Complex Singularity Analysis for a Nonlinear PDE

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Abstract

We introduce a method of rigorous analysis of complex singularities for nonlinear higher order PDEs with prescribed initial data. The method is applied to determine rigorously the asymptotic structure of singularities of the modified Harry-Dym equation

$$H_t + H_y = -\frac{1}{2}H^3 + H^3 H_{yyy} \quad : \quad H(y,0) = y^{-1/2}$$

for small time at the boundaries of the sector of analyticity. Previous work [7], [5] shows existence, uniqueness and Borel summability of solutions of general PDEs. It is shown that the solution to the above initial value problem is represented convergently by a series in a fractional power of t down to a small annular neighborhood of a singularity of the leading order equation. We deduce that the exact solution has a singularity nearby having, to leading order, the same type.

1 Introduction

The singularity structure of solutions of nonlinear partial differential equations in the complex plane is not a well understood subject. The goal of the present paper is to develop a relatively general and constructive technique to address this issue for equations that admit formal asymptotic similarity solutions. Such formal asymptotic similarity solutions could be applicable for small or large time, or when one approaches the finite blow-up time of a similarity solution of a PDE. We prove that complex singularities of these formal asymptotic solution actually correspond to singularities of the solution of the full PDE.

The motivation for understanding complex singularity formation of PDEs, aside from intrinsic interest, is that in some cases of physical interest [1], [4], [2], there is evidence that singularities that appear in the real physical domain after a finite time can be traced to the complex plane.

Given the complexity of singularities of nonlinear PDEs it appears to us impractical to formulate and prove a general result at this stage. Instead, we describe the method and analyze concrete problems; it will be however transparent that the method is much more general.

The procedure consists in the following steps: (i) an early time asymptotic expansion in powers of t, the validity of which is justified for the modified Harry-Dym equation in [5], (ii) introduction of appropriately scaled "inner" dependent and independent variables beyond the region of validity of the expansion (i), (iii) determination of singularities of the leading order equation and (iv) proof that a secondary expansion in scaled time, involving inner-variables, is convergent in a domain encircling a singularity of the leading order solution. Insofar as analysis of the leading order equation (in step (iii) above) is concerned, which (typically a nonlinear ODE), formal calculations have been used before (see [3], and references in [14]). These can now be rigorously derived from the general theory introduced in [14].

The present paper justifies the above four step procedure for the modified Harry-Dym equation This equation arises in the small surface tension limit of Hele-Shaw interfacial evolution [3] in the neighborhood of an initial zero of the derivative of an associated conformal map. The justification of singularity formation is a crucial first-step to understanding "daughter"-singularity phenomena where a smoothly evolving interface corresponding to a zero-surface tension solution is singularly perturbed in $O(1)$ time by arbitrarily small surface tension effect.

Consider the following initial value problem for the modified Harry-Dym equation:

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial y} - H^3 \frac{\partial^3 H}{\partial y^3} - \frac{H^3}{2} = 0 \ \text{ with } \ H(y,0) = y^{-1/2} \tag{1}$$

Theorem 36 and Corollary 37 in [5] imply that for any $t \in [0,T]$, for large enough $|y-t|/t^{2/9}$ with $\arg(y-t) \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$, there exists a unique solution $H(y,t)$ to (1) with $H(y,t) \sim y^{-1/2}$ as $|y-t|/t^{2/9} \to \infty$, with the following asymptotic expansion for $t \ll 1$:

$$H(y,t) = (y-t)^{-1/2} \sum_{n=0}^{\infty} P_n \left(\frac{t}{(y-t)^{9/2}}, \frac{t}{(y-t)} \right) \tag{2}$$

where $P_0 = 1$ and P_n is a homogeneous polynomial determined recursively in terms of P_{n-1}, P_{n-2},...P_1. The first two polynomials are

$$P_1(a,b) = -\frac{15}{8}a - \frac{1}{2}b, \qquad P_2(a,b) = \frac{25875}{128}a^2 + \frac{195}{32}ab + \frac{3}{8}b^2 \tag{3}$$

Further, if we introduce the scaled variables

$$\eta = \frac{x-t}{t^{2/9}} \ \ ; \ \ \tau = t^{7/9} \ \ ; \ \ H(y(\eta,t),t) = t^{1/9} G(\eta,\tau), \tag{4}$$

then, according to Corollary 37 in [5][1], for $|\eta|$ sufficiently large, with $\arg \eta \in \left(-\frac{4}{9}\pi, \ \frac{4}{9}\pi\right)$, the function G has a convergent series expansion in τ:

$$G(\eta,\tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta) \tag{5}$$

In this paper, among other results, it will be shown that the convergence of the series (5) actually holds in an extended domain in η that includes at least a region close to a singularity $\hat{\eta}_s$ of $G_0(\eta)$ in a neighborhood of the boundary $\arg \eta \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$ for large $|\hat{\eta}_s|$.

[1]The variable variable $\zeta = \eta^{3/2}$ is used there.

Substituting (4) into (1), we obtain the following equation for $G(\eta, \tau)$:

$$-\frac{G}{9} - \frac{2}{9}\eta G_\eta + \frac{7}{9}\tau G_\tau + \frac{\tau}{2}G^3 - G^3 G_{\eta\eta\eta} = 0 \tag{6}$$

From (5), it follows that the leading order solution G_0 satisfies

$$\frac{1}{9}G_0 + \frac{2}{9}\eta G_0' + G_0^3 G_0''' = 0 \tag{7}$$

In order for G in (5) to match the asymptotic expansion (2) we need to require that

$$G_0(\eta) = \eta^{-1/2}(1 + o(1)); \quad |\eta| \text{ large}, \ \arg\eta \in \left(-\frac{4\pi}{9}, \frac{4\pi}{9}\right) \tag{8}$$

The solution $G_0(\eta)$ to the leading order ODE (7) with asymptotic condition (8) have been studied before. Numerical solutions were found [3] and computational evidence suggested that there is a cluster of singularities $\hat{\eta}_s$, where $G_0(\eta) \sim \ e^{i\pi/3}\left(\frac{\eta_s}{3}\right)^{1/3}(\eta - \hat{\eta}_s)^{2/3}$. Using the fact that $G_0(\eta)$ is indeed a similarity solution to the Harry-Dym equation, which is integrable, it was shown [8] that (7) can be transformed to Painlevé $\mathrm{P_{II}}$. Isomonodromic methods were used to prove existence and uniqueness of sectorially analytic solution for $\arg\eta \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$ that satisfies far-field condition (8). Outside this sector, the behavior of the transformed equation solution is given by elliptic functions, whose singularities can be related to the two-thirds singularity of $G_0(\eta)$, as above.

However, unlike the isomonodromic method, the method based on generalized Borel summation summation [13], [15] applies to initially small solutions of non-integrable equations as well. To apply this analysis in our example, which does not satisfy all the conditions in [14], small adaptations of the proofs are needed (see Appendix). One can determine that for large η, uniformly in the sector $\arg\eta \ \in \ \left[-\frac{4\pi}{9} - \delta, \ \frac{4\pi}{9} - \delta\right]$ for some $\delta \ \in \left(0, \frac{2}{9}\pi\right)$, except for an exponentially small region around singularity of G_0, the asymptotic series of $G_0(\eta)$ is of the form

$$G_0(\eta) \ \sim \ \eta^{-1/2}U(\zeta) + O(\eta^{-5}) \tag{9}$$

where

$$\zeta = -\log\ C + \frac{9}{8}\ \log\ \eta + \frac{i4\sqrt{2}}{27}\eta^{9/4} + (2\hat{n} - 1)i\pi \tag{10}$$

with the principal branch of the log, where C is a Stokes constant of G_0 in the large η expansion for $\arg\eta \in \left[-\frac{4}{9}\pi + \delta, \frac{4}{9}\pi - \delta\right]$

$$G_0(\eta) \sim \eta^{-1/2}\left[1 + \sum_{k=1}^{\infty} a_k \eta^{-9k/2}\right] \tag{11}$$

The function $U(\zeta)$ is determined implicitly from the equation:

$$\zeta = \log 4 - 2 - i\pi - 2\sqrt{U} - \ln\left(\frac{1-\sqrt{U}}{1+\sqrt{U}}\right) \tag{12}$$

and $U(\zeta)$ has a singularity (where $U = 0$) at $\zeta = \zeta_s \equiv \log 4 - 2 - i\pi$, corresponding to a string of singularities at $\eta = \eta_s$, where

$$\frac{i4\sqrt{2}}{27}\eta_s^{9/4} + \frac{9}{8}\log\eta_s = -2 + \log 4 - 2\hat{n}i\pi + \log C \tag{13}$$

where $\hat{n} \in \mathbb{N}$ has to be large for η_s to be large. For large $|\eta_s|$(large $\hat{n}$), it is to be noted that $\arg\eta_s$ is close to $-\frac{4\pi}{9}$, the anti-Stokes line. There is similarly another quasi-periodic array of singularities close to $\arg\eta = \frac{4\pi}{9}$, but our focus will be only on the ones in the lower-half plane. It can be shown that for large $|\eta_s|$ the singularities of G_0 lie within an exponentially small distance of η_s and, to leading order, are of the same type. This can be further verified directly from the equation for G_0.

Remark 1 *It is easy to check that G_0 cannot be zero, except at a singularity $\eta = \hat{\eta}_s$. Furthermore , in any domain $\mathcal{D}$ that excludes a neighborhood of the singularities of G_0, and extends to ∞ so that $\arg\eta \in \left[-\frac{4}{9}\pi + \delta, \frac{4}{9}\pi + \delta\right]$, it follows from differentiability of the asymptotics of solutions of ODEs [16] that*

$$\sup_{\eta\in\mathcal{D}} |\eta^{1/2}G_0(\eta)|, \quad \sup_{\eta\in\mathcal{D}} |\eta^{7/2}G_0'''(\eta)| < C \tag{14}$$

Remark 2 *By (12), near the singularity $\eta = \eta_s$ we have*

$$\frac{2}{3}U^{3/2} + O(U^{5/2}) = \zeta - \zeta_s \;=\; \frac{1}{\eta_s}\left\{\frac{9}{8} + \frac{i\sqrt{2}}{3}\eta_s^{9/4}\right\}(\eta - \eta_s)\left[1 + O\left(\eta_s^{-1}(\eta - \eta_s)\right)\right] \tag{15}$$

and hence for $\eta - \eta_s = o(\eta_s^{-5/4})$, for large enough $|\eta_s|$,

$$U \;\sim\; e^{i\pi/3}\left(\frac{\eta_s^{5/6}}{2^{1/3}}\right)\left(1 - \frac{27i}{8\sqrt{2}}\eta_s^{-9/4}\right)^{2/3}(\eta - \eta_s)^{2/3} \tag{16}$$

Note that if $r_i|\eta_s|^{-5/4} < |\eta - \eta_s| < r_0|\eta_s|^{-5/4}$, with $r_0 > r_i$ small, then there exists upper and lower bounds for $|U|$, independent of η_s for large $|\eta_s|$. Since the singularity $\hat{\eta}_s$ of G_0 is exponentially close to η_s, it follows that the lower bound of G_0 in this annular region is also independent of $|\eta_s|$.

Given these leading order singularities for $G_0(\eta)$, we investigate the series expansion (5), known to converge for large enough $|\eta|$ in any compact subset of $\arg\eta \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$, in a neighborhood of a singularity of G_0.

Remark 3 *The domain $\mathcal{D}$ in the next theorem, sketched in Fig. 1, is specified in Definition 4. It contains a small annular region of a singularity η_s of U (cf. (13), (12)) for large $\hat{n} \in \mathbb{N}$ and a sectorial region $\arg\eta \in \left(-\frac{2\pi}{9} + \delta, \frac{2\pi}{9} - \delta\right)$, for δ small and large η.*

Comparing powers of τ^k (for $k \geq 1$) obtained by substituting power series (5) into (6) results in:

$$G_0^3\mathcal{L}_k G_k = R_k \tag{17}$$

where the linear operator $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k u = u''' + \frac{2}{9G_0^3}\eta u' - \left(\frac{\beta_k}{G_0^3} + \frac{3G_0'''}{G_0}\right) u \quad \text{where} \quad \beta_k = \frac{7k-1}{9} \tag{18}$$

and the terms R_k on the right side of (17) are given by

$$R_k(\eta) = \frac{1}{2} \sum_{\sum k_i = k-1} G_{k_1} G_{k_2} G_{k_3} - \sum_{k_j < k, \sum k_j = k} G_{k_1} G_{k_2} G_{k_3} G_{k_4}''' \tag{19}$$

In order to match to the asymptotic expansion expansion (2), we require

$$G_k(\eta) \sim \frac{A_k}{\eta^{k+1/2}}; \quad |\eta| \text{ large, } \arg\eta \in \left(-\frac{4\pi}{9}, \frac{4\pi}{9}\right) \tag{20}$$

for some specific constants A_k ($A_1 = -1/2$, $A_2 = \frac{3}{8}$, $A_3 = -\frac{5}{16}$,...). As explained later, it is not necessary to impose (20); any solution G_k which approaches 0 as $|\eta| \to \infty$ with $\arg\eta \in \left(-\frac{2}{9}\pi - \delta, \frac{2}{9}\pi + \delta\right)$ at a rate faster than $\eta^{-1/2}$ must necessarily have the asymptotic behavior (20) (See Remark 6).

Theorem 1 *The expansion (5) is convergent in $\mathcal{D}$ for all sufficiently small τ. In particular, for any singularity $\hat{\eta}_s$ of $G_0(\eta)$ near the anti-Stokes line $\arg\eta = -\frac{4}{9}\pi$ with $|\hat{\eta}_s|$ sufficiently large, there is a singularity of $G(\eta,\tau)$ for small τ, to leading order of the same type, approaching it as $\tau \to 0^+$.*

Remark 4 *The convergence of the Taylor expansion in τ and the bounds on G_k and G_k' suffice to show that $G(\eta,\tau)$ has the singularities close to those of $G_0(\eta)$ since for a circle S_{ϵ_1} of radius ϵ_1 around η_s we have*

$$\frac{1}{2\pi i}\oint_{S_{\epsilon_1}} \frac{G_\eta}{G} d\eta \sim \frac{1}{2\pi i}\oint_{S_{\epsilon_1}} \frac{G_0'}{G_0} d\eta \sim \frac{2}{3} + O\left(\epsilon_1^{1/3}, \tau\right) \tag{21}$$

For small τ, $G(\eta,\tau)$ thus has, to leading order, a branch-point of algebraic order $2/3$.

Remark 5 *The convergence of the series (5) in $\mathcal{D}$ is a corollary of the following lemma.*

Lemma 2 *There exist constants A and B independent of $j \geq 1$, with $A > 1$, $0 < B < 1$, so that*

$$\|\eta^{3/2} G_j\|_{\infty,\mathcal{D}} \leq \frac{BA^j}{j^3} \tag{22}$$

$$\|\eta^{5/2} G_j'\|_{\infty,\mathcal{D}} \leq \frac{BA^j}{j^2} \tag{23}$$

$$\|G_j'''\|_{\infty,\mathcal{D}} \leq \frac{BA^j}{j^2} \tag{24}$$

Remark 6 *The proof of this key Lemma that leads to the proof of Theorem 1 is given at the end of §6. First, we prove a Lemma bounding the $R_k(\eta)$. This provides bounds of G_k using a suitable inversion of $\mathcal{L}_k$ in (18). The estimates suffice for our purpose but are not sharp, as (20) implies a faster decay rate in η. The uniqueness of the solution $G(\eta,\tau)$ in the regime $|\eta| \gg 1$ for $\arg\eta \in \left(-\frac{4}{9}\pi, \frac{4}{9}\pi\right)$ with $G(\eta,\tau) \sim \eta^{-1/2}$ is shown in [5].*

The proof of Lemma 2 is by induction; we first prove an general result for sums of type (19).

