mathbb{Z}) \to \mathbb{R}$ induces the one dimensional representation $\rho = \rho_{[\omega]} : \pi_1(M,p) \to \mathrm{GL}_1(\mathbb{R})$ defined by the composition $\pi_1(M,p) \to H_1(M;\mathbb{Z}) \xrightarrow{[\omega]} \mathbb{R} \xrightarrow{\exp} \mathbb{R}_+ = \mathrm{GL}_1(\mathbb{R})$. The representation ρ provides a flat rank one vector bundle $\xi_\rho : E_\rho \to M$ with the fiber above p identified with $\mathbb{R}$. This bundle is the quotient of trivial flat bundle $\widetilde{M} \times \mathbb{R} \to \widetilde{M}$ by the group Γ which acts diagonally on $\widetilde{M} \times \mathbb{R}$. Here $\widetilde{M}$ denotes the principal Γ–covering associated with $[\omega]$ and constructed canonically with respect to p (from the set of continuous paths originating from p). Note that $\tilde{M}$ is equipped with a base point $\tilde{p}$ corresponding to the constant path in p. The group Γ acts freely on $\widetilde{M}$ with quotient space M. The action of Γ on $\mathbb{R}$ is given by the representation $\Gamma \to \mathbb{R} \xrightarrow{\exp} \mathbb{R}_+ = \mathrm{GL}_1(\mathbb{R})$.[21]

There is a bijective correspondence between the closed one forms ω in the cohomology class represented by ρ and the Hermitian structures μ in the vector bundle ξ_ρ which agree with a given Hermitian structure on the fiber above p (identified to $\mathbb{R}$).

Given ω in the cohomology class $[\omega] \in H^1(M;\mathbb{R})$, one constructs a Hermitian structure $\tilde{\mu}_\omega$ on the trivial bundle $\widetilde{M} \times \mathbb{R} \to \widetilde{M}$ which is Γ–invariant. Therefore, by passing to quotients one obtains a Hermitian structure μ_ω in ξ_ρ. The Hermitian structure $\tilde{\mu}_\omega$ is defined as follows:

(i) Observe that the pull back $\tilde{\omega}$ of ω on $\widetilde{M}$ is exact and therefore equal $d\tilde{h}$ where $\tilde{h} : \widetilde{M} \to \mathbb{R}$ is the unique function with $\tilde{h}(\tilde{p}) = 0$ and $d\tilde{h} = \tilde{\omega}$.
(ii) Define $\tilde{\mu}(\tilde{x})$ by specifying the length of the vector $1 \in \mathbb{R}$. We put $||1_{\tilde{x}}||_{\tilde{\mu}(\tilde{x})} := e^{\tilde{h}(\tilde{x})}$.

Given a Hermitian structure μ one construct a closed one form ω_μ as follows: Denote by $(\tilde{E}_\rho \to \tilde{M}, \tilde{\mu})$ the pair consisting of the flat line bundle $\tilde{E}_\rho \to \tilde{M}$ and the Hermitian structure $\tilde{\mu}$ the pullback of the pair $(E_\rho \to M, \mu)$ to $\tilde{M}$ by the map $\widetilde{M} \to M$. Let $\overline{\mu}$ be the Hermitian structure obtained by parallel transporting the scalar product $\tilde{\mu}_{\tilde{p}}$. Denote by $\alpha : \tilde{M} \to \mathbb{R}$ the function $\alpha(\tilde{x}) := ||v||_{\tilde{\mu}(\tilde{x})}/||v||_{\overline{\mu}(\tilde{x})}$ for v a nonzero vector in $\tilde{E}_{\tilde{x}}$.

[21]Note that instead of $\tilde{M} \to M$ one can use the universal covering $\hat{M} \to M$ which is a π–principal covering. One ends up with the same $E_\rho \to M$.

Define $\tilde{\omega}_\mu := d\log(\alpha)$ and observe that this is a Γ invariant closed one form, hence descends to a closed one form ω on M.