Lemma 3 *With G_0 the same as before, there is a constant K_3 so that for any $A \in (1,\infty)$, $B \in (0,1)$, $\mathbb{N} \ni k \geq 2$ and $\{G_j\}_{j=1,\ldots,k-1}$ defined in $\mathcal{D}$ which satisfy (22)-(24) we have in (19),*

$$\|\eta^{3/2} R_k\|_{\infty,\mathcal{D}} \;\leq\; \frac{K_3}{k^2}\,\left(B^2 A^k + B A^{k-1}\right) \tag{25}$$

PROOF. It is convenient to break up R_k as:

$$R_k = R_{0,k} + R_{1,k}$$

where for $k=1$,

$$R_{0,k} = \frac{G_0^3}{2}$$

and for $k>1$,

$$R_{0,k} = \frac{3}{2} G_0 \sum_{*,k-1} G_{k_1} G_{k_2} + \frac{3}{2} G_0^2 G_{k-1} - 3 G_0 G_0''' \sum_{*,k} G_{k_1} G_{k_2} - G_0''' \sum_{*,k} G_{k_1} G_{k_2} G_{k_3}$$
$$- 3 G_0^2 \sum_{*,k} G_{k_1} G_{k_2}''' - 3 G_0 \sum_{*,k} G_{k_1} G_{k_2} G_{k_3}'''$$

where $\sum_{*,\alpha}$ denotes summation over $k_i \geq 1$ with $\sum_i k_i = \alpha$ and

$$R_{1,k}(\eta) = \frac{1}{2} \sum_{*,k-1} G_{k_1} G_{k_2} G_{k_3} - \sum_{*,k} G_{k_1} G_{k_2} G_{k_3} G_{k_4}'''$$

The proof follows by using the upper bounds on G_j, G_j' and G_j''' in (22)-(24) for $k-1 \geq j \geq 1$, using (14) and noting that

$$\sup_k \left\{ \sum_{*,k-1} \frac{k^3}{k_1^3 k_2^3},\; \sum_{*,k} \frac{k^2}{k_1^3 k_2^2},\; \sum_{*,k} \frac{k^2}{k_1^3 k_2^3 k_3^2},\; \sum_{*,k} \frac{k^3}{k_1^3 k_2^3 k_2^3},\; \sum_{*,k} \frac{k^2}{k_1^3 k_2^3 k_3^3 k_4^2} \right\} < \infty$$

□

2 Proofs

The proofs rely on bounding G_k in (5). For given k_0 and $1 \le k \le k_0$, it can be seen that the solution G_k to $\mathcal{L}_k G_k = \frac{R_k}{G_0^3}$ that goes to 0 as $\eta \to \infty$ in the sector $\arg \eta \in \left(-\frac{2}{9}\pi + \delta, \frac{2}{9}\pi - \delta\right)$, with $0 < \delta < \frac{\pi}{63}$, is given by

$$G_k(\eta) = \sum_{j=1}^{3} u_j(\eta) \int_{\infty e^{\theta_j}}^{\eta} v_j(\eta') \frac{R_k(\eta')}{G_0^3(\eta')} d\eta' \tag{26}$$

Here $\theta_1 = -\frac{2}{9}\pi + \delta$, $\theta_2 = \frac{2}{9}\pi - \delta$ and $\theta_3 = 0$. Also, in (26), u_1, u_2 and u_3 are three independent solutions of $\mathcal{L}_k u = 0$, with the following asymptotic behavior for large η (see [16]):

$$u_1(\eta) \sim \eta^{-15/8} \exp\left[i\frac{4\sqrt{2}}{27}\eta^{9/4}\right]$$

$$u_2(\eta) \sim \eta^{-15/8} \exp\left[-i\frac{4\sqrt{2}}{27}\eta^{9/4}\right]$$

$$u_3(\eta) \sim \eta^{\frac{9}{2}\beta_k}$$

where β_k is defined in (18), $(v_1, v_2, v_3)^T$ is the third column of Φ^{-1} and

$$\Phi(\eta) = \begin{bmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{bmatrix}$$

It is easily seen that for large $|\eta|$ we have

$$v_1(\eta) \sim -\frac{9}{4}\eta^{-5/8} \exp\left[-i\frac{4\sqrt{2}}{27}\eta^{9/4}\right]$$

$$v_2(\eta) \sim -\frac{9}{4}\eta^{-5/8} \exp\left[i\frac{4\sqrt{2}}{27}\eta^{9/4}\right]$$

$$v_3(\eta) \sim \frac{9}{2}\eta^{-\frac{9}{2}\beta_k - \frac{5}{2}}$$

and that v_1, v_2, v_3 are three independent solutions to the adjoint third order linear equation $\mathcal{L}_k^+ v = 0$, where the coefficients are regular when $G_0 \ne 0$. The G_k determined from (26) are bounded in any domain that excludes the singularities of G_0 (the only places where $G_0 = 0$), and it is easily seen that the bounds (22)-(24) are valid for $1 \le k \le k_0$ for large A and B (depending on k_0).

To prove the bounds (22)-(24) in Lemma 2 for all k, it is sufficient to prove them for sufficiently large k (large β_k).

Note. We have to treat separately two regimes of η and β_k : (a) $|\eta| = O(\epsilon \beta_k^{4/9})$ or larger and (b) $|\eta| = o(\beta_k^{4/9})$. These require different integral representation of G_k and choice of domain.

2.1 Control in the regime (a), $|\eta| >$ constant $\beta_k^{4/9}$

It is convenient to define

$$\chi = \beta_k^{-4/9}\eta \text{ and } z_k(\chi) = G_k(\beta_k^{4/9}\chi)$$

Then, using variation of parameters (see §5), we have

$$z_k(\chi) = \tilde{\mathcal{V}}\left[\hat{R}\right](\chi)$$

where

$$\tilde{\mathcal{V}}[\hat{R}](\chi) \equiv \sum_{j=1}^{3} \frac{1}{\beta_k^2} \int_{\infty e^{i\theta_j}}^{\chi} e^{\beta_k\ [P_j(\chi)-P_j(\tilde{\chi})]+W_j(\chi)\ -\ W_j(\tilde{\chi})} n_{j,3}(\tilde{\chi})\hat{R}(\tilde{\chi})d\tilde{\chi} \tag{27}$$

where $\hat{R}$ depends on R_k and z_k; $n_{j,3}$, and W_j are given functions of χ, whose exact expression is irrelevant, with behavior $n_{j,3} = O(\chi^{-5/2})$, $W_1 = -\frac{15}{8}\ln\ \chi + o(1)$, $W_2 = -\frac{15}{8}\ln\ \chi + o(1)$ and $W_3 = o(1)$ for large χ; and $P_1'(\chi)$, $P_2'(\chi)$ and $P_3'(\chi)$ are the three roots of the cubic

$$\alpha^3 + \frac{2}{9}\chi^{5/2}\alpha - \chi^{3/2} = 0 \tag{28}$$

with the following asymptotic behavior for large χ:

$$\begin{aligned} P_1 &= \frac{4\sqrt{2}}{27}i\chi^{9/4} - \frac{9}{4}\ln\ \chi + o(1) \\ P_2 &= -\frac{4\sqrt{2}}{27}i\chi^{9/4} - \frac{9}{4}\ln\ \chi + o(1) \\ P_3 &= \frac{9}{2}\ln\ \chi + o(1) \end{aligned}$$

as $\chi \to \infty$ for $\arg\chi \in \left(-\frac{2}{9}\pi + \delta, \frac{2}{9}\pi - \delta\right)$, with $0 < \delta < \frac{\pi}{63}$. In (27), $\theta_1 = -\frac{2}{9}\pi + \delta$, $\theta_2 = \frac{2}{9}\pi - \delta$ and $\theta_3 = 0$.

It is necessary that the operators $\tilde{\mathcal{V}}$ be defined in a suitable domain $\mathcal{E}$ in the χ-plane containing the integration path where the bounds for the previous z_j, $j = 1, ..., k-1$ are available to estimate R_k. Also $\tilde{\mathcal{V}}$ need to be bounded for large β_k. To satisfy the latter requirement for each j ($j = 1, 2, 3$), any point $\chi \in \mathcal{E}$ must have the property that it can be connected to $\infty e^{i\theta_j}$ along a path $\tilde{\mathcal{C}}_j$ entirely in $\mathcal{E}$ so that on the path $\tilde{\chi}(s)$, parameterized by the arclength s increasing towards ∞,

$$\frac{d}{ds}\Re P_j(\tilde{\chi}(s)) \geq C|\tilde{\chi}(s)|^{5/4} > 0\ , \quad \text{for } j = 1, 2$$

$$\frac{d}{ds}\Re P_3(\tilde{\chi}(s)) \geq C\ |\tilde{\chi}(s)|^{-1} > 0\ ,$$

where C is a constant independent of χ. It is shown in §3 that these properties are ensured if we choose

$$\mathcal{E} = \left\{\chi :\ \chi \text{ to the right of } \partial\mathcal{E}_L,\ \arg\chi \in \left(-\frac{2}{9}\pi + \delta,\ \frac{2}{9}\pi - \delta\right)\right\}$$

where $\partial\mathcal{E}_L$ is the polygonal line connecting χ_1, χ_3 and χ_2, and where

$$\chi_3 = \epsilon\ ,\quad \chi_2 = \chi_3 + \tilde{\rho}e^{i2\pi/3}\ ,\quad \chi_1 = \chi_3 + \tilde{\rho}e^{-i2\pi/3}$$

Here $\tilde{\rho}$ is chosen so that $\arg\chi_1 = -\frac{2\pi}{9}+\delta$ and $\arg\chi_2 = \frac{2}{9}\pi-\delta$ and ϵ is suitably small, independent of k, so that $\tilde{\delta}$ appearing in the proof of Theorem 34 in §6 is smaller than $\frac{1}{2}$. The domain $\mathcal{E}$ is sketched in Fig. 3. Corresponding to $\mathcal{E}$, we define the domain $\mathcal{E}_k$ (Figure 2)

$$\mathcal{E}_k = \left\{ \eta \ : \ \beta_k^{-4/9}\eta = \chi \in \mathcal{E} \right\}$$

2.2 Control in regime (b), $\eta = o\left(\beta_k^{4/9}\right)$:

In this case, as shown in §6, we can write

$$G_k(\eta) = \mathcal{V}\left[\hat{R}_k\right] + \sum_{j=1}^{3} a_j g_j(\eta)$$

where

$$\mathcal{V}\left[\hat{R}_k\right] \equiv \sum_{j=1}^{3} \frac{\beta_k^{-2/3}}{3}\omega_j G_0(\eta) \int_{\eta_{j,k}}^{\eta} G_0(\eta')\hat{R}_k(\eta')\mathrm{e}^{\omega_j \beta_k^{1/3}[P(\eta)-P(\eta')]}d\eta' \tag{29}$$

$g_j = G_0\ \mathrm{e}^{\omega_j\beta_k^{1/3}P}$, $\hat{R}_k$ involves R_k, G_k and G_k' and

$$P(\eta) = \int_{\eta_i}^{\eta} \frac{1}{G_0(\eta')}d\eta' \ , \ \text{for some } \eta_i \in \mathcal{D} \tag{30}$$

while $\omega_1 = \mathrm{e}^{i2\pi/3}$, $\omega_2 = \mathrm{e}^{-i2\pi/3}$ and $\omega_3 = 1$ (the three cubic roots of unity). In (29) the limits of integration satisfy $\eta_{j,k} \equiv \beta_k^{4/9}\chi_j$, where χ_1, χ_2 and χ_3 are as defined in the last subsection. The choice of the domain $\mathcal{D}_k$ for η is subject to the conditions below.

1. $\mathcal{D}_k$ must contain a region $\mathcal{S}_0$ that winds around η_s, near $\arg\eta = -\frac{4}{9}\pi$, excluding a $O(\eta_s^{-5/4})$ neighborhood of η_s itself. Since the singularity $\hat{\eta}_s$ of G_0 is within $\mathrm{e}^{-C|\eta_s|}$ distance of η_s and a singularity of G_0 is the only point where $G_0 = 0$, this condition ensures a lower bound on G_0 and provides the contour integration $\oint_{S_{\epsilon_1}}$ in Remark 2.

2. Any point $\eta \in \mathcal{D}_k$ can be connected to $\eta_{j,k}$ along a contour $\mathcal{C}_j$ that lies entirely within $\mathcal{D}_k$ such that $\Re\left[\omega_j P\right]$ is increasing monotonically when the points $\eta_{j,k}$ are approached. The integration contour $\mathcal{C}_j$ in (29) is chosen to be such a path. Monotonicity ensures there is no exponential growth in k (β_k). A fortiori, the points $\eta_{j,k}$, as defined above, are points of maximum of $\Re\left[\omega_j P\right]$ in $\mathcal{D}_k$.

3. We must have for $k \geq k_0$, the property $\mathcal{D}_{k+1} \cup \mathcal{E}_{k+1} \subset \mathcal{D}_k \cup \mathcal{E}_k$. For $k \leq k_0$, all G_k can be determined through the representation (26) on a common domain $\mathcal{D}_{k_0} \cup \mathcal{E}_{k_0}$. The necessity of this condition comes from the fact that the R_k, needed to determine G_k in the domain $\mathcal{D}_k \cup \mathcal{E}_k$, involve G_1, G_2, ..., G_{k-1}.

4. For any k, $\mathcal{D}_k \cup \mathcal{E}_k$ must contain the region $\mathcal{S}_0$ around the singularity η_s. Also, for large enough $|\eta|$ in this domain, we must have $\arg\eta \in \left[-\frac{2}{9}\pi + \delta, \frac{2}{9}\pi - \delta\right]$. We shall furthermore ensure a nonempty common domain $\mathcal{D} = \cap_{k\geq k_0}\left[\mathcal{D}_k \cup \mathcal{E}_k\right]$.

5. To simplify the analysis, we seek domains so that $\mathcal{D}_k \cup \mathcal{E}_k$ does not contain turning points (occurring when $\arg\eta = \pm\frac{2}{9}\pi$) of the WKB solutions for large k in (27), $e^{\beta_k P_j(\chi(\eta))+W_j(\chi(\eta))}$ (see §4).

3 Properties of $P(\eta)$ and choice of the domains $\mathcal{D}$ and $\mathcal{D}_k$

This section is devoted to the construction of the domains $\mathcal{D}$ and $\mathcal{D}_k$ corresponding to a particular η_s, determined from (13) for large enough $\hat{n} \in \mathbb{N}$. The domains clearly depend on the choice of η_s. The construction is relatively involved since monotonicity of $\Re[\omega_j P\,]$ must be ensured, while $P(\eta)$ is only implicitly known through (9)–(12) and (30). Also, such a domain has to contain an annular region around singularity η_s.

Remark 7 *In this section, constants such as C, K, δ, r_i, r_0, etc are positive and independent of η and η_s.*

First we define $\mathcal{D}_A$, part of the region where $G_0(\eta) \sim \ \eta^{-1/2}$.

Definition 4 *For small $\delta \in \left(0, \frac{\pi}{63}\right)$ we have*

$$\mathcal{D}_{A_1} = \left\{\eta : |\eta| > \frac{1}{2}|\eta_s|, \ \ \arg\eta \in \left(-\frac{2}{9}\pi + \delta, \frac{2}{9}\pi - \delta\right)\right\}$$

$$\mathcal{D}_{A_2} = \left\{\eta : \ \theta = \arg\eta \in \left(-\frac{4}{9}\pi + \delta, -\frac{2}{9}\pi + \delta\right], |\eta_1(\theta))| > |\eta| > \frac{1}{2}|\eta_s|, \right\}$$

where $\eta_1(\theta(r)) = 2|\eta_s|e^{-i\frac{4}{9}\pi + i\delta} + re^{-i\frac{\pi}{9}}$ for $r \geq 0$. We also define $M = |\eta_1\left(-\frac{2}{9}\pi + \delta\right)|$ and

$$\mathcal{D}_A = \mathcal{D}_{A_1} \cup \mathcal{D}_{A_2}$$

(See Figure 1).

Lemma 5 *For any point $\eta \in \mathcal{D}_A$, there exist three piecewise smooth paths from η to ∞, $\tilde{\eta} := \tilde{\eta}_j$, for $j = 1, 2, 3$, contained in $\mathcal{D}_A$ so that on any smooth segment we have*

$$\frac{d}{ds}\Re\left(\omega_j P[\tilde{\eta}(s)]\right) > C|\tilde{\eta}(s)|^{1/2} > 0$$

where s is the arclength. Furthermore,

$$|\tilde{\eta}(s)| > C_1|\eta| > 0$$

PROOF. On the line segment $\tilde{\eta}(s) = \eta_0 + se^{i\phi}$ in $\mathcal{D}_A$, (30) and largeness of $|\tilde{\eta}(s)|$ and $|\eta_s|$, together with the asymptotic behavior $G_0(\tilde{\eta}) \sim \tilde{\eta}^{-1/2}$, imply

$$\frac{d}{ds}\ \Re(\omega_j P) = \ \Re\left[\frac{\omega_j}{G_0(\tilde{\eta}(s))}\tilde{\eta}'(s)\right] \ \sim \ |\tilde{\eta}(s)|^{1/2}\cos\left(\theta_j + \frac{\theta_{\tilde{\eta}}}{2} + \phi\right)$$

where $\theta_j = \ \arg\omega_j \in \left\{\pm\frac{2}{3}\pi, 0\right\}$, $\theta_{\tilde{\eta}} = \arg\tilde{\eta}(s)$. For suitable ϕ and η_0, it is easy to see that for any $\eta \in \mathcal{D}_A$ and $j = 1, 2, 3$, there exists a polygonal line so that $\cos\left(\theta_j + \frac{1}{2}\theta_{\tilde{\eta}} + \phi\right) > C > 0$. Further, the line can be chosen so that $|\tilde{\eta}(s)| \geq |\eta|$. □

Definition 6

$$L = \left\{ \eta : \arg \eta = -\frac{4}{9}\pi + \delta, \frac{1}{2}|\eta_s| < |\eta| < 2|\eta_s| \right\}$$

Definition 7 *Let $\mathcal{S}_0$ be a region around η_s (the singularity of $U(\zeta(\eta))$ in (12)) defined by*

$$\mathcal{S}_0 = \left\{ \eta : \; r_i|\eta_s|^{-5/4} < |\eta - \eta_s| < r_o|\eta_s|^{-5/4}, \; \arg(\eta - \eta_s) \; \in \; \left(-\pi + \frac{\pi}{18}, \frac{\pi}{18} + \pi \right) \right\} \tag{31}$$

with $0 < r_i < r_0$, small enough to ensure $\arg U_0^{1/2} \in \left[-\frac{2}{5}\pi, \frac{2}{5}\pi\right]$ (see relation (16)).