To simplify the writing below we denote (by a slight abuse of notation):

(i) $\rho(t) := \rho_{t\omega}$
(ii) $\mu(t) := \mu_{t\omega}$

Remark 9. The cochain complex $(\Omega^*(M), d_\omega(t))$ equipped with the scalar product induced from g is isometric to the cochain complex $(\Omega^*(M, \rho(t))$ equipped with the scalar product induced from g and $\mu(t) = \mu$ as defined in [3].

In particular we have

$$\log T_{\mathrm{an}}^{\omega,g}(\omega, t) = \log T_{\mathrm{an}}(M, \rho(t), g, \mu(t))$$

where $\log T_{\mathrm{an}}(M, \rho, g, \mu)$ is the analytic torsion considered in [3] and associated with the Riemannian manifold (M, g) the representation ρ and the Hermitian structure μ in the flat bundle induced from ρ.

Remark 10. Let $\xi \in H^1(M; \mathbb{R})$ and ω a closed one form representing ξ. Suppose $X = -\operatorname{grad}_{g''} f$ where $\tau = (f, g'')$ is a generalized triangulation. Choose orientations $\mathcal{O}$ for the unstable manifolds of X. The morphism

$$\mathrm{Int}^*_{X,\mathcal{O},\omega,}(t) : (\Omega^*(M), d_\omega(t)) \to \mathbb{C}^*(X, \mathcal{O}, \omega)(t)$$

defined in (5) where $(\Omega^*(M)$ is equipped with the scalar product induced from g and $\mathrm{Maps}(\mathcal{X}_q, \mathbb{R})$ is equipped with the obvious scalar product, i.e. associated with the base $\{E_x\}$, is isometrically conjugate to

$$\mathrm{Int}^* : (\Omega^*(M, \rho(t)), d_{\rho(t)}) \to (C^*(\tau, \rho(t)), \delta_{\mathcal{O},\rho(t)})$$

defined in [3] where $(\Omega^*(M, \rho(t))$ is equipped with the scalar product induced by $(g, \mu(t))$ and $C^*(\tau, \rho(t))$ is equipped with the scalar product induced from τ and $\mu(t)$.

6.5. **Proof of Theorem 4.** We begin with a triple (g, g', ω) with $X^1 = X = -\operatorname{grad}_{g'} \omega$ as in the hypothesis of Theorem 4. We choose orientations $\mathcal{O}^1$ for the unstable manifolds of X^1 We also choose $X^2 = -\operatorname{grad}_{g''} f$ so that $\tau = (f, g'')$ is a generalized triangulation and choose orientations $\mathcal{O}^2$ for the unstable manifolds of X^2.

For simplicity of the writing we will use the following abbreviations: $I_1(t) := \mathrm{Int}^*_{X^1,\mathcal{O}^1,\omega}(t)|_{\Omega^*_{\mathrm{sm}}(M)}$ and $I_2(t) := \mathrm{Int}^*_{X^2,\mathcal{O}^2,\omega}(t)|_{\Omega^*_{\mathrm{sm}}(M)}$.

In view of Proposition 17(ii) applied to $I_1(t)$ one obtains

$$\begin{aligned} \log(\mathbb{V}(t)) &= \log T(\mathbb{C}(X^1, \mathcal{O}^1, \omega)(t), \langle\cdot,\cdot\rangle_1) \\ &\quad - \log T^{\omega,g}_{\mathrm{an,sm}}(t) + \log \mathrm{Vol}(H^*(I_1(t)) \end{aligned} \tag{39}$$

where $\langle\cdot,\cdot\rangle_1$ is the scalar product induced from the canonical base $\{E_x\}$, $x \in \mathcal{X}^1$.

In view of Theorem 7 and the Remarks 9 and 10 in section 6.4 one has

$$\begin{aligned}\log T^{\omega,g}_{\mathrm{an}}(t) &= \log T(\mathbb{C}(X^2,\mathcal{O}^2,\omega)(t),\langle\cdot,\cdot\rangle_2) \\ &\quad +\log \mathrm{Vol}(H^*(I_2(t)) - tR(\omega,g,X^2)\end{aligned} \tag{40}$$

where $\langle\cdot,\cdot\rangle_2$ is the scalar product induced from the canonical base $\{E_x\}$, $x \in \mathcal{X}^2$.