Definition 8 *We define*

$$\mathcal{D}_{T,1} = \left\{ \eta : |\eta^{3/2} - \eta_0^{3/2}| < 3\left(\frac{1+B_0}{1-B_0}\right)^2 r \quad \text{for } \eta_0 \in \mathcal{S}_0 \text{ and} \right.$$
$$\left. \left| \frac{1-\sqrt{U}}{1+\sqrt{U}} \right| < B_0 \mathrm{e}^{-K_4|\eta_s|^{3/4} r} < 1, \text{ for some } r \in [0, |\eta_s|^{1/4}] \right\}$$

For $r \in \left[0, \frac{2\delta}{K_3}|\eta_s|^{3/2}\right]$, we define

$$\mathcal{D}_{T2,r} := \left\{ \eta : \quad (1) \; \arg \eta \in \left(\arg \eta_s - K_5|\eta_s|^{-5/4} + K_3|\eta_s|^{-3/2} r, -\frac{4}{9}\pi + \delta, \right] \right.$$
$$(2) \; |\eta|^{3/2} \in \left(|\eta_s|^{3/2} - K_5|\eta_s|^{1/4} - 3r, |\eta_s|^{3/2} + K_5|\eta_s|^{1/4} + 3r \right)$$
$$\left. (3) \; |U - 1| < 5B_0 \mathrm{e}^{-K_4|\eta_s|} \mathrm{e}^{-K_1|\eta_s|^{3/4} r} \right\}$$

$$\begin{aligned} \mathcal{D}_{T2} &:= \bigcup_{r \in I} \mathcal{D}_{T2,r}, \quad \text{where} \quad I = \left[0, \frac{2\delta}{K_3}|\eta_s|^{3/2}\right] \\ \mathcal{D}_T &:= \mathcal{D}_{T,1} \cup \mathcal{D}_{T,2} \\ \mathcal{D} &:= \mathcal{D}_A \cup \mathcal{D}_T \end{aligned}$$

Theorem 9 *For large $|\eta_s|$, for any point $\eta \in \mathcal{D}$, there exist B_0, K_i and piecewise smooth paths from η to ∞ $\tilde{\eta}_j =: \tilde{\eta}$ $(j = 1, 2, 3)$ contained in $\mathcal{D}$, so that on any smooth subsegment we have*

$$\frac{d}{ds} \Re \left(\omega_j P[\tilde{\eta}(s)] \right) > C|\tilde{\eta}(s)|^{1/2} > 0 \tag{32}$$

Furthermore

$$|\tilde{\eta}(s)| > C_1|\eta| > 0 \tag{33}$$

For the proof, given at the end of §2, we need a few more definitions, constructions and lemmas.

Remark 8 *Note that by Lemma 5, it is enough to show that for any $\eta \in \mathcal{D}_T$, we can choose a path for each of $j = 1, 2, 3$ connecting η to $\eta_L \in L$, entirely within $\mathcal{D}_T$ so that the monotonicity property (32) is satisfied. Noting also that since the ratio of any two values of $\eta \in \mathcal{D}_T$ is bounded by a constant independent of η_s, the second part of Theorem 9 follows.*

Definition 10 *For $k \geq k_0$, we define*

$$\eta_{1,k} = \eta_{3,k} + \rho_0 \mathrm{e}^{-i\frac{2}{3}\pi}, \eta_{2,k} = \eta_{3,k} + \rho_0 \mathrm{e}^{i\frac{2}{3}\pi}, \eta_{3,k} = \epsilon \beta_k^{4/9}$$

where ρ_0 is chosen so that $\arg \eta_{1,k} = -\frac{2\pi}{9} - \delta$, $\arg \eta_{2,k} = \frac{2\pi}{9} + \delta$ for $0 < \delta < \frac{\pi}{63}$. The parameter ϵ is small, but independent of k, as needed in Lemma 25, and k_0 is chosen large enough so that for $k \geq k_0$, we have $\epsilon \beta_k^{4/9} > M$, for M as defined in Definition 4. We define a boundary $\partial E_k = \partial E_k^- \cup \partial E_k^+$ where ∂E_k^- is the straight line joining $\eta_{3,k}$ with $\eta_{1,k}$ and ∂E_k^+ is the straight line joining $\eta_{3,k}$ to $\eta_{2,k}$. We then define $\mathcal{D}_k$ (See Fig. 2)

$$\mathcal{D}_k = \mathcal{D} \backslash \mathcal{E}_k$$

Lemma 11 *Given $j = 1, 2$ or 3, for any $\eta \in \partial E_k$, the path $\tilde{\eta}(s)$ from η to $\eta_{j,k}$ along ∂E_k satisfies the monotonicity property (32).*

Proof. On ∂E_k^+ we note that

$$\frac{d}{d\rho} \Re \left[\omega_3 P(\eta_{3,k} + \rho \mathrm{e}^{i2\pi/3}) \right] \sim \Re \left[\mathrm{e}^{i2\pi/3} \eta^{1/2} \right] = -|\eta|^{1/2} \sin \left(\frac{\pi}{6} + \frac{1}{2} \arg \eta \right) < -C|\eta|^{1/2}$$

for some positive constant C. On ∂E_k^- we note that

$$\frac{d}{d\rho} \Re \left[\omega_3 P(\eta_{3,k} + \rho \mathrm{e}^{-i2\pi/3}) \right] \sim \Re \left[\mathrm{e}^{-i2\pi/3} \eta^{1/2} \right] = -|\eta|^{1/2} \sin \left(\frac{\pi}{6} + \frac{1}{2} \arg \eta \right) < -C|\eta|^{1/2}$$

for some positive constant C. It is therefore clear that the path $\tilde{\eta}_3(s)$ from $\tilde{\eta}$ to $\eta_{3,k}$ satisfies the monotonicity property (32). In a similar manner, it is seen that $\Re[\omega_1 P]$ and $\Re[\omega_2 P]$ satisfy (32) on a path from η to $\eta_{j,k}$ for $j = 1, 2$ along ∂E_k. □

The lemma above, together with Theorem 9 prove the following Corollary:

Corollary 12 *(**Property 1:**) For all sufficiently large k, given any point $\eta \in \mathcal{D}_k$, there exists a piecewise smooth path $\mathcal{C}_j$ for each $j = 1, 2, 3$ from η to $\eta_{j,k}$ such that the path is entirely in $\mathcal{D}_k$ and*

$$\frac{d}{ds} \Re \left(\omega_j P(\tilde{\eta}(s)) \geq C|\tilde{\eta}|^{1/2} > 0 \right. \tag{34}$$

Furthermore, if $\tilde{\eta} \in \mathcal{C}_j$, we then have $|\tilde{\eta}| > C|\eta|$.

Remark 9 *To prove Theorem 9, we introduce three autonomous flows as follows.*

Definition 13 *Let $\eta_j(t, \eta_0)$ be the solution to the differential equation*

$$\dot{\eta} = i\mathrm{e}^{-i\phi_j} \omega_j^{-1} G_0(\eta), \text{ where } \omega_1 = \mathrm{e}^{i2\pi/3},\ \omega_2 = \mathrm{e}^{-i2\pi/3},\ \omega_3 = 1 \tag{35}$$

with initial condition $\eta_j(0,\eta_0)=\eta_0$*, where* ϕ_j *are given by:*

$$\phi_1 = \frac{\pi}{3},\ \phi_2 = \frac{6\pi}{7},\ \phi_3 = \frac{2}{3}\pi \tag{36}$$

Remark 10 *We note from (30) that for any choice* $\phi_j \in (0,\pi)$*,*

$$\frac{d}{dt}\ \Re\left[\omega_j P(\eta_j(t,\eta_0))\right] = \ \cos\left(\phi_j - \frac{\pi}{2}\right) > 0$$

Hence using arclength parameterization we have

$$\frac{d}{ds}\ \Re\left[\omega_j P(\tilde{\eta}(s)\right] = \frac{\cos\left(\phi_j - \frac{\pi}{2}\right)}{|G_0(\tilde{\eta}(s))|} > C|\tilde{\eta}(s)|^{1/2}$$

when $\tilde{\eta}(s)\in\mathcal{D}$*. Thus, the differential equation (35) generates ascent paths for* $\Re[\omega_j P]$*.*

Lemma 14 *There exists a* B_0 *so that* $\mathcal{S}_0 \subset \mathcal{D}_T$*.*

Proof. Since for $\eta\in\mathcal{S}_0$, the corresponding U determined from (12) has upper and lower bounds independent of η_s, as discussed in Remark 2. Also, from (16), for $\eta\in\mathcal{S}_0$, $\arg U^{1/2}\in\left[-\frac{2}{5}\pi, \frac{2}{5}\pi\right]$. Thus, it follows that for $\eta\in\mathcal{S}_0$, we have $|1-\sqrt{U}|/|1+\sqrt{U}| < B_0$ for some $B_0<1$. Thus, for some B_0, we have $\mathcal{S}_0\subset\mathcal{D}_{T,1}\subset\mathcal{D}_T$. □

Definition 15 *It is convenient to define, see (35) and (36),*

$$\nu_j = i\mathrm{e}^{-i\phi_j}\omega_j^{-1}$$

Remark 11 *It follows that*

$$\arg\nu_1 = -\frac{\pi}{2},\quad \arg\nu_2 = \frac{13\pi}{42},\quad \arg\nu_3 = -\frac{\pi}{6} \tag{37}$$

The specific choice of ϕ_j *(and thus of* ν_j*) is unimportant, but it is essential that* ϕ_j*,* $\arg\nu_j$ *remain in compact subintervals of* $(0,\pi)$ *and* $\left(-\frac{2}{3}\pi, \frac{\pi}{3}\right)$ *respectively, independent of* η_s *and* δ*.*

In order to study the solution to (35) near η_s, it is convenient to think of $U(t)=U(\eta(t))$ as an unknown together with $\eta(t)$. Using (10), (12) and (35), it follows that

$$\frac{2}{3}\frac{d}{dt}\eta^{3/2} = \nu_j U\left[1+E_1(\eta)\right] \quad \text{where} \quad E_1(\eta) = \frac{\eta^{1/2}G_0(\eta)-U}{U} \tag{38}$$

$$\frac{d}{dt}U = -\alpha_j|\eta_s|^{3/4}\sqrt{U}(U-1)\left[1+E_1\right]\left[1+E_2\right],\ \text{where}$$

$$E_2(\eta) = \left[\frac{\nu_j}{\alpha_j|\eta_s|^{3/4}}\left(\frac{i\sqrt{2}}{3}\eta^{3/4}+\frac{9}{8\eta^{3/2}}\right)-1\right] \tag{39}$$

where E_1, E_2 will be shown to be small for large $|\eta_s|$ in the range of integration and

$$\alpha_j = \frac{\nu_j}{|\eta_s|^{3/4}} \left[\frac{i\sqrt{2}}{3} \eta_s^{3/4} + \frac{9}{8\eta_s^{3/2}} \right]$$

The initial condition U_0 satisfies

$$\frac{i4\sqrt{2}}{27}\eta_0^{9/4} - \frac{i4\sqrt{2}}{27}\eta_s^{9/4} + \frac{9}{8} \ln\left(\frac{\eta_0}{\eta_s}\right) = -\ln \frac{1-\sqrt{U_0}}{1+\sqrt{U_0}} - 2\sqrt{U_0} \tag{40}$$

Remark 12 *It is to be noted that with ϕ_j given by (36) and using the fact that as $\hat{n} \to \infty$ (i.e. as $|\eta_s| \to \infty$), we get $\arg \eta_s \to -\frac{4\pi}{9}$. It follows that in this limit,*

$$\arg \alpha_1 \to -\frac{\pi}{3}, \ \ \arg \alpha_2 \to \frac{10\pi}{21}, \ \ \arg \alpha_3 \to 0 \tag{41}$$

It is important for us that $\arg \alpha_j \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Lemma 16 *For suitable K_i and $\delta < \frac{K_3}{24}$, if $\eta_{0,0} \in \mathcal{D}_{T_2}$, then for sufficiently large $|\eta_s|$ and some $t \in \left(0, \frac{2\delta}{K_3}|\eta_s|^{3/2}\right)$, $\eta_j(t;\eta_{0,0})$, leaves $\mathcal{D}_{T_2}$ through L.*

Proof. The differential equation satisfied by η and the corresponding U for $0 \le t \le 2\frac{\delta}{K_3}|\eta_s|^{3/2}$ is given by:

$$\frac{2}{3}\frac{d}{dt}\eta^{3/2} = \nu_j[1+E_3] \ \ ; \ \ \frac{d}{dt}U = -\nu_j \left(\frac{i\sqrt{2}}{3}\eta^{3/4} + \frac{9}{8\eta^{3/2}} \right)(U-1)(1+E_4)$$

where

$$E_3 = E_1 U + (U-1), \ \ E_4 = (\sqrt{U}-1) + \sqrt{U}E_1,$$

where E_1, E_2 are defined in (38) and (39). It follows that

$$\eta^{3/2} = \eta_{0,0}^{3/2} + \frac{3}{2}\nu_j \int_0^t (1+E_3)dt \tag{42}$$

$$(U-1) = (U_{0,0}-1)\exp\left[-\nu_j \int_0^t \left(\frac{i\sqrt{2}}{3}\eta^{3/4} + \frac{9}{8\eta^{3/2}} \right)(1+E_4)dt \right] \tag{43}$$

where $U_{0,0}$ is obtained from (40) by substituting $\eta_0 = \eta_{0,0}$. It is convenient to define the leading order equations

$$\tilde{\eta}^{3/2} = \eta_{0,0}^{3/2} + \frac{3}{2}\nu_j t \ \ ; \ \ \tilde{U} - 1 = (U_{0,0}-1)\exp\left[-\nu_j \int_0^t \left(\frac{i\sqrt{2}}{3}\tilde{\eta}^{3/4} + \frac{9}{8\tilde{\eta}^{3/2}} \right) dt \right] \tag{44}$$

From (42), (43) and (44) it follows that

$$\eta^{3/2} - \tilde{\eta}^{3/2} = \frac{3}{2}\nu_j \int_0^t E_3 dt \tag{45}$$

$$U-\tilde{U}=(U_{0,0}-1)\Bigg\{\exp\left[-\nu_j\int_0^t\left(\frac{i\sqrt{2}}{3}\eta^{3/4}+\frac{9}{8\eta^{3/2}}\right)(1+E_4)dt\right] \\ -\exp\left[-\nu_j\int_0^t\left(\frac{i\sqrt{2}}{3}\tilde{\eta}^{3/4}+\frac{9}{8\tilde{\eta}^{3/2}}\right)dt\right]\Bigg\} \quad (46)$$

From (44) it follows that

$$\arg\eta_{0,0}+3t|\eta_s|^{-3/2}\geq\arg\tilde{\eta}\geq\arg\eta_{0,0}+\frac{3}{2}K_3t|\eta_s|^{-3/2}\quad\text{where } 2K_3=\min_{j=1,2,3}\sin\left(\frac{2}{3}\pi+\nu_j\right)$$

$$|\tilde{\eta}|^{3/2}\in\left(|\eta_{0,0}|^{3/2}-\frac{3}{2}t,|\eta_{0,0}|^{3/2}+\frac{3}{2}t\right)$$

Using these relations in (44) we have

$$|(\tilde{U}-1)|=|(U_{0,0}-1)|\mathrm{e}^{-2K_1|\eta_s|^{3/4}t}$$

where $2K_1$ is a lower bound (independent of δ) of

$$\Re\left\{\frac{\nu_j}{2|\eta_s|^{3/4}}\left[\frac{i\sqrt{2}}{3}\tilde{\eta}^{3/4}+\frac{9}{8\tilde{\eta}^{3/2}}\right]\right\}$$

for $\tilde{\eta}$ restricted to the domain $|\tilde{\eta}|>\frac{1}{2}|\eta_s|$, $\arg\tilde{\eta}\in\left[-\frac{4}{9}\pi-\delta,-\frac{4}{9}\pi+4\delta\right]$. Thus, for some t in $0\leq t\leq\frac{3\delta}{2K_3}|\eta_s|^{3/2}$, $\tilde{\eta}$ leaves the domain $\mathcal{D}_{T_2}$ through the segment of L, when $\frac{15}{32}|\eta_s|<|\tilde{\eta}|<\frac{3}{2}|\eta_s|$.

Now, we show that η is close to $\tilde{\eta}$ and hence has roughly the same behavior. We define

$$(\zeta,V)=\left(\frac{\eta^{3/2}-\tilde{\eta}^{3/2}}{t_1},\frac{U-\tilde{U}}{U_{0,0}-1}\right)$$

on the interval $[0,t_1]$, for $0<t_1\leq\frac{2\delta}{K_3}|\eta_s|^{3/2}$. We introduce the norm

$$\|(\zeta,V)\|_\infty=\sup_{0\leq t\leq t_1}|\zeta(t)|+\sup_{0\leq t\leq t_1}\mathrm{e}^{\frac{3}{2}K_1|\eta_s|^{3/4}t}|V(t)|$$

and consider the right side of (45) and (46) as the mapping

$$(\mathcal{F}_1(\zeta,V),\mathcal{F}_2(\zeta,V))$$

of the ball

$$\mathcal{B}=\{(\zeta(t),V(t)):\|(\zeta,V)\|_\infty<\epsilon_1\}$$

for some small ϵ_1 in the Banach space of pair of continuous functions $(\zeta(t),V(t))$ of t in the interval $[0,t_1]$ for $t_1<\frac{2\delta}{K_3}|\eta_s|^{3/2}$.

Using the smallness of E_3 and E_4 for large η it can be checked directly that

$$(\mathcal{F}_1(\zeta,V),\mathcal{F}_2(\zeta,V))\in\mathcal{B}$$

and that

$$\|\,(\mathcal{F}_1(\zeta_1,V_1),\mathcal{F}_2(\zeta_1,V_1))-(\mathcal{F}_1(\zeta_2,V_2),\mathcal{F}_2(\zeta_2,V_2))\,\|_\infty\le\epsilon_2\|(\zeta_1,V_1)-(\zeta_2,V_2)\|_\infty$$

for some $\epsilon_2<1$ and the map is contractive. Thus, there is a unique solution to the integral system (45) – (46) for $(\zeta(t),V(t))$ in $\mathcal{B}$. In particular, this implies that

$$|U(t)-1|\le|U_{0,0}-1|\mathrm{e}^{-K_1|\eta_s|^{3/4}t},\ \ |(\eta(t))^{3/2}-\eta_{0,0}^{3/2}|\le 3t \tag{47}$$

Hence, with r as in the definition of $\mathcal{D}_{T_2,r}$ we have

$$|U-1|\le 5B_0\mathrm{e}^{-K_4|\eta_s|}\mathrm{e}^{-K_1|\eta_s|^{3/4}(t+r)}$$

$$\arg\eta\ge\arg\eta_{0,0}+K_3|\eta_s|^{-3/2}t\ge\arg\eta_s-K_5|\eta_s|^{-5/4}+K_3|\eta_s|^{-3/2}(t+r)$$

$$|\eta^{3/2}|\in\Big(|\eta_s|^{3/2}-K_5|\eta_s|^{1/4}-3(t+r),|\eta_s|^{3/2}+K_5|\eta_s|^{1/4}+3(t+r)\Big)$$

Therefore, from the definition of $\mathcal{D}_{T,2}$, for small enough $t+r$, we have $\eta\in\mathcal{D}_{T,2}$, while from continuity, there exists some larger $t+r\le\frac{2\delta}{K_3}|\eta_s|^{3/2}$ for which $\eta\in L$ as it exits $\mathcal{D}_{T_2}$. □

Lemma 17 *Let $\eta_{0,0}\in\mathcal{D}_{T,1}$. Define*

$$\hat\eta=\eta_j(t,\eta_{0,0})$$

Then, there exist B_0 and K_i so that $\hat\eta\in\mathcal{D}_{T_1}\cup\mathcal{D}_{T_2}$ for large $|\eta_s|$ and $0\le t\le|\eta_s|^{1/4}$.

Proof. Note that for any $0\le t\le|\eta_s|^{1/4}$, we write (38) and (39) as

$$\eta^{3/2}=\eta_{0,0}^{3/2}+\frac{3}{2}\int_0^t\nu_jU(1+E_1)dt\ ,\ \ \frac{1-\sqrt{U}}{1+\sqrt{U}}=b_{0,0}\exp\left\{-\alpha_j|\eta_s|^{3/4}\int_0^t(1+E_2)(1+E_1)dt\right\} \tag{48}$$

where $|b_{0,0}|<B_0\mathrm{e}^{-K_4|\eta_s|^{3/4}r}$, with B_0 chosen in accordance to Lemma 14 and $2K_4:=\min_j\cos\left(\frac{\pi}{6}+\nu_j\right)$. We introduce $\tilde\eta(t)$ and $\tilde U(t)$ (describing leading behavior) by

$$\frac{1-\sqrt{\tilde U}}{1+\sqrt{\tilde U}}=b_{0,0}\mathrm{e}^{-\alpha_j|\eta_s|^{3/4}t}\quad,\quad\tilde\eta^{3/2}=\eta_{0,0}^{3/2}+\frac{3}{2}\nu_j\int_0^t\tilde U(t')dt'$$

It is to be noted that

$$\eta^{3/2}-\tilde\eta^{3/2}=\frac{3}{2}\int_0^t\nu_j[(U-1)+UE_1]dt \tag{49}$$

$$\frac{1-\sqrt{U}}{1+\sqrt{U}}-\frac{1-\sqrt{\tilde U}}{1+\sqrt{\tilde U}}=b_{0,0}\mathrm{e}^{-\alpha_jt|\eta_s|^{3/4}}\left[\exp\left\{-\alpha_j|\eta_s|^{3/4}\int_0^t[(1+E_2)(1+E_1)-1]dt\right\}-1\right] \tag{50}$$

We note that $\frac{5}{3}K_4$ is a lower bound for $\Re\,[\alpha_j]$ for $|\eta_s|$ large. It is convenient to define the pair of continuous functions,

$$(\zeta(t),V(t))=\left(\eta^{3/2}(t)-\tilde\eta^{3/2}(t),\frac{1-\sqrt{U(t)}}{1+\sqrt{U(t)}}-\frac{1-\sqrt{\tilde U(t)}}{1+\sqrt{\tilde U(t)}}\right)$$

and the norm

$$\|(\zeta,V)\|_\infty = \sup_{0\le t\le t_1} |\zeta(t)| + \sup_{0\le t\le t_1} \mathrm{e}^{\frac{3}{2}K_4|\eta_s|^{3/4}t}|V(t)|$$

for $t_1 \in (0, |\eta_s|^{1/4})$. Consider the right hand side of (49) and (50) as a mapping $(\mathcal{F}_1(\zeta,V), \mathcal{F}_2(\zeta,V))$ on the ball

$$\mathcal{B} = \{(\zeta,V) : \|(\zeta,V)\|_\infty < \epsilon_1 t_1\}$$

Using smallness of E_1, E_2 and their derivatives with respect to η, it can be readily checked that $(\mathcal{F}_1,\mathcal{F}_2)$ is a contractive mapping of the ball $\mathcal{B}$ into itself; hence the solution (ζ,V) satisfying (49) and (50) is in $\mathcal{B}$ for large $|\eta_s|$. In particular, since $\Re\alpha_j > \frac{5}{3}K_4$ we have

$$\left|\frac{1-\sqrt{U(t)}}{1+\sqrt{U(t)}}\right| \le B_0 \mathrm{e}^{-K_4|\eta_s|^{3/4}(t+r)}, \ |[\eta(t)]^{3/2} - \eta_0^{3/2}| \le 3\left(\frac{1+B_0}{1-B_0}\right)^2 (t+r)$$

There are two cases: if $t+r \le |\eta_s|^{1/4}$, then clearly $\eta \in \mathcal{D}_{T_1}$. If $|\eta_s|^{1/4} \le t+r \le 2|\eta_s|^{1/4}$, from the definition of $\mathcal{D}_{T_2}$, it follows $\eta \in \mathcal{D}_{T,2}$, with $K_5 = 6\left(\frac{1+B_0}{1-B_0}\right)^2$. $\square$

Proof of Theorem 9. From Lemmas 5, 17, 16 (see Remark 8 as well), it is clear that that the domain $\mathcal{D} = \mathcal{D}_T \cup \mathcal{D}_A$ is invariant under the flows $\tilde{\eta}_j(s)$. From Remark 10, Theorem 9 follows.

4 Properties of $P_j(\chi)$ and choice of domain $\mathcal{E}$

Remark 13 *The WKB solution for large β_k of the homogenous equation $\mathcal{L}_k u = 0$ (see §2.2, item 5) is not uniformly valid in the domain $\mathcal{D}$ for large η. To invert the operator $\mathcal{L}_k$ in the regime $\eta = O(\beta_k^{4/9})$, we introduce the scaled variables:*

$$\chi = \beta_k^{-4/9}\eta \tag{51}$$

The WKB solution to the homogeneous equation is then of the form

$$\mathrm{e}^{\beta_k P_j(\chi)+W_j(\chi)} \text{ where } \alpha = P_j' \text{ are roots of the cubic } \alpha^3 + \frac{2}{9}\alpha\chi^{5/2} - \chi^{3/2} = 0 \tag{52}$$

We now choose a domain $\mathcal{E}$ where the WKB solution is valid. First, we define a boundary $\partial\mathcal{E}_L$, which corresponds in the χ plane to ∂E_k (see Definition 10).

Definition 18 *Let $\partial\mathcal{E}_L = \left\{\chi : \eta = \beta_k^{4/9}\chi \in \partial E_k\right\}$. We define $\partial\mathcal{E}_L^+$ and $\partial\mathcal{E}_L^-$ analogously in terms of ∂E_k^+ and ∂E_k^-, (see Definition 10).*

Definition 19 *We let*

$$\mathcal{E} \ = \ \left\{\chi : \chi \text{ to the right of } \partial E_L, \quad \arg\eta \in \ \left[-\frac{2\pi}{9}+\delta, \ \frac{2\pi}{9}-\delta\right]\right\}$$

(See Fig. 3.) It is also convenient to define

$$\mathcal{E}_k = \ \left\{\eta : \beta_k^{-4/9}\eta = \chi \ \in \ \mathcal{E}\right\}$$

Remark 14 *Note that for large k we have the following properties : $\mathcal{D} \subset \mathcal{D}_k \cup \mathcal{E}_k$ and $\mathcal{D}_{k+1} \cup \mathcal{E}_{k+1} \subset \mathcal{D}_k \cup \mathcal{E}_k$. This follows from the construction of $\mathcal{D}_k$ and $\mathcal{E}_k$. Our strategy is to prove the bounds in Lemma 2 in the domain $\mathcal{D}_k \cup \mathcal{E}_k$ based on bounds on all previous G_j, $j = 1,2...,(k-1)$ established on the domains $\mathcal{D}_j \cup \mathcal{E}_j$ (which contain $\mathcal{D}_k \cup \mathcal{E}_k$). The large k requirement is not restrictive, since for any fixed k_0 it is possible to choose A large enough so that the bounds in Lemma 2 hold for $1 \le j \le k_0$.*

The main theorem in this section is the following.

Theorem 20 *For any $\chi \in \mathcal{E}$, it is possible to choose a path $\mathcal{C}_j$ connecting χ to $\infty e^{i\theta_j}$, where $\theta_1 = -\frac{2\pi}{9} + \delta$, $\theta_2 = \frac{2}{9}\pi - \delta$ and $\theta_3 = 0$ so that, except for a finite set of points,*

$$\frac{d}{ds}\Re\left[P_{1,2}(\tilde{\chi}(s))\right] \;\geq\; C|\tilde{\chi}(s)|^{5/4} \;>\; 0$$

and

$$\frac{d}{ds}\Re\left[P_3(\tilde{\chi}(s))\right] \;\geq\; \frac{C}{|\tilde{\chi}(s)|} \;>\; 0$$

where s is the arc-length increasing towards ∞ and the (different) constants C above are independent of χ. Furthermore, for $|\chi|$ sufficiently large in $\mathcal{E}$, and with $\tilde{\chi} \in \mathcal{C}_j$ as above, we have $|\tilde{\chi}| > C\,|\chi|$ for $C > 0$ independent of χ and $\tilde{\chi}$.

PROOF. This follows, after a few Lemmas, at the end of §3. □

Remark 15 *Though the domain $\mathcal{E}$ restricts the size of $|\chi|$ (it is bounded below), it is convenient to first consider the properties of P_j on an enlarged domain $\mathcal{E}_0$ with no restriction on $|\chi|$ and larger width:*

Definition 21

$$\mathcal{E}_0 = \left\{\chi : \ \arg\chi \in \left[-\frac{2}{9}\pi,\ \frac{2}{9}\pi\right]\right\}$$

It is convenient to associate each P_j with a first order differential equation as follows. Note from (52) that with $P_j' := \chi^{5/4}\psi$ we have

$$\chi^{-9/4} = \psi^3 + \frac{2}{9}\psi \tag{53}$$

Now, we consider the trajectory in the complex χ plane generated by the differential equation

$$\frac{d\chi}{dt} = \frac{1}{P_j'(\chi)} \quad \text{implying} \quad \frac{4}{9}\frac{d\chi^{9/4}}{dt} = \frac{1}{\psi} \tag{54}$$

The solution with initial value χ_0 will be denoted by $\chi_j(t;\chi_0)$. Using (53), it follows that

$$\frac{d\psi}{dt} = -\frac{\psi(2+9\psi^2)^2}{4(2+27\psi^2)} \tag{55}$$

For large $\chi \in \mathcal{E}$ it is clear from (53) that the three possible behaviors of ψ are $\psi \sim i\sqrt{\frac{2}{9}}$, $\psi = -i\sqrt{\frac{2}{9}}$ and $\psi \sim \frac{9}{2}\chi^{-9/4}$. We associate these behaviors with P_1', P_2' and P_3' respectively, so that

$$P_1' \sim i\sqrt{\frac{2}{9}}\chi^{5/4}\ ,\ P_2' \sim -i\sqrt{\frac{2}{9}}\chi^{5/4}\ ,\ P_3' \sim \frac{9}{2}\chi^{-1} \tag{56}$$

Remark 16 *Note that $ds = |\frac{d\tilde{\chi}}{dt}|dt$, and so on a trajectory generated by the differential equation (54), we have*

$$\frac{d}{ds}\ \Re P_j(\tilde{\chi}) = |P_j'(\tilde{\chi})|$$

and hence one of the two conditions in Theorem 20 is satisfied by the path $\mathcal{C}_j = \{\tilde{\chi} : \tilde{\chi} = \chi_j(t, \chi)\}$, provided it remains within $\mathcal{E}$.

Lemma 22 *$\Re P_1$ increases monotonically on the boundary of $\mathcal{E}_0$ counterclockwise from $\infty e^{i\frac{2}{9}\pi}$ to $\infty e^{-i\frac{2}{9}\pi}$ with*

$$\frac{d}{ds}\ \Re P_1(\chi(s)) > C|\chi(s)|^{5/4},$$

while $\Re P_2$ increases monotonically on the boundary of $\mathcal{E}_0$ clockwise from $\infty e^{-i\frac{2}{9}\pi}$ to $\infty e^{i\frac{2}{9}\pi}$ with

$$\frac{d}{ds}\ \Re P_2(\chi(s)) > C|\chi(s)|^{5/4}$$

s being arc-length on $\mathcal{E}_0$.

PROOF. Consider the solution to (55), with initial condition on the imaginary ψ-axis slightly above $\psi = i\sqrt{\frac{2}{9}}$. This corresponds to starting at $\chi = \infty e^{i2\pi/9}$ with $P_1'(\chi)$ and tracing the Stokes line where Im $P_1 = 0$ and $\Re P_1$ is increasing. From the equation it is clear that ψ remains on the imaginary axis and approaches $i\infty$, implying that $\arg\chi = \frac{2}{9}\pi$ is a Stokes line where $\Re P_1$ is increasing monotonically all the way to the origin in the χ-plane. This also means that locally near $\chi = 0$, $P' \sim \omega_1\chi^{1/2}$ and $P_1 \sim \frac{2}{3}\omega_1\chi^{3/2}$, since this is the only root of the cubic (53) which is real on $\chi = re^{i2\pi/9}$. This corresponds to $\psi \sim \omega_1\chi^{-3/4}$ as $\chi \to 0$. Now, taking the initial condition slightly above $\psi = i\sqrt{\frac{2}{27}}$, it is clear from the differential equation (55) that ψ remains on the positive imaginary ψ-axis and approaches $\psi = i\sqrt{\frac{2}{9}}$ from below. This corresponds to the fact that $\arg\chi = -\frac{2}{9}\pi$ is a Stokes line beyond the turning point $\chi = \chi_s = \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9} e^{-i2\pi/9}$, with $\Re P_1$ increasing monotonically towards $\infty e^{-i\frac{2}{9}\pi}$ and for large r, $\frac{d}{dr}\ \Re P_1 \geq Cr^{5/4}$. Now, consider the segment $\chi = r\ e^{-i2\pi/9}$, where $0 < r < \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}$. If we introduce the transformations

$$\psi = i\ \Psi; \quad \chi = r\ e^{-i2\pi/9}$$

into (52), then

$$\Psi^3 - \frac{2}{9}\Psi + q^{-1} = 0, \text{where } \ q = r^{9/4}$$

The roots of the cubic that corresponds to P_j' are:

$$\Psi = \Psi_j = -\frac{(2916)^{1/3}}{18q^{1/3}} J^{1/3}\omega_j^{-1} \quad - \quad \frac{4q^{1/3}}{3(2916)^{1/3}J^{1/3}}\omega_j \ \text{ where } \ J = 1 - \sqrt{1 - \frac{96q^2}{59049}} \tag{57}$$

(the principal branch is used). The asymptotic behavior of Ψ_j for small r is given by

$$\Psi_1 \sim \mathrm{e}^{i\pi/3} r^{-3/4}, \Psi_2 \sim -r^{-3/4}, \Psi_3 \sim \mathrm{e}^{-i\pi/3} r^{-3/4} \tag{58}$$

From (57), it follows that on the line $\chi = r\mathrm{e}^{-i2\pi/9}$, for $0 < r < \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}$, we have

$$\frac{d}{dr}\Re P_1(r\mathrm{e}^{-i2\pi/9}) = r^{5/4}\Re\Psi_1 > Cr^{1/2} > 0$$

Thus, for all r, we have $\frac{d}{dr}\Re P_1\left(r\mathrm{e}^{-i2\pi/9}\right) > Cr^{5/4}$. From the reflection-symmetry between P_1 and P_2 on the positive real χ-axis, the statement for P_2 follows. □

Lemma 23 *$\Re P_3$ decreases monotonically on the boundary of $\mathcal{E}_0$ counter-clockwise from $\infty\mathrm{e}^{\pm i\frac{2}{9}\pi}$ to 0, and*

$$\frac{d}{ds}\Re P_3(\chi(s)) > \frac{C|\chi(s)|^{1/2}}{|\chi(s)|^{3/2}+1}$$

s being the arc-length towards ∞. In this, the positive real χ-axis is a Stokes line with $\Re P_3$ increasing towards $\infty\mathrm{e}^{i0}$ and satisfying the above monotonicity condition.