Combining with (39) and (40) one obtains

$$\begin{aligned}&\log(\mathbb{V}(t)) - \log T^{\omega,g}_{\mathrm{an,la}}(t) = \\ &= \log \mathrm{Vol}(H^*(I_1(t)) - \log \mathrm{Vol}(H^*(I_2(t)) \\ &\quad + \log \tau(\mathbb{C}(X^1,\mathcal{O}^1,\omega)(t),\langle\cdot,\cdot\rangle_1) \\ &\quad - \log T(\mathbb{C}(X^2,\mathcal{O}^2,\omega)(t),\langle\cdot,\cdot\rangle_2) + tR(X^2,\omega,g)\end{aligned} \tag{41}$$

First consider the case that $X = X^1$ has exponential growth and $H^*(M,t[\omega]) = 0$ for t large enough. Note that X^2 has exponential growth too by Proposition 16. Clearly then $\log \mathrm{Vol}(H^*(I_1(t)) = \log \mathrm{Vol}(H^*(I_2(t)) = 0$. By (33), (36) and (38) we have

$$\begin{aligned}&\log T(\mathbb{C}(X^1,\mathcal{O}^1,\omega)(t),\langle\cdot,\cdot\rangle_1) - \log T(\mathbb{C}(X^2,\mathcal{O}^2,\omega)(t),\langle\cdot,\cdot\rangle_2) = \\ &= \log(\mathrm{ev}_t(\tau(X^1,\xi,e_1^*)) \\ &\quad - \log(\mathrm{ev}_t(\tau(X^2,\xi,e_2^*)) - tI(X^1,X^2,\omega)\end{aligned} \tag{42}$$

By Theorem 8 in section 6.3 we have

$$\begin{aligned}&\log(\mathrm{ev}_t(\tau(X^1,\xi,e_1^*)) - \log(\mathrm{ev}_t(\tau(X^2,\xi,e_2^*)) = \\ &= \log(\mathrm{ev}_t(e^{\mathbb{Z}_{X^1}} \cdot e^{-\mathbb{Z}_{X^2}})) \\ &= \log(\mathrm{ev}_t\, e^{-\mathbb{Z}_{X^1}}) \\ &= -L(\mathbb{Z}_{X^1})(t)\end{aligned} \tag{43}$$

Combining (41), (42) and (43) one obtains the result.

Second consider the case X has (strong) exponential growth property. Then choose a homotopy $\mathbb{X}$ which satisfy: $\rho(\xi,\mathbb{X}) < \infty$. Then for t big enough, we have the following (algebraically) homotopy commutative diagram of finite dimensional cochain complexes whose arrows induce isomorphisms in cohomology.

$$\begin{array}{ccc}(\Omega^*_{\mathrm{sm}}(M)(t), d_\omega(t)) & \xrightarrow{\mathrm{Id}} & (\Omega^*_{\mathrm{sm}}(M)(t), d_\omega(t)) \\ \Big\downarrow{\scriptstyle \mathrm{Int}^*_{X^1,\mathcal{O},\omega_1}(t)} & & \Big\downarrow{\scriptstyle \mathrm{Int}^*_{X^2,\mathcal{O},\omega_2}(t)} \\ \mathbb{C}^*(X^1,\mathcal{O}_1,\omega)(t) & \xrightarrow{u^*_{\mathbb{X},\mathcal{O}_1,\mathcal{O}_2,\omega}(t)} & \mathbb{C}^*(X^2,\mathcal{O}_2,\omega)(t)\end{array}$$