PROOF. Consider (55) starting with ψ on the positive imaginary axis, slightly below $\psi = i\sqrt{\frac{2}{27}}$, corresponding to $\chi = \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}\mathrm{e}^{-i2\pi/9}$. The differential equation implies that ψ remains on the positive imaginary axis as it moves towards the origin. This corresponds to $\chi = \infty\mathrm{e}^{-i\frac{2}{9}\pi}$, since $\psi \sim \frac{9}{2}\chi^{-9/4}$ for large χ, where $P_3' \sim \frac{9}{2\chi}$. Thus, the segment $\chi = r\mathrm{e}^{-i\frac{2}{9}\pi}$, $r > \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}$ is a Stokes line with

$$\frac{d}{dr}\Re P_3\left(r\mathrm{e}^{-i\frac{2}{9}\pi}\right) > \frac{C}{r}$$

From the symmetry about the real χ-axis, the same argument can be repeated for $\chi = r\mathrm{e}^{i\frac{2}{9}\pi}$ for $r > \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}$ to show that this segment is also part of the Stokes line with $\Re P_3$ increasing with r.

For $r < \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9}$, an examination of Ψ_3 in (57) shows that $\Re P_3(r\mathrm{e}^{\pm i\frac{2}{9}\pi})$ continues to decrease monotonically with decreasing r, though these segments are not part of any Stokes line. Near the origin, given the asymptotics of Ψ_3 in (58), it follows that $P_3(\chi) \sim \frac{2}{3}\chi^{3/2}$. Hence a corresponding inequality follows, incorporating this behavior at the origin, while at the same time satisfying condition for large χ

$$\frac{d}{dr}\Re P_3\left(r\mathrm{e}^{\pm i\frac{2}{9}\pi}\right) > \frac{Cr^{1/2}}{r^{3/2}+1}$$

which implies the inequality in the Lemma. Furthermore, when $\arg\chi = 0$, it is easily seen that P_3' is real and positive and so P_3 increases monotonically to ∞ as we approach $\infty\mathrm{e}^{i0}$. □

Lemma 24 *For any $\delta \in (0, \frac{\pi}{63})$ there exists R_0 independent of δ so that*

1. $$\frac{d}{ds} \Re P_1(\chi(s)) \geq C|\chi(s)|^{5/4}$$

   for $C > 0$ independent of any parameter, where s is the arc-length representation of part of the boundary of $\mathcal{E}$ for which $|\chi(s)| > R_0$; s is increasing in r when $\chi = r\mathrm{e}^{-i\frac{2}{9}\pi + i\delta}$ and decreasing when $\chi = r\mathrm{e}^{i\frac{2}{9}\pi - i\delta}$.

2. $$\frac{d}{ds} \Re P_2(\chi(s)) \geq C|\chi(s)|^{5/4}$$

   for $C > 0$ independent of any parameter, where s is the arc-length representation of part of the boundary of $\mathcal{E}$ for which $|\chi(s)| > R_0$; s is increasing in r when $\chi = r\mathrm{e}^{i\frac{2}{9}\pi - i\delta}$ and decreasing when $\chi = r\mathrm{e}^{-i\frac{2}{9}\pi + i\delta}$.

3. $$\frac{d}{ds} \Re P_3(\chi(s)) \geq C|\chi(s)|^{-1}$$

   for $C > 0$ independent of any parameter, where s is the arc-length representation of part of the boundary of $\mathcal{E}$ for which $|\chi(s)| > R_0$; s is increasing in r when $\chi = r\exp\left\{\pm i\left[\frac{2}{9}\pi - \delta\right]\right\}$.

Proof. This follows from the asymptotic behavior of P_1', P_2' and P_3' for large χ in (56) after noting that

$$\frac{d}{dr} \Re P_j(r\mathrm{e}^{i\theta}) = \Re\left[\mathrm{e}^{i\theta} P_j'(r\mathrm{e}^{i\theta})\right]$$

□

Lemma 25 *For $0 < \epsilon_1 \leq r \leq R_0$. There exists a small enough $\delta > 0$, independent of any parameter, so that*

$$\frac{d}{dr} \Re P_j\left(r\mathrm{e}^{-i\frac{2}{9}\pi + i\delta}\right) > C > 0 \text{ for } j = 1, 3$$

while

$$-\frac{d}{dr} \Re P_2\left(r\mathrm{e}^{-i\frac{2}{9}\pi + i\delta}\right) > C > 0$$

with C independent of δ. Again, for $\epsilon_1 \leq r \leq R_0$, there is a $\delta > 0$, independent of any parameter so that

$$\frac{d}{dr} \Re P_j\left(r\mathrm{e}^{i\frac{2}{9}\pi - i\delta}\right) > C > 0 \text{ for } j = 2, 3$$

while

$$-\frac{d}{dr} \Re P_1\left(r\mathrm{e}^{i\frac{2}{9}\pi - i\delta}\right) > C > 0$$

for some C independent of δ.

Proof. From the lemmas about the behavior of P_j on $\partial\mathcal{E}$, the statements are clearly true for $\delta = 0$. From continuity, it follows that the same is true (adjusting C) for all sufficiently small δ and hence the lemma follows. □

Definition 26

$$\partial\mathcal{E}_L = \partial\mathcal{E}_L^+ \cup \mathcal{E}_L^-$$

where

$$\partial\mathcal{E}_L^+ = \left\{\chi = \chi_3 + r\mathrm{e}^{i2\pi/3} \text{ for } 0 \le r \le |\chi_2 - \chi_3|\right\}$$
$$\partial\mathcal{E}_L^- = \left\{\chi = \chi_3 + r\mathrm{e}^{-i2\pi/3} \text{ for } 0 \le r \le |\chi_2 - \chi_1\right\}$$

Lemma 27 *$\Re P_3$ increases in r on $\partial\mathcal{E}_L^+$ and $\partial\mathcal{E}_L^-$. $\Re P_1$ decreases in r on $\partial\mathcal{E}_L^+$, but increases in r on $\partial\mathcal{E}_L^-$. $\Re P_2$ increases in r on $\partial\mathcal{E}_L^+$ and decreases in r on $\partial\mathcal{E}_L^-$ and in all cases, we have on $\partial\mathcal{E}_L$,*

$$\left|\frac{d}{dr}\Re P_j(\chi(r))\right| \ge C > 0$$

where C only depends on the choice of $|\chi_j|$. $\Re P_j$ attains a maximum on $\partial\mathcal{E}_L$ at the corresponding χ_j.

Proof. We note that since $|\chi_3|$ is small, we have

$$-\frac{d}{dr}\Re P_3(\chi(r)) = -\Re\left[P_3'(\chi(r))\mathrm{e}^{i2\pi/3}\right] \sim |\chi(r)|^{1/2}\sin\left(\frac{\pi}{6}+\frac{\theta}{2}\right) > C > 0$$

where $\arg\chi = \theta \in \left[-\frac{2\pi}{9}+\delta, \frac{2\pi}{9}-\delta\right]$. By symmetry we also get for χ on $\partial\mathcal{E}_L^-$

$$-\frac{d}{dr}\Re P_3(\chi(r)) = -\Re\left[P_3'(\chi(r))\mathrm{e}^{-i2\pi/3}\right] \sim |\chi(r)|^{1/2}\sin\left(\frac{\pi}{6}-\frac{\theta}{2}\right) > C > 0$$

For P_1 we find that for $\chi \in \partial\mathcal{E}_L^+$,

$$-\frac{d}{dr}\Re P_1(\chi(r)) \sim |\chi(r)|^{1/2}\cos\left(\frac{\pi}{3}+\frac{\theta}{2}\right) > C > 0$$

On $\partial\mathcal{E}_L^-$, we obtain

$$\frac{d}{dr}\Re P_1(\chi(r)) \sim |\chi(r)|^{1/2}\cos\left(\frac{\theta}{2}\right) > C > 0$$

Thus, on $\partial\mathcal{E}_L$, $\Re P_1$ increases monotonically from top to bottom with $\frac{d}{ds}\Re P_1(\chi(s)) > C > 0$. On this boundary P_2 increases monotonically from bottom to top by a similar argument. On the other hand, P_3 is maximum at χ_3; it decreases as we move up or down. □

Lemma 28 *On the boundary of $\mathcal{E}$, $\Re P_1$ increases monotonically with s as we traverse the boundary counterclockwise and:*

$$\frac{d}{ds}\Re P_1(\chi(s)) \ge C|\chi(s)|^{5/4} > 0$$

whereas $\Re P_2$ increases monotonically with the arclength s as this boundary is traversed clockwise and

$$\frac{d}{ds}\Re P_2(\chi(s)) \ge C|\chi(s)|^{5/4} > 0$$

On the other hand at the upper part of $\partial\mathcal{E}$, i.e. on $\partial\left(\mathcal{E}\cap\{\chi:\operatorname{Im}\chi>0\}\right)$,

$$\frac{d}{ds}\Re P_3(\chi(s))\geq C|\chi(s)|^{-1}>0$$

where the boundary is traversed counterclockwise. For the lower part of $\partial\mathcal{E}$, i.e. on $\partial\left(\mathcal{E}\cap\{\chi:\operatorname{Im}\chi<0\}\right)$ we have

$$\frac{d}{ds}\Re P_3(\chi(s))\geq C|\chi(s)|^{-1}>0$$

where the boundary is now traversed clockwise.

PROOF. The proof follows from Lemmas 54-57. □

Proof of Theorem 20. Any $\chi\in\partial\mathcal{E}$ can be joined to $\infty e^{i\theta_j}$ along $\partial\mathcal{E}$ so that $\frac{d}{ds}\Re P_j(\tilde{\chi}(s))$ satisfies the lower bounds given in Lemma 28. If $\chi\in\mathcal{E}$, we choose steepest ascent paths for $\Re P_j$ until (i) it goes to ∞, or (ii) it intersects $\partial\mathcal{E}$, from which point we continue along the ascent paths of $\partial\mathcal{E}$. The proof is complete.

5 Estimates on the solution G_k in the domain $\mathcal{E}_k$

The main theorem proved in this section is the following.

Theorem 29 *For $\eta\in\mathcal{E}_k$ we have*

$$\|\eta^{3/2}G_k\|_{\infty,\mathcal{E}_k}\leq\frac{K}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{E}_k}$$

$$\|\eta^{5/2}G_k'\|_{\infty,\mathcal{E}_k}\leq K\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{E}_k}$$

$$\|G_k''\|_{\infty,\mathcal{E}_k}\leq K\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{E}_k}$$

where the constant K is independent of k (and therefore of β_k).

Remark 17 *The proof comes at the end of §5, after a few lemmas. It is convenient to derive an integral equation for G_k and its first two derivatives. We exploit the largeness of k to control the solution of the integral equation. The asymptotic behavior of the solution of the homogeneous equation $\mathcal{L}_k u=0$ is different in the regimes $\eta\ll k^{4/9}$ and $|\eta|\gtrsim k^{4/9}$. Consequently, different integral equations will be used in $\mathcal{E}_k$ and $\mathcal{D}_k$ (analyzed in the next section).*

In $\mathcal{E}_k$, it is convenient to introduce scaled variables:

$$\eta=\beta_k^{4/9}\chi,\ G_k(\beta_k^{4/9}\chi)=z_k(\chi)\tag{59}$$

Then, (17) becomes

$$\hat{\mathcal{L}}_k z_k=-\frac{2\Psi_0}{9\chi^2}z_k'+\beta_k\left(\frac{\Psi_0}{\chi^3}+\frac{\Psi_1}{\beta_k\chi^3}\right)z_k+\frac{R_k}{G_0^3}\beta_k^{4/3}=\mathcal{R}(\chi),\tag{60}$$

where

$$\hat{\mathcal{L}}_k u := u''' + \frac{2}{9}\beta_k^2\chi^{5/2}u' - \beta_k^3\chi^{3/2}u, \tag{61}$$

and Ψ_0 and Ψ_1 are defined by

$$\frac{1}{G_0^3} - \eta^{3/2} = -\frac{\Psi_0}{\eta^3}, \quad -\frac{3G_0^2 G_0'''}{G_0^3} = -\frac{\Psi_1}{\eta^3} \tag{62}$$

From the large $\eta \in \mathcal{E}_k$ behavior of G_0 we see that Ψ_0 and Ψ_1 are bounded for large β_k as well as for large χ. Let v be the solution for $\chi \in \mathcal{E}$ of

$$\hat{\mathcal{L}}_k v = \mathcal{R} \tag{63}$$

Using rigorous WKB results [16], it follows that for large β_k, there exist three independent solutions of the associated homogeneous equation, with leading behavior v_1, v_2, v_3 where

$$v_j(\chi) = \mathrm{e}^{\beta_k P_j(\chi) + W_j(\chi)} \tag{64}$$

where $\alpha = P_j'$ are the three roots of the cubic equation

$$\alpha^3 + \frac{2}{9}\chi^{5/2}\alpha - \chi^{3/2} = 0 \tag{65}$$

Note that two roots of (65) coincide iff $\alpha^2 + \frac{2}{27}\chi^{5/2} = 0$ i.e. iff

$$\chi = \chi_s = \left(\frac{81\sqrt{3}}{4\sqrt{2}}\right)^{4/9} \mathrm{e}^{\pm i2\pi/9}$$

only possible *outside* $\mathcal{E}$. Hence the $v_i, i = 1, 2, 3$ are independent in $\mathcal{E}$. The corresponding W_j are given by

$$W_j' = -\frac{3P_j' P_j''}{3P_j'^2 + \frac{2}{9}\chi^{5/2}} \tag{66}$$

and the P_j are uniquely determined by the following asymptotic conditions for large χ:

$$P_1 = \frac{4\sqrt{2}}{27} i\chi^{9/4} - \frac{9}{4}\ln\chi + o(1), P_2 = -\frac{4\sqrt{2}}{27} i\chi^{9/4} - \frac{9}{4}\ln\chi + o(1), \ P_3 = \frac{9}{2}\ln\chi + o(1) \tag{67}$$

$$W_1 = -\frac{15}{8}\ln\chi + o(1), \ W_2 = -\frac{15}{8}\ln\chi + o(1), \ W_3 = o(1) \tag{68}$$

We now use the v_i to write an integral equation for v, equivalent to (63), with appropriate decay conditions at ∞. First, we have

$$\mathcal{M} := \begin{bmatrix} v_1 & v_2 & v_3 \\ \beta_k^{-1}v_1' & \beta_k^{-1}v_2' & \beta_k^{-1}v_3' \\ \beta_k^{-2}v_1'' & \beta_k^{-2}v_2'' & \beta_k^{-2}v_3'' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{bmatrix} \tag{69}$$

where for $j = 1, 2, 3$

$$m_{2j} = P_j' + \frac{W_j'}{\beta_k}$$

$$m_{3j} = \left(P_j' + \frac{W_j'}{\beta_k}\right)^2 + \frac{1}{\beta_k}\left(P_j'' + \frac{W_j''}{\beta_k}\right)$$

From the asymptotic properties of P_j and W_j, it follows that for large β_k we have $m_{ij} = O(1)$ for all i, j. Furthermore, for large χ, we also have

$$m_{21} = O(\chi^{5/4}), m_{22} = O(\chi^{5/4}), m_{23} = O(\chi^{-1}) \tag{70}$$

$$m_{31} = O(\chi^{5/2}),\ m_{32} = O(\chi^{5/2}),\ m_{33} = O(\chi^{-2}) \tag{71}$$

Let

$$Q_1 = (\mathcal{M}' - Q_2\mathcal{M})\mathcal{M}^{-1}, \text{ where } Q_2 = \beta_k \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \chi^{3/2} & -\frac{2}{9}\chi^{5/2} & 0 \end{bmatrix} \tag{72}$$

Then $\mathcal{M}$ satisfies the differential equation

$$\mathcal{M}' - (Q_2 + Q_1)\mathcal{M} = 0 \tag{73}$$

Denoting

$$\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}^{-1} \tag{74}$$

and

$$\Delta = m_{22}m_{33} - m_{23}m_{32} - m_{21}m_{33} + m_{21}m_{32} - m_{31}m_{22} + m_{31}m_{23}$$

we have

$$n_{1,3} = (m_{23} - m_{22})/\Delta;\quad n_{2,3} = (m_{21} - m_{23})/\Delta;\quad n_{3,3} = (m_{21} - m_{22})/\Delta \tag{75}$$

The first two rows of $\mathcal{M}' - Q_2\mathcal{M}$ are zero. Hence, the same is true for the first two rows of Q_1. Therefore,

$$Q_1 = \beta_k^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{76}$$

Defining r_j so that

$$\hat{\mathcal{L}}_k v_j = \beta_k r_j v_j, \tag{77}$$

we have

$$r_j = 3P_j'W_j'^2 + 3P_j'W_j'' + 3W_j'P_j'' + P_j''' + \beta_k^{-1}\left(W_j'^3 + 3W_j'W_j'' + W_j'''\right) \tag{78}$$

We note that $r_j = O(1)$ for large β_k. For large χ we have

$$r_1 = O(\chi^{-3/4}),\quad r_2 = O(\chi^{-3/4}),\quad r_3 = O(\chi^{-5}) \tag{79}$$

Also, with $\Delta_1 = (m_{21} - m_{22})(m_{32} - m_{33}) - (m_{22} - m_{23})(m_{31} - m_{32})$we have

$$b_{3,2} = [(r_1 - r_2)(m_{32} - m_{33}) - (r_2 - r_3)(m_{31} - m_{32})]/\Delta_1 \tag{80}$$

$$b_{3,3} = -[(r_1 - r_2)(m_{22} - m_{23}) - (r_2 - r_3)(m_{21} - m_{22})]/\Delta_1 \tag{81}$$

$$b_{3,1} = r_3 - b_{3,2}m_{2,3} - b_{3,3}m_{3,3} \tag{82}$$

From the asymptotics of r_j and $m_{i,j}$ for large β_k we get $b_{3,j} = O(1)$. For large $\chi \in \mathcal{E}$ we have

$$b_{3,2} = O(\chi^{-2}),\ b_{3,3} = O(\chi^{-2}), \quad \text{and} \quad b_{3,1} = O(\chi^{-3}) \tag{83}$$

Then, for large $\chi \in \mathcal{E}$, it follows that

$$n_{1,3} = O(\chi^{-5/2}), \quad n_{2,3} = O(\chi^{-5/2}) \ \text{ and } \ n_{3,3} = O(\chi^{-5/2}) \tag{84}$$

In the domain $\mathcal{E}$ the $b_{3,j}$ are analytic in χ, bounded for large β_k and decay for large χ, (see (83)). Furthermore, (73) implies

$$\mathcal{L}_{WKB} v_j := v_j''' - \frac{b_{33}}{\beta_k} v_j'' + \left[\frac{2}{9}\beta_k^2 \chi^{5/2} - b_{3,2}\right] v_j' - \left[\beta_k^3 \chi^{3/2} + \beta_k b_{3,1}\right] v_j = 0 \tag{85}$$

Using variation of parameters, we see that one solution of $\tilde{\mathcal{L}}_k v = \mathcal{R}$ satisfies:

$$v(\chi) = \tilde{\mathcal{V}}\left[\hat{R}\right](\chi) \ \ ; \ \ v'(\chi) = \tilde{\mathcal{V}}'\left[\hat{R}\right](\chi) \ \ ; \ \ v''(\chi) = \tilde{\mathcal{V}}''\left[\hat{R}\right](\chi) \ \ ; \tag{86}$$

where

$$\hat{R}(\chi) = \mathcal{R}(\chi) - \beta_k^{-1} b_{3,3} v'' - b_{3,2} v'(\chi) - \beta_k b_{3,1} v(\chi), \tag{87}$$

and the operators $\tilde{\mathcal{V}}$, $\tilde{\mathcal{V}}'$ and $\tilde{\mathcal{V}}''$ are defined by:

$$\tilde{\mathcal{V}}[\hat{R}](\chi) = \sum_{j=1}^{3} \frac{1}{\beta_k^2} \int_{\infty e^{i\theta_j}}^{\chi} e^{\beta_k \ [P_j(\chi) - P_j(\tilde{\chi})] + W_j(\chi) \ - \ W_j(\tilde{\chi})} n_{j,3}(\tilde{\chi}) \hat{R}(\tilde{\chi}) d\tilde{\chi} \tag{88}$$

$$\tilde{\mathcal{V}}'[\hat{R}](\chi) = \sum_{j=1}^{3} \frac{m_{2,j}(\chi)}{\beta_k} \int_{\infty e^{i\theta_j}}^{\chi} e^{\beta_k \ [P_j(\chi) - P_j(\tilde{\chi})] + W_j(\chi) \ - \ W_j(\tilde{\chi})} n_{j,3}(\tilde{\chi}) \hat{R}(\tilde{\chi}) d\tilde{\chi} \tag{89}$$

$$\tilde{\mathcal{V}}''[\hat{R}](\chi) = \sum_{j=1}^{3} m_{3,j}(\chi) \int_{\infty e^{i\theta_j}}^{\chi} e^{\beta_k \ [P_j(\chi) - P_j(\tilde{\chi})] + W_j(\chi) \ - \ W_j(\tilde{\chi})} n_{j,3}(\tilde{\chi}) \hat{R}(\tilde{\chi}) d\tilde{\chi} \tag{90}$$

where $\theta_1 = -\frac{2}{9}\pi + \delta$, $\theta_2 = \frac{2}{9}\pi - \delta$ and $\theta_3 = 0$, and the paths of integration $\mathcal{C}_j$ are chosen to be the ascent paths for $\Re P_j$ of Theorem 20. Also, note that for large χ, $W_j(\chi)$ grows at most logarithmically with χ implying that, uniformly in $\mathcal{E}$, we have $W_j = o(\beta_k P_j)$. As we shall see, there is a unique solution v of (86) that decays as $\chi \to \infty$ in $\mathcal{E}$, with $\hat{R}$ having similar decay properties. The way we show this is by proving contractivity of the integral system in a suitable space of decaying functions. (In fact, there can be no other decaying solutions, since the associated homogeneous equation does not have nonzero decaying solutions in $\mathcal{E}$.)

Lemma 30 *If the P_j satisfy Property 1 in $\mathcal{E}$, then for sufficiently large k (or, which amounts to the same, large β_k) we have*

$$\|\chi^{3/2} \hat{\mathcal{V}}[\hat{R}]\|_{\infty,\mathcal{E}} \leq \frac{C}{\beta_k^3} \|\hat{R}\|_\infty \tag{91}$$

$$\|\chi^{5/2}\hat{\mathcal{V}}'[\hat{R}]\|_{\infty,\mathcal{E}} \le \frac{C}{\beta_k^2}\,\|\hat{R}\|_\infty \tag{92}$$

$$\|\hat{\mathcal{V}}''[\hat{R}]\|_{\infty,\mathcal{E}} \le \frac{C}{\beta_k}\,\|\hat{R}\|_\infty \tag{93}$$

where the constant C is independent of $\hat{R}$ and β_k.

PROOF. Theorem 20 shows that on $\mathcal{C}_j$ (defined before Lemma 30) we have $|\tilde{\chi}| > C|\chi|$ and

$$\frac{d}{ds}\Re P_{1,2}(\tilde{\chi}(s)) > C|\tilde{\chi}|^{5/4},\ \frac{d}{ds}\Re P_3(\tilde{\chi}(s)) > C|\tilde{\chi}(s)|^{-1}$$

Since W_j'/P_j' is bounded, this implies that for sufficiently large β_k we have

$$\frac{d}{ds}\Re\left[P_{1,2}+\frac{W_{1,2}}{\beta_k}\right](\tilde{\chi}(s)) > \frac{C}{2}|\tilde{\chi}|^{5/4}$$

$$\frac{d}{ds}\Re\left[P_3+\frac{W_3}{\beta_k}\right](\tilde{\chi}(s)) > \frac{C}{2}|\tilde{\chi}(s)|^{-1}$$

Also, from (70) and (71),

$$|m_{2,1}| < C\,|\chi|^{5/4}, |m_{2,2}| < C\,|\chi|^{5/4},\ |m_{2,3}| < C\,|\chi|^{-1},$$

$$|m_{3,1}| < C\,|\chi|^{5/2},\ |m_{3,2}| < C\,|\chi|^{5/2},\ |m_{3,3}| < C\,|\chi|^{-2},$$

while from (84), $|n_{3,j}| < C|\chi|^{-5/2}$. Then,

$$\frac{1}{\beta_k^2}\left|\int_{\infty e^{i\theta_j}}^{\chi} \exp\left[\beta_k(P_j(\chi)-P_j(\tilde{\chi})\right] n_{3,j}R(\tilde{\chi})e^{W_j(\chi)-W_j(\tilde{\chi})}d\tilde{\chi}\right|$$
$$\le \frac{C\|\hat{R}\|_\infty|\chi|^{-3/2}}{\beta_k^3}\int_0^1 d\left[\exp\left(\beta_k[\Re P_j(\chi)-\Re P_j(\tilde{\chi})]\right)\right] \tag{94}$$

The bounds for $\hat{\mathcal{V}}$ follow; $\hat{\mathcal{V}}'$ and $\mathcal{V}''$ are bounded similarly. □

Corollary 31 *Define the operator $\mathcal{T}_k$ acting on triples (z_k, z_k', z_k'') as follows:*

$$\mathcal{T}_k\left(z_k, z_k', z_k''\right)(\chi) = -\beta_k^{-1}b_{3,3}z_k'' - \left(\frac{2\Psi_0}{9\chi^2}+b_{3,2}\right)z_k' + \beta_k\left(\frac{\Psi_0}{\chi^3}+\frac{\Psi_1}{\beta_k\chi^3}-b_{3,1}\right)z_k \tag{95}$$

Then, it follows

$$\|\chi^{3/2}\hat{\mathcal{V}}\left[\mathcal{T}_k\left(z_k, z_k', z_k''\right)\right]\|_{\infty,\mathcal{E}} \le C\left[\beta_k^{-4}\|z_k''\|_{\infty,\mathcal{E}} + \beta_k^{-3}\|\chi^{5/2}z_k'\|_{\infty,\mathcal{E}} + \beta_k^{-2}\|\chi^{3/2}z_k\|_{\infty,\mathcal{E}}\right]$$

$$\|\chi^{5/2}\hat{\mathcal{V}}'\left[\mathcal{T}_k\left(z_k, z_k', z_k''\right)\right]\|_{\infty,\mathcal{E}} \le C\left[\beta_k^{-3}\|z_k''\|_{\infty,\mathcal{E}} + \beta_k^{-2}\|\chi^{5/2}z_k'\| + \beta_k^{-1}\|\chi^{3/2}z_k\|_{\infty,\mathcal{E}}\right]$$

$$\|\hat{\mathcal{V}}''\left[T_k\left(z_k, z_k', z_k''\right)\right]\|_{\infty,\mathcal{E}} \le C\left[\beta_k^{-2}\|z_k''\|_{\infty,\mathcal{E}} + \beta_k^{-1}\|\chi^{5/2}z_k'\| + \|\chi^{3/2}z_k\|_{\infty,\mathcal{E}}\right]$$

PROOF. This follows from Lemma 30 and bounds on $b_{3,j}$ in (83) and those on Ψ_0, Ψ_1 that follow from (62). $\square$

Lemma 32

$$\left\| \chi^{3/2} \hat{\mathcal{V}} \left[\beta_k^{4/3} \frac{R_k}{G_0^3} (\beta_k^{4/9} \chi) \right] \right\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k^3} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

$$\left\| \chi^{5/2} \hat{\mathcal{V}}' \left[\beta_k^{4/3} \frac{R_k}{G_0^3} (\beta_k^{4/9} \chi) \right] \right\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k^2} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

$$\left\| \hat{\mathcal{V}}'' \left[\beta_k^{4/3} \frac{R_k}{G_0^3} (\beta_k^{4/9} \chi) \right] \right\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

PROOF. This is a consequence of Lemma 30, with $\hat{R}$ replaced by $\beta_k^{4/3} R_k/G_0^3$. $\square$

Lemma 33 *For $\left\| R_k G_0^{-3} \right\|_{\infty,\mathcal{E}} < \infty$, and for β_k sufficiently large, the system (86) has a unique solution $(z_k(\chi), z'(\chi), z''(\chi))$ in $\mathcal{E}$, which satisfies the bounds*

$$\|\chi^{3/2} z_k\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k^3} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

$$\|\chi^{5/2} z_k'\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k^2} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

$$\|z_k''\|_{\infty,\mathcal{E}} \leq \frac{C\beta_k^{4/3}}{\beta_k} \left\| \frac{R_k}{G_0^3} \right\|_{\infty,\mathcal{E}}$$

PROOF. Define the Banach space $\mathcal{F}$ of triples of functions (u, v, w) analytic in the interior of $\mathcal{E}$ and continuous in its closure in the norm

$$\|(u,v,w)\|_{\mathcal{F}} = \beta_k^{5/3} \|\chi^{3/2} u\|_{\infty,\mathcal{E}} + \beta_k^{2/3} \|\chi^{5/2} v\|_{\infty,\mathcal{E}} + \beta_k^{-1/3} \|w\|_{\infty,\mathcal{E}}$$

We associate z_k, z_k' and z_k'' with u, v and w respectively, and consider $\hat{R}$ as depending on u, v and w for fixed R_k/G_0^3. We define the linear operator $\mathbf{L} : \mathcal{F} \to \mathcal{F}$ by

$$\mathbf{L}\left[(u,v,w)\right] := \left(\hat{\mathcal{V}} \left[\hat{R}(u,v,w) \right], \hat{\mathcal{V}}' \left[\hat{R}(u,v,w) \right], \hat{\mathcal{V}}'' \left[\hat{R}(u,v,w) \right] \right)$$

where $\hat{\mathcal{V}}, \hat{\mathcal{V}}', \hat{\mathcal{V}}''$ are now thought of as acting on $u = z_k$, $v = z_k'$ $w = z_k''$ for fixed $R_k G_0^{-3}$. From (60), (86) and (87), the definition of $\mathcal{T}_k$ in (95), and the estimates in Corollary 31, it is easily seen that

$$\begin{aligned} &\|\mathbf{L}\left[(u,v,w)\right] - \mathbf{L}\left[(\tilde{u},\tilde{v},\tilde{w})\right]\|_{\mathcal{F}} \\ &\quad = \|\mathcal{V}\left[\mathcal{T}_k(u-\tilde{u}, v-\tilde{v}, w-\tilde{w})\right], \hat{\mathcal{V}}'\left[\mathcal{T}_k(u-\tilde{u}, v-\tilde{v}, w-\tilde{w})\right], \hat{\mathcal{V}}''\left[\mathcal{T}_k(u-\tilde{u}, v-\tilde{v}, w-\tilde{w})\right]\| \\ &\qquad \leq \frac{C}{\beta_k^2} \|(u-\tilde{u}, v-\tilde{v}, w-\tilde{w})\|_{\mathcal{F}} \end{aligned}$$

Hence **L** is contractive and the system (86) has a unique solution (z_k, z_k', z_k''). The estimates on z_k, z_k', z_k'' follow easily from Lemma 32. □

Proof of Theorem 29. This is a consequence of Lemma 33, noting that

$$\eta^{3/2}G_k(\eta) = \beta_k^{2/3}\chi^{3/2}z_k(\chi),\ \eta^{5/2}G_k'(\eta) = \beta_k^{2/3}\chi^{5/2}z_k'(\chi),\ G_k''(\eta) = \beta_k^{-8/9}z_k''(\chi)$$

6 Estimate of G_k for large k in $\mathcal{D}_k$

In this section we prove the following.

Theorem 34 *In $\mathcal{D}_k$ (see Definition 10) we have*

$$\|\eta^{3/2}G_k(\eta)\|_{\infty,\mathcal{D}_k} \le \frac{K_{10}}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \tag{96}$$

$$\|\eta^{5/2}G_k'(\eta)\|_{\infty,\mathcal{D}_k} \le K_{11}\epsilon^{3/2}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \tag{97}$$

Remark 18 *The proof of theorem (34) is completed at the end of §6, after a few lemmas establishing the properties of $\mathcal{L}_k^{-1}$.*

We first find a representation of the solution to

$$\mathcal{L}_k G_k = G_k''' + \frac{2}{9G_0^3}\eta G_k' - \frac{7k-1}{9G_0^3}G_k + \frac{3G_0'''}{G_0}G_k = \frac{R_k}{G_0^3} \tag{98}$$

for large k for $\eta \in \mathcal{D}_k$ where $|\eta|$ is small compared to $\beta_k^{4/9}$. Again following [16], there exist three independent solutions u_1, u_2, u_3 to the homogeneous equation $\mathcal{L}_k u = 0$ such that, for large β_k we have

$$u_j(\eta) \sim g_j(\eta) = G_0(\eta)\mathrm{e}^{\omega_j\beta_k^{1/3}P(\eta)},\ \text{where } \omega_1 = \mathrm{e}^{i2\pi/3},\ \omega_2 = \mathrm{e}^{-i2\pi/3},\ \omega_3 = 1 \text{ and}$$
$$P(\eta) = \int_{\eta_0}^{\eta}\frac{1}{G_0(\eta')}d\eta' \text{ for fixed } \eta_0 \in \mathcal{D}_k \tag{99}$$

We use g_1, g_2, g_3 to find a suitable integral equation for the solution u to (98). As in §5, it is convenient to define

$$\mathcal{M} := \begin{bmatrix} g_1 & g_2 & g_3 \\ \beta_k^{-1/3}g_1' & \beta_k^{-1/3}g_2' & \beta_k^{-1/3}g_3' \\ \beta_k^{-2/3}g_1'' & \beta_k^{-2/3}g_2'' & \beta_k^{-2/3}g_3'' \end{bmatrix} \tag{100}$$

and

$$Q_1 := (\mathcal{M}' - Q_2\mathcal{M})\mathcal{M}^{-1},\ \text{where } Q_2 := \beta_k^{1/3}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{9G_0^3} - \frac{3G_0'''}{\beta_k G_0} & -\frac{2\eta}{9\beta_k^{2/3}G_0^3} & 0 \end{bmatrix} \tag{101}$$

We get

$$\mathcal{M}' - (Q_2 + Q_1)\mathcal{M} = 0 \tag{102}$$

Using (99) we see that

$$Q_1 = \beta_k^{-2/3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{G_0'^3}{G_0^3} - 2\frac{G_0' G_0''}{G_0^2} + 4\frac{G_0'''}{G_0} & \left[-\frac{G_0'^2}{G_0^2} + \frac{2G_0''}{G_0} + \frac{2\eta}{9G_0^3} \right] \beta_k^{1/3} & 0 \end{bmatrix} \tag{103}$$

The columns of $\mathcal{M}$ also satisfy (102); it follows that for j=1,2,3 we have

$$\hat{\mathcal{L}}_k g_j := g_j''' + \left(\frac{G_0'^2}{G_0^2} - \frac{2G_0''}{G_0} \right) g_j' + \left(-\frac{\beta_k}{G_0^3} - \frac{G_0'^3}{G_0^3} - \frac{G_0'''}{G_0} + \frac{2G_0' G_0''}{G_0^2} \right) g_j = 0 \tag{104}$$

We note that

$$\hat{\mathcal{L}}_k u = \mathcal{L}_k u - b_{3,2} u' - b_{3,1} u \tag{105}$$

where

$$b_{3,2} = \left(\frac{2G_0''}{G_0} + \frac{2\eta}{9G_0^3} - \frac{G_0'^2}{G_0^2} \right)$$

$$b_{3,1} = \left(\frac{G_0'^3}{G_0^3} + 4\frac{G_0'''}{G_0} - \frac{2G_0' G_0''}{G_0^2} \right)$$

For large $|\eta|$ in $\mathcal{D}_k$ we find

$$b_{3,2} = O(\eta^{5/2}), \ b_{3,1} = O(\eta^{-3}) \tag{106}$$

Also, $b_{3,1}$ and $b_{3,2}$ are analytic in $\mathcal{D}_k$. It follows that the G_k in (98) also satisfy the integral equation

$$G_k(\eta) = \mathcal{V}\left[\hat{R}_k\right](\eta) + \sum_{j=1}^{3} a_j g_j(\eta) \tag{107}$$

where

$$\hat{R}_k(\eta) = \frac{R_k}{G_0^3} - b_{3,2} G_k' - b_{3,1} G_k \tag{108}$$

The constants a_j are defined in (110) in terms of $G_k(\eta_{1,k})$, $G_k(\eta_{2,k})$ and $G_k(\eta_{3,k})$ and the operator $\mathcal{V}$ is defined by

$$\mathcal{V}[\hat{R}_k](\eta) = \sum_{j=1}^{3} \frac{\beta_k^{-2/3}}{3} \omega_j G_0(\eta) \int_{\eta_j}^{\eta} G_0(\eta') \hat{R}_k(\eta') \mathrm{e}^{\omega_j \beta_k^{1/3} [P(\eta) - P(\eta')]} d\eta' \tag{109}$$

The contours of integration chosen in (109) are ascent paths of $\Re\left[\omega_j P\right]$, see Corollary 12). Given $G_k(\eta_{1,k})$, $G_k(\eta_{2,k})$ and $G_k(\eta_{3,k})$ we define a_1, a_2, a_3 by

$$\begin{bmatrix} g_1(\eta_{1,k}) & g_2(\eta_{1,k}) & g_3(\eta_{1,k}) \\ g_1(\eta_{2,k}) & g_2(\eta_{2,k}) & g_3(\eta_{2,k}) \\ g_1(\eta_{3,k}) & g_2(\eta_{3,k}) & g_3(\eta_{3,k}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} G_k(\eta_{1,k}) - \mathcal{V}[\hat{R}][\eta_{1,k}] \\ G_k(\eta_{2,k}) - \mathcal{V}[\hat{R}][\eta_{2,k}] \\ G_k(\eta_{3,k}) - \mathcal{V}[\hat{R}][\eta_{3,k}] \end{bmatrix} \tag{110}$$

Again,

$$G_k'(\eta) = \mathcal{V}'[\hat{R}_k](\eta) + \beta_k^{1/3} \sum_{j=1}^{3} a_j h_j g_j(\eta) \tag{111}$$

where

$$\mathcal{V}'\left[R\right](\eta) = \sum_{j=1}^{3} \frac{\beta_k^{-1/3}}{3} \omega_j h_j(\eta) G_0(\eta) \int_{\eta_{j,k}}^{\eta} G_0(\eta') R(\eta') \mathrm{e}^{\omega_j \beta_k^{1/3}[P(\eta)-P(\eta')]} d\eta', \tag{112}$$

and

$$h_j(\eta) = \frac{\omega_j}{G_0} + \frac{G_0'}{\beta_k^{1/3} G_0} \tag{113}$$

It is to be noted that

$$|\eta^{-1/2} h_j(\eta)| < C$$

for some constant C independent of β_k.

A few properties of $\mathcal{V}$ and $\mathcal{V}'$ follow from Property 1 of $P(\eta)$ (established in §3).

Lemma 35 *Assume* $\|R\|_{\infty,\mathcal{D}_k} < \infty$. *Then,*

$$\|\eta^{3/2} \mathcal{V}\left[R\right](\eta)\|_\infty \le \frac{K_1}{\beta_k} \|R\|_\infty$$

for a constant K_1 *independent of* β_k.

Proof. Note that on any of the contours $\mathcal{C}_j$, from Property 1, there exists a constant $C > 0$ so that $\tilde{\eta} > C|\eta|$ for $\tilde{\eta} \in \mathcal{C}_j$ and

$$\frac{d}{ds} \Re\left\{\omega_j P(\tilde{\eta}(s))\right\} > C_1 |\eta(s)|^{1/2} > 0$$

where s is the arc length. Therefore the proof follows from the estimate

$$\left| \beta_k^{-2/3} G_0(\eta) \int_{\eta_{j,k}}^{\eta} G_0(\eta') R(\eta') \mathrm{e}^{\beta_k^{1/3}(\omega_j[P(\eta)-P(\eta')])} d\eta' \right|$$
$$\le \int_0^1 d\left[\exp[\beta_k^{1/3}(\Re(\omega_j[P(\eta) - P(\eta')]))\right] \frac{C}{\beta_k |\eta|^{3/2}} \|R\|_{\infty,\mathcal{D}_k}$$

□

Lemma 36 *Assume* $\|R\|_{\infty,\mathcal{D}_k} < \infty$. *Then*

$$\|\eta^{5/2} \mathcal{V}'\left[R\right](\eta)\|_\infty \le \frac{K_2 |\eta_{3,k}|^{3/2}}{\beta_k^{2/3}} \|R\|_{\infty,\mathcal{D}_k}$$

where K_2 *is a constant independent of* β_k.

PROOF. As before, there exists a constant $C > 0$ so that on the contour $\mathcal{C}_j$ we have $C|\eta| < \eta'$ and

$$\frac{d}{ds}\Re\omega_j P(\eta'(s)) > C|\eta'(s)|^{1/2} > 0$$

where s is the arc length. Thus

$$\left|\beta_k^{-1/3} h_j(\eta) G_0(\eta) \int_{\eta_{j,k}}^{\eta} G_0(\eta') R(\eta') \mathrm{e}^{\omega_j \beta_k^{1/3} P(\eta) - P(\eta')} d\eta'\right|$$

$$\leq \int_0^1 d\left\{\exp[\beta_k \Re(P(\eta) - P(\eta'))]\right\} \frac{C|\eta|^{3/2}}{\beta_k^{2/3}|\eta|^{5/2}} \|R\|_{\infty,\mathcal{D}_k}$$

The Lemma follows by noting that in $\mathcal{D}_k$ we have $|\eta| \leq |\eta_{3,k}|$. □

Corollary 37 *We have*

$$\|\eta^{3/2}\mathcal{V}\left[b_{3,2}G_k' + b_{3,1}G_k\right](\eta)\|_{\infty,\mathcal{D}_k} \leq \frac{K_3}{\beta_k}\left[\|\eta^{5/2}G_k'\|_{\infty,\mathcal{D}_k} + \|\eta^{3/2}G_k\|_{\infty,\mathcal{D}_k}\right]$$

PROOF. This follows from Lemma 35, and the bounds on $b_{3,2}$ and $b_{3,1}$ in (106). □

Corollary 38 *We have*

$$\|\eta^{5/2}\mathcal{V}'\left[b_{3,2}G_k' + b_{3,1}G_k\right](\eta)\|_{\infty,\mathcal{D}_k} \leq K_4 \frac{|\eta_{3,k}|^{3/2}}{\beta_k^{2/3}}\left[\|\eta^{5/2}G_k'\|_{\infty,\mathcal{D}_k} + \|\eta^{3/2}G_k\|_{\infty}\right]$$

PROOF. This follows from Lemma 36, and the bounds on $b_{3,2}$ and $b_{3,1}$ in (106). □

Corollary 39 *The following inequality holds*

$$\left\|\eta^{3/2}\mathcal{V}\left[\frac{R_k}{G_0^3}\right](\eta)\right\|_{\infty,\mathcal{D}_k} \leq \frac{K_5}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k}$$

for a constant K_5 independent of k.

PROOF. This follows from Lemma 35. □

Corollary 40 *We have*

$$\left\|\eta^{5/2}\mathcal{V}'\left[\frac{R_k}{G_0^3}\right](\eta)\right\|_{\infty,\mathcal{D}_k} \leq \frac{K_5 \eta_{3,k}^{3/2}}{\beta_k^{2/3}}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k}$$

PROOF. This follows from Lemma 36, after noting that for $\eta \in \mathcal{D}_k$, $|\eta| \leq \eta_{3,k}$. □

Definition 41 *Define the linear operators $\mathcal{T}_1$ and $\mathcal{T}_2$ by*

$$\mathcal{T}_1\left[G_k, G_k'\right](\eta) = \sum_{j=1}^{3} a_j g_j(\eta)$$

$$\mathcal{T}_2\left[G_k, G_k'\right](\eta) = \beta_k^{1/3} \sum_{j=1}^{3} a_j h_j(\eta) g_j(\eta)$$

(see (110)) since $\eta_{j,k} \in \partial\mathcal{E}_k$, $G_k(\eta_{j,k})$ are known from the previous section.

Lemma 42 *We have*

$$\|\eta^{3/2}\mathcal{T}_1\left[G_k, G_k'\right]\|_{\infty,\mathcal{D}_k} \le \frac{2(K+K_1)}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} + \frac{2K_8}{\beta_k}\left[\|\eta^{3/2}G_k\|_\infty + \|\eta^{5/2}G_k'\|_\infty\right]$$

PROOF. From (110), since $\eta_{j,k}$ are large and therefore $g_j(\eta_{j',k})/g_j(\eta_{j,k})$ are exponentially small in β_k for $j' \ne j$, it is clear that

$$a_j g_j(\eta_{j,k}) \sim \mathcal{V}\left[\frac{R_k}{G_0^3} - b_{3,2}G_k' - b_{3,1}G_k\right](\eta_{j,k}) - G_k(\eta_{j,k})$$

From Lemma (35), Corollaries 37 and 38, it follows that

$$|a_j\eta_{j,k}^{3/2}g_j(\eta_{j,k})| < 2|\eta_{j,k}|^{3/2}|G_k(\eta_{j,k})| + \frac{2K_1}{\beta_k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k} + \frac{\hat{K}_8}{\beta_k}\left[\|\eta^{3/2}G_k\|_{\infty,\mathcal{D}_k} + \|\eta^{5/2}G_k'\|_{\infty,\mathcal{D}_k}\right] \tag{114}$$

Now, we conclude from Theorem 29 that

$$|\eta_{j,k}|^{3/2}|G_k(\eta_{j,k})| \le \frac{K}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{E}_k} \tag{115}$$

Since $\eta^{3/2}g_j(\eta)/(\eta_{j,k}^{3/2}g_j(\eta_{j,k}))$ are bounded independently of β_k and the proof follows. □

Lemma 43

$$\|\eta^{5/2}\mathcal{T}_2\left[G_k, G_k'\right]\|_{\infty,\mathcal{D}_k} \le \frac{|{\eta_{j,k}}^{3/2}|}{\beta_k^{2/3}}\left\{\frac{2(K+K_1)}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} + \frac{2\hat{K}_8}{\beta_k}\left[\|\eta^{3/2}G_k\|_\infty + \|\eta^{5/2}G_k'\|_\infty\right]\right\}$$

PROOF. Taking into account the behavior of $h_j(\eta)$ for large η we note that

$$|\beta_k^{1/3}\eta_{j,k}^{5/2}h_j(\eta_{j,k})a_j g_j(\eta_{j,k})| \le C\beta_k^{1/3}|\eta_{j,k}|^{3/2}|\eta_{j,k}^{3/2}a_j g_j(\eta_{j,k})|$$

Using (114) and (115), the proof follows. □

Proof of Theorem 34. We consider the space Banach $\mathcal{B}$ of pairs of analytic functions (u,v) in the interior of $\mathcal{D}_k$ continuous in its closure with the norm

$$\|(u,v)\| = \|\eta^{3/2}u\|_{\infty,\mathcal{D}_k} + \|\eta^{5/2}u'\|_{\infty,\mathcal{D}_k}$$

Associating G_k and G'_k in (107) and (111) with u and v, we define the linear operator L from $\mathcal{B}$ to $\mathcal{B}$ by

$$L[(u,v)] = \left(\mathcal{V}\left[\hat{R}_k[u,v]\right] + \mathcal{T}_1[u,v], \mathcal{V}'\left[\hat{R}_k[u,v]\right] + \mathcal{T}_2[u,v]\right)$$

where $\hat{R}_k$ is now thought of as an operator on (u,v) for fixed $\frac{R_k}{G_0^3}$ such that $\hat{R}_k[G_k, G'_k](\eta)$ equals the right hand side of (108).

It is a simple application of Lemmas 35-36, 42-43 and Corollaries 37 and 40 that

$$\|L[(u,v) - (\tilde{u},\tilde{v})]\| \le \tilde{\delta}\|(u,v) - (\tilde{u},\tilde{v})\|$$

where

$$\tilde{\delta} = \max\left\{\frac{K_3}{\beta_k}, \frac{\hat{K}_8}{\beta_k}, \frac{K_4\eta_{3,k}^{3/2}}{\beta_k^{2/3}}, \frac{\hat{K}_8\eta_{3,k}^{3/2}}{\beta_k^{2/3}}\right\} < \frac{1}{2}$$

for sufficiently large β_k and small ϵ. Contractivity of L implies that it has a unique fixed point. The estimates in the Lemma follow from (107) and (111).

Proof of Lemma 2. First, for $k = 1, ..., k_0$, the statement in the Lemma holds if A is sufficiently large (depending on k_0) in a common domain $\mathcal{D}_{k_0} \cup \mathcal{E}_{k_0}$, chosen to contain $\mathcal{D}_{k_0+1} \cup \mathcal{E}_{k_0+1}$. Assume therefore that $k > k_0$ where $k_0 + 1$ is large enough to ensure contractivity in Theorems 29 and 34. Assume the statement holds $j = 1, ..., k_0$ in a common domain $\mathcal{D}_{k_0} \cup \mathcal{E}_{k_0}$ and for $j = k_0 + 1$, ..., $k - 1$ in a corresponding sequence of domains $\mathcal{D}_j \cup \mathcal{E}_j$. It follows from the construction of these domains that it then holds in $\mathcal{D}_k \cup \mathcal{E}_k$. We then get the estimates on R_k needed in Theorems 29 and 34, which imply

$$\|\eta^{3/2}G_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \le \frac{K_{10}}{k}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k}$$

$$\|\eta^{5/2}G'_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \le K_{11}\left\|\frac{R_k}{G_0^3}\right\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k}$$

and therefore, from the estimates on $\|R_k\eta^{3/2}\|$ in (25), we get

$$\|\eta^{3/2}G_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \le \frac{K_{10}K_3}{k^3}(B^2A^k + BA^{k-1})$$

$$\|\eta^{5/2}G'_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \le \frac{K_{11}K_3}{k^2}(B^2A^k + BA^{k-1})$$

Using eq. (17) and the bounds on R_k, it follows that

$$\begin{aligned}\|G'''_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} &\le K_{12}\|\eta^{3/2}G_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} + K_{13}\|\eta^{5/2}G'_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} + K_{14}\|\eta^{3/2}R_k\|_{\infty,\mathcal{D}_k\cup\mathcal{E}_k} \\ &\le \left(K_{12}K_{11} + \frac{K_{13}}{k}K_{10} + K_{14}\right)\frac{K_3}{k^2}(B^2A^k + BA^{k-1})\end{aligned}$$

It is clear that for B sufficiently small and A sufficiently large, the estimates (22)-(24) on G_k, G'_k and G'''_k follow. The result follows now by induction.

Proof of Theorem 1. Now this follows easily from Lemma 2 since the estimates guarantee convergence of the Taylor series (5) for sufficiently small τ.

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8 Appendix: Singularities of nonlinear ODEs

We first mention briefly a number of results in [14] and then allow for slight modifications in the assumptions, to adjust for the equation of G_0.

8.1 Setting of [14] and generalizations

We adopt, with few exceptions that we mention, the same conditions, notations and terminology as [15] and [14]; the results on formal solutions and their generalized Borel summability are also taken from [15].

The differential system considered has the form

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n, \quad x \in \mathbb{C} \tag{116}$$

where
(i) $\mathbf{f}$ is *analytic* in a neighborhood $\mathcal{V}_x \times \mathcal{V}_{\mathbf{y}}$ of $(0, \mathbf{0})$, under the genericity conditions that:
(ii) the eigenvalues λ_j of the matrix $\hat{\Lambda} = -\left\{\frac{\partial f_i}{\partial y_j}(0,\mathbf{0})\right\}_{i,j=1,2,\ldots n}$ are linearly independent over $\mathbb{Z}$ (in particular $\lambda_j \neq 0$) and such that $\arg \lambda_j$ are all different.

We now allow for the same assumptions, except we replace (ii) by
(ii') There is at most one zero eigenvalue of $\hat{\Lambda}$ and all the other λ_j are linearly independent over $\mathbb{Z}$ (in particular $\lambda_j \neq 0$) and such that $\arg \lambda_j$ are all different.

By elementary changes of variables, the system (116) can be brought to the *normalized form* [15].

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{117}$$

where $\hat{\Lambda} = \text{diag}\{\lambda_j\}$, $\hat{A} = \text{diag}\{\alpha_j\}$ are constant matrices, $\mathbf{g}$ is analytic at $(0, \mathbf{0})$ and $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$ as $x \to \infty$ and $\mathbf{y} \to 0$.

As in [15] we normalize the system so that $\Re(\alpha_j) > 0$.

Performing a further transformation of the type $\mathbf{y} \mapsto \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ (which takes out M terms of the formal asymptotic series solutions of the equation), makes

$$\mathbf{g}(|x|^{-1}, \mathbf{y}) = O(x^{-M-1}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|) \quad (x \to \infty;\ \mathbf{y} \to 0) \tag{118}$$

where

$$M \geq \max_j \Re(\alpha_j)$$

and $O(a; b; c)$ means (at most) of the order of the largest among a, b, c.

Our analysis applies to solutions $\mathbf{y}(x)$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some arbitrary direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. A movable singularity of $\mathbf{y}(x)$ is a point $x \in \mathbb{C}$ with $x^{-1} \in \mathcal{V}_x$ where $\mathbf{y}(x)$ is not analytic. The point at infinity is an irregular singular point of rank 1; it is a fixed singular point of the system since, after the substitution $x = z^{-1}$ the r.h.s of the transformed system, $\frac{dy}{dz} = -z^{-2}\mathbf{f}(z, \mathbf{y})$ has, under the given assumptions, a pole at $z = 0$.

An n-parameter formal solution of (117) (under the assumptions mentioned) as a combination of powers and exponentials is found in the form

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k} \in (\mathbb{N} \cup \{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda} \cdot \mathbf{k} x} x^{\boldsymbol{\alpha} \cdot \mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{119}$$

where $\tilde{\mathbf{s}}_{\mathbf{k}}$ are (usually factorially divergent) formal power series: $\tilde{\mathbf{s}}_{\mathbf{0}} = \tilde{\mathbf{y}}_{\mathbf{0}}$ and in general

$$\tilde{\mathbf{s}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^r} \tag{120}$$

that can be determined by formal substitution of (119) in (117); $\mathbf{C} \in \mathbb{C}^n$ is a vector of parameters[2](we use the notations $\mathbf{C}^{\mathbf{k}} = \prod_{j=1}^{n} C_j^{k_j}$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, $|\mathbf{k}| = k_1 + ... + k_n$).

Note the structure of (119): an infinite sum of (generically) divergent series multiplying exponentials. They are called *formal exponential power series* [16].

From the point of view of correspondence of these formal solutions to actual solutions it was recognized that not all expansions (119) should be considered meaningful; also they are defined relative to a sector (or a direction).

Given a direction d in the complex x-plane the *transseries* (on d), introduced by Écalle [10], are, in our context, those exponential series (119) which are formally *asymptotic* on d, i.e. the terms $\mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda} \cdot \mathbf{k} x} x^{\boldsymbol{\alpha} \cdot \mathbf{k}} x^{-r}$ (with $\mathbf{k} \in (\mathbb{N} \cup \{0\})^n$, $r \in \mathbb{N} \cup \{0\}$) form a well ordered set with respect to $\gg$ on d (see also [15]).[3] (For example, this is the case when the terms of the formal expansion become (much) smaller when $\mathbf{k}$ becomes larger.)

We recall that the *antistokes lines* of (117) are the $2n$ directions of the x-plane $i\overline{\lambda_j}\,\mathbb{R}_+$, $-i\overline{\lambda_j}\,\mathbb{R}_+$, $j = 1, ..., n$, i.e. the directions along which some exponential $\mathrm{e}^{-\lambda_j x}$ of the general formal solution (119) is purely oscillatory.

In the context of differential systems with an irregular singular point, asymptoticity should be (generically) discussed relative to a direction towards the singular point; in fact, under the present assumptions (of non-degeneracy) asymptoticity can be defined on sectors.

Let d be a direction in the x-plane which is not an antistokes line. The solutions $\mathbf{y}(x)$ of (117) which satisfy

$$\mathbf{y}(x) \to 0 \quad (x \in d;\ |x| \to \infty) \tag{121}$$

are analytic for large x in a sector containing d, between two neighboring antistokes lines and have the same asymptotic series

[2]In the general case when some assumptions made here do not hold, the general formal solution may additionally logs iterated exponentials, and powers [10]. The present paper only discusses equations in the setting explained at the beginning of the present section.

[3]We note here a slight difference between our transseries and those of Écalle, in that we are allowing complex constants.

$$\mathbf{y}(x) \sim \tilde{\mathbf{y}}_{\mathbf{0}} \quad (x \in d;\ |x| \to \infty) \tag{122}$$

In the context of (117), a generalized Borel summation $\mathcal{LB}$ of transseries (119) is defined in [15].

The formal solutions (119) are determined by the equation (117) that they satisfy, except for the parameters $\mathbf{C}$. Then a correspondence between actual and formal solutions of the equation is an association between solutions and constants $\mathbf{C}$. This is done using a generalized Borel summation $\mathcal{LB}$.

The operator $\mathcal{LB}$ constructed in [15] can be applied to any transseries solution (119) of (117) (valid on its open sector S_{trans}, assumed non-empty) on any direction $d \subset S_{trans}$ and yields an actual solution $\mathbf{y} = \mathcal{LB}\tilde{\mathbf{y}}$ of (117), analytic in a domain S_{an}. Conversely, any solution $\mathbf{y}(x)$ satisfying (122) on a direction d is represented as $\mathcal{LB}\tilde{\mathbf{y}}(x)$, on d, for some unique $\tilde{\mathbf{y}}(x)$:

$$\mathbf{y}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x) = \sum_{\mathbf{k}\geq 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{M}\cdot\mathbf{k}} \mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \mathcal{LB}\tilde{\mathbf{y}}(x) \tag{123}$$

for some constants $\mathbf{C} \in \mathbb{C}^n$, where $M_j = \lfloor \Re\alpha_j \rfloor + 1$ ($\lfloor\cdot\rfloor$ is the integer part), and

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^{-\mathbf{k}\boldsymbol{\alpha}'+r}} \qquad (\boldsymbol{\alpha}' = \boldsymbol{\alpha} - \mathbf{M}) \tag{124}$$

(for technical reasons the Borel summation procedure is applied to the series

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = x^{\mathbf{k}\boldsymbol{\alpha}'} \tilde{\mathbf{s}}_{\mathbf{k}}(x) \tag{125}$$

rather than to $\tilde{\mathbf{s}}_{\mathbf{k}}(x)$ cf. (119),(120)).

The modification necessary to extend (123) to the case $\lambda_0 = 0$ is outlined in §8.3.

8.2 Normal form of Eq. (7)

We first give some detail on the normalization procedure, in the limit $|x| \to \infty$. It can be checked that there is a one-parameter family of formal solutions to (7) in the form $Cx^{-1/2} - \frac{15C^4}{8}x^{-5} + \ldots$. The physical problem requires $C = 1$; this suggests the substitution $G_0 = x^{-1/2} + h(x)$ where h is expected to behave like $-\frac{15}{8}x^{-5}$.

The normalizing substitution produces an equation with solutions in the form (123), where the terms with $\mathbf{k} > 0$ contain exponentials with argument linear in the final variable; the type of the exponenential in the equation for h can be found by linear perturbation theory around a solution h_0; with $h - h_0 = \delta$, the leading order equation for δ is

$$\delta''' + \frac{2}{9}x^{5/2}\delta' + \frac{1}{9}x^{2/3}\delta = 0 \tag{126}$$

where the substitution of the form $\delta = A(x)\mathrm{e}^{bx^p}$ shows that $p = 9/4$ implying that the natural variable is $x^{9/4}$.

Taking $G_0 = x^{-1/2} + x^{-1/2}g(x^{9/4})$, $\xi = x^{9/4}$ in (7) we obtain

$$g''' + \frac{1}{\xi}g'' + \left(\frac{11}{81\xi^2} + \frac{32}{729}\frac{1}{(1+g)^3}\right)g' = \frac{40}{243}\left(\frac{1}{\xi^3} + \frac{g}{\xi^3}\right) \tag{127}$$

which, written as a system, becomes

$$\begin{pmatrix} g'' \\ g' \\ g \end{pmatrix}' = \begin{pmatrix} 0 & -\frac{32}{729} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} g'' \\ g' \\ g \end{pmatrix} - \frac{1}{\xi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g'' \\ g' \\ g \end{pmatrix} + O(g^2, \xi^{-2}) \tag{128}$$

The eigenvalues of the first matrix on the rhs of (128), $\{0, \pm\frac{4i\sqrt{2}}{27}\}$, are the values of λ in (123). The fact that one eigenvalue is zero requires a slight modification in the proofs of [15].

8.3 Extension of the proofs in [15] to the assumption (ii')

In an attempt to minimize the possibility of confusion with the setting in [15] we assume that the order of the system is $n+1$, we count dimensions starting with zero, and take $\lambda_0 = 0$. There is no contribution from λ_0 to the general formally decreasing transseries (119); this is due to the normalization $\Re(\alpha_j) > 0$.

The convolution equations satisfied by $\mathbf{Y} = \mathcal{B}\mathbf{y}$ and $\mathbf{Y_k} = \mathcal{B}\mathbf{y_k}$ are given still given by equations (1.13) and (1.16) as in [15] (with the notation $\hat{A} = -\hat{B}$ used there):

$$-p\mathbf{Y} = \mathbf{F}_0 - \hat{\Lambda}\mathbf{Y} - \hat{B}\mathcal{P}\mathbf{Y} + \mathcal{N}(\mathbf{Y}) \tag{129}$$

$$\left(-p + \hat{\Lambda} - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{Y_k} + \left(\hat{B} + \mathbf{k}\cdot\mathbf{m}\right)\mathcal{P}\mathbf{Y_k} + \sum_{|\mathbf{j}|=1} \mathbf{d_j} * \mathbf{Y_k^{*j}} = \mathbf{T_k} \tag{130}$$

The only difference relevant to [14] with respect to the analysis in [15] is in the study of $\mathbf{Y}$, and once the analog results are obtained, the analysis of $\mathbf{Y_k}$ is virtually identical. By the normalization choice, we have $\mathbf{F_0} = p^M\mathbf{H}(p)$ where $\mathbf{H}$ is analytic at zero. In the equation (2.35) of [15]

$$\mathbf{Y} = \left(\hat{\Lambda} - p\right)^{-1}\left(\mathbf{F}_0 - \hat{B}\mathcal{P}\mathbf{Y} + \mathcal{N}(\mathbf{Y})\right) = \mathcal{M}(\mathbf{Y}) \tag{131}$$

we separate the zeroth component which is apparently singular (as was done in the study of $\mathbf{Y}_1$ in[15] §2.2.2; here the analysis is simpler):

$$-p(\mathbf{Y})_0 - \alpha_0\mathcal{P}(\mathbf{Y})_0 = \mathbf{F}_{0;0} + (\mathcal{N}(\mathbf{Y}))_0 := \mathbf{R}_0 \tag{132}$$

or

$$-p(\mathbf{Y})_0' - (\mathbf{Y})_0 - \alpha_0(\mathbf{Y})_0 = \mathbf{R}_0' \tag{133}$$

which we rewrite as an integral equation, which after integration by parts reads:

$$(\mathbf{Y})_0 = -\mathbf{F}_{0;0} + (1+\alpha_0)\int_0^1 \mathbf{F}_{0;0}(tp)dt - (\mathcal{N}(\mathbf{Y}))_0 + (1+\alpha_0)\int_0^1 (\mathcal{N}(\mathbf{Y}))_0 (tp)dt = \mathcal{M}_0^{[1]}(\mathbf{Y}) \quad (134)$$

The system is of the form (131)

$$\mathbf{Y} = \mathcal{M}^{[1]}(\mathbf{Y}) \tag{135}$$

with $\mathcal{M}^{[1]} = \mathcal{M}$ for all components other than the zeroth one defined in (134). The equation (135) is contractive in the ball $B = \{\mathbf{Y} : \{p : |p| < \epsilon\} : \|\mathbf{Y}\|_\infty < 2\epsilon\}$ for small enough ϵ, and also in the focusing algebra (3a) in §2.1.1 in [15] for $\beta_k = 1$ (allowed by the normalization of $\mathbf{F}_0$) as follows from immediate estimates.

No other nontrivial adaptations are needed in the proofs in [15].

8.4 Results of [14] as extended in §8.3

The map $\tilde{\mathbf{y}} \mapsto \mathcal{LB}(\tilde{\mathbf{y}})$ depends on the direction d, and (typically) is discontinuous at the finitely many Stokes lines, see [15], Theorem 4.

For linear equations only the directions $\overline{\lambda_j}\,\mathbb{R}_+$, $j = 1, ..., n$ are Stokes lines, but for nonlinear equations there are also other Stokes lines, recognized first by Écalle. $\mathcal{LB}$ is only discontinuous because of the jump discontinuity of the vector of "constants" $\mathbf{C}$ across Stokes directions (Stokes' phenomenon); between Stokes lines $\mathcal{LB}$ does not vary with d.

The function series in (123) is uniformly *convergent* and the functions $\mathbf{y_k}$ are analytic on domains S_{an} (for some $\delta > 0$, $R = R(\mathbf{y}(x), \delta) > 0$.

Theorem 44 *There exists $\delta_1 > 0$ so that for $|\xi| < \delta_1$ the power series*

$$\mathbf{F}_m(\xi) = \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;m}, \quad m = 0, 1, 2, ... \tag{136}$$

converge. Furthermore

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m} \mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{S}_{\delta_1},\ x \to \infty) \tag{137}$$

uniformly in $\mathcal{S}_{\delta_1}$, and the asymptotic representation (137) is differentiable.

The functions $\mathbf{F}_m$ are uniquely defined by (137), the requirement of analyticity at $\xi = 0$, and $\mathbf{F}_0'(0) = \mathbf{e}_1$.

Remark 19 *A direct calculation shows that the functions $\mathbf{F}_m$ are solutions of the system of equations*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_0 = \xi^{-1}\left(\hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0)\right) \tag{138}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_m + \hat{N}\mathbf{F}_m = \alpha_1 \frac{\mathrm{d}}{\mathrm{d}\xi}\mathbf{F}_{m-1} + \mathbf{R}_{m-1} \qquad \text{for } m \geq 1 \tag{139}$$

where $\hat{N}$ is the matrix

$$\xi^{-1}(\partial_{\mathbf{y}}\mathbf{g}(0,\mathbf{F}_0)-\hat{\Lambda}) \tag{140}$$

and the function $\mathbf{R}_{m-1}(\xi)$ depends only on the $\mathbf{F}_k$ with $k<m$:

$$\xi\mathbf{R}_{m-1}=-\left[(m-1)I+\hat{A}\right]\mathbf{F}_{m-1}-\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z;\sum_{j=0}^{m-1}z^j\mathbf{F}_j\right)\Bigg|_{z=0} \tag{141}$$

8.5 Formal arguments for thin-film equation

Consider the particular initial value problem in one space dimension:

$$h_t+\left(h^3h_{xxx}\right)_x=0 \quad , \quad h(x,0)=\frac{1}{1+x^2} \tag{142}$$

This is a special case of $h_t+(h^nh_{xxx})_x=0$. Global existence proofs are available only for $n>3.5$; numerical solutions suggest finite-time singularity for $n=1$ [17].

For the problem (142) and variations of it, the complex region for which existence is expected, at least for small t, includes the real x-axis. For the specific initial value problem, we change variables:

$$h(x,t)=H(1+x^2,t), \qquad \xi=1+x^2$$

and obtain a nonlinear PDE for $H(\xi,t)$. A formal asymptotic expansion in powers of t results in

$$H(\xi,t)=\frac{1}{\xi}\sum_{j=0}^{\infty}P_{2j}\left(t^{1/2}\xi^{-7/2},t^{1/2}\xi^{-5/2}\right) \tag{143}$$

where P_{2j} are homogenous polynomials of order $2j$. With appropriate changes of variables, we expect the regularity theorem [5] to be adaptable to prove short term existence for a complex ξ sector that includes $(1,\infty)$ (i.e., $x\in\mathbb{R}$), and show further the validity of (143) for $\xi\gg t^{1/7}$ in this sector.

Asymptotics (143) fails when $\xi=O(t^{1/7})$. Introducing scaled variables,

$$\eta=\xi t^{-1/7},\tau=t^{1/7},H(\xi(\eta,\tau),t(\tau))=\xi^{-1}F(\eta,\tau),$$

gives a formal solution as an expansion in integer powers of τ,

$$F(\eta,\tau)=\sum_{k=0}^{\infty}\tau^kF_k(\eta) \tag{144}$$

We expect this series to be convergent. The equation of F_0 can be integrated once by using far-field matching condition to give:

$$F_0^3F_0'''-\frac{6}{\eta^3}F_0^4-\frac{\eta^4}{112}F_0+\frac{6}{\eta^2}F_0^3F_0'-\frac{3}{\eta}F_0^3F_0''+\frac{\eta^4}{112}=0$$

With the further transformation $F_0(\eta)=1+y(\eta^{7/3})$, the equation for y is in a form to which the general theory [14] applies. From the leading order singularity of the ODE, and the expected convergence of (144), as for modified Harry-Dym, we expect to show that the thin-film equation has singularities at points close to $x_s(t)$ with $1+x_s^2=\eta_st^{1/7}$.

References

[1] D.W. Moore, The spontaneous appearance of a singularity in the shape of an evolving vortex sheet, *Proc. R. Soc. Lond.*, A **365**, 105 (1979).

[2] S.J. Cowley, G.R. Baker & S. Tanveer, On the formation of Moore curvature singularities in vortex sheets, *J. Fluid Mech.*, **378**, 233 (1999).

[3] S. Tanveer, Evolution of Hele-Shaw interface for small surface tension, *Proc. Roy. Soc. Lond*, A **343**, 155 (1993).

[4] S. Tanveer, Singularities in classical Rayleigh-Taylor flow: Formation and subsequent motion. *Proc. R