For simplicity we write $u^*(t) := u^*_{\mathbb{X},\mathcal{O}^1,\mathcal{O}^2,\omega}(t)$ and observe that in view of the homotopy commutativity of the above diagram and of Proposition 17(ii) we have

$$\begin{aligned}\log T(u^*(t), \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) = \\ = \ \log \mathrm{Vol}(H^*(I_1(t)) - \log \mathrm{Vol}(H^*(I_2(t)) \\ + \log T(\mathbb{C}(X^2, \mathcal{O}^1, \omega)(t), \langle\cdot,\cdot\rangle_1) \\ - \log T(\mathbb{C}(X^2, \mathcal{O}^1, \omega)(t), \langle\cdot,\cdot\rangle)\end{aligned} \tag{44}$$

As noticed $(\mathbb{C}^*(X^2, \mathcal{O}^2, \omega)(t), \langle\cdot,\cdot\rangle)$ is isometric to $(\mathbb{C}(M, \tau, \rho(t), \mu(t))$.

By (33), (37) and Proposition 18 combined with the observations that X^2 has no closed trajectories we have

$$\begin{aligned}\log T(u^*(t), \langle\cdot,\cdot\rangle_1, \langle\cdot,\cdot\rangle_2) &= \log \tau(u^*(t), e_1^*, e_2^*) \\ &= \mathrm{ev}_t(\mathbb{Z}_{X^1}) + I(X^1, X^2, t\omega)\end{aligned} \tag{45}$$

Combining (41) and (45) we obtain

$$\log \mathbb{V}(t) + \mathrm{ev}_t(\mathbb{Z}_{X^1}) + tI(X^1, X^2, \omega) = \log T^{\omega,g}_{\mathrm{an,la}}(t) + tR(X^2, \omega, g) \tag{46}$$

which in view of Proposition 12 implies the result.

When $H^*(M; t\Lambda_\xi)$ is acyclic we do not need the morphism u^* and a simple consequence of Corollary 5 implies the result. It turns out the *strong exponential growth* can be weaken to the (apparently) weaker hypothesis $H^*_{\mathrm{sing}}(M; \Lambda_{\xi,\rho})$ is a free module over $\Lambda_{\xi,\rho}$ for some ρ. As in acyclic case one can circumvent the morphism $u^*(t)$.

Appendix A

Lemma 11. *Suppose $a_1, \dots, a_n \in \mathbb{R}$, $a_i \neq a_j$ for $i \neq j$. Suppose $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $\lambda_i \neq 0$. Consider the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(t) := \sum_{i=1}^{n} \lambda_i e^{ta_i}. \tag{47}$$

Then there are only finitely many $t \in \mathbb{R}$ for which $f(t) = 0$.

Proof. W.l.o.g. $a_1 > a_i$ for $i \neq 1$. There certainly exists $T_+ \in \mathbb{R}$, such that

$$\frac{1}{n-1}|\lambda_1 e^{ta_1}| > |\lambda_i e^{ta_i}|, \quad \text{for all } t \geq T_+ \text{ and for all } i \neq 1.$$

Using $|x + y| \geq |x| - |y|$ we derive

$$|f(t)| \geq |\lambda_1 e^{ta_1}| - \sum_{i=2}^{n} |\lambda_i e^{ta_i}| > |\lambda_1 e^{ta_1}| - \sum_{i=2}^{n} \frac{1}{n-1}|\lambda_1 e^{ta_1}| = 0,$$

for all $t \geq T_+$. Similarly (using the smallest a_i instead of the largest) one finds $T_- \in \mathbb{R}$, such that $|f(t)| > 0$ for all $t \leq -T_-$. Setting $T := \max\{T_+, T_-\}$ we have $f(t) \neq 0$ for all $|t| \geq T$ and since the set of zeros has to be discrete there can only be a finite number of them. □

Corollary 6. *Let ω be a closed one form and let $\beta^i(t)$ denote the dimension of $H^i(\Omega^*(M), d_\omega(t))$, where $t \in \mathbb{R}$ and $i = 0, 1, \dots, \dim(M)$. Then there are finitely many $t_i \in \mathbb{R}$, $t_0 < t_1 < \cdots < t_N$ and positive integers β^i so that $\beta^i(t) = \beta^i$, for $t \neq t_1, \dots, t_N$ and so that $\beta^i(t_k) > \beta^i$ for all k.*

Proof. Take any generalized (in particular smooth) triangulation τ and consider X an Euler vector field i.e. a vector field with isolated rest points and whose unstable sets identify to the open cells (simplexes) of the triangulation. For any $t \in \mathbb{R}$

$$\mathrm{Int}^*_{X,\mathcal{O},\omega}(t) : (\Omega^*(M), d_\omega(t)) \to \mathbb{C}^*(X, \mathcal{O}, \omega)(t)$$

is well defined and provides an isomorphism in cohomology by deRham's theorem.

Note that the differential $\delta^*_{X,\mathcal{O},\omega}(t)$ of the complex $\mathbb{C}^*(X, \mathcal{O}, \omega)(t)$ with respect to the canonical base of $\mathbb{C}^*(X, \mathcal{O}, \omega)$ is given by a matrix whose entries are functions in t of the form (47), see (19). The condition for the change of the dimension of $H^i(\mathbb{C}(X, \mathcal{O}, \omega)(t))$ can be expressed as vanishing of finitely many minors of the matrix representing $\delta^*_{X,\mathcal{O},\omega}(t)$. These minors are functions of the form (47) and hence have only finitely many zeros in view of previous lemma. □

References

[1] J.M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller.* Astérisque **205**, Société Mathématique de France, 1992.

[2] D. Burghelea, *Removing Metric Anomalies from Ray–Singer Torsion*, Lett. Math. Phys. **47**(1999), 149–158.

[3] D. Burghelea, L. Friedlander and T. Kappeler, *Relative Torsion*, Commun. Contemp. Math. **3**(2001), 15–85.

[4] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. **6**(1996), 751–859.

[5] D. Burghelea and S. Haller, *On the topology and analysis of closed one form. I (Novikov theory revisited)*, Monogr. Enseign. Math. **38**(2001), 133–175.

[6] D. Burghelea and S. Haller, *A Riemannian invariant, Euler structures and some topological applications*, preprint OSU(2003). See also `arXiv.math.DG/0310154`.

[7] P. Hartman, *Ordinary Differential Equations.* Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA]. Classics in Applied Mathematics **38**,

Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[8] M. Hutchings, *Reidemeister torsion in generalized Morse Theory*, Forum Math. **14**(2002), 209–244.

[9] M. Hutchings, Y.J. Lee, *Circle–valued Morse Theory, Reidemeister torsion and Seiberg–Witten invariants of* 3 *manifolds*, Topology **38**(1999), 861–888.

[10] J. Marcsik, *Analytic torsion and closed one forms.* Thesis OSU **8**(1998).

[11] V. Mathai and D. Quillen, *Superconnections, Thom Classes, and Equivariant differential forms*, Topology **25**(1986), 85–110.

[12] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72**(1966), 358–426.

[13] S.P. Novikov, *Quasiperiodic structures in topology*, in Topological methods in modern mathematics, Proceedings of the symposium in honor of John Milnor's sixtieth birthday held at the State University of New York, Stony Brook, New York, 1991, eds L. R. Goldberg and A. V. Phillips, Publish or Perish, Houston, TX, 1993, 223–233.

[14] S.P. Novikov, *On the exotic De-Rham cohomology. Perturbation theory as a spectral sequence*, preprint `arXiv.math-ph/0201019`.

[15] A. Pajitnov, *Counting closed orbits of gradients of circle-valued maps*, preprint `arXiv.math.DG/0104273`.

[16] M.M. Peixoto, *On an Approximation Theorem of Kupka and Smale*, J. Differential Equations **3**(1967), 214–227.

[17] J.P. Serre, *A course in arithmetic.* Translated from the French. Graduate Texts in Mathematics **7**, Springer Verlag, New York–Heidelberg, 1973.

[18] D.V. Widder, *The Laplace Transform.* Princeton Mathematical series **6**, Princeton University Press, Princeton, N.J., 1941.

[19] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17**(1982), 661–692.

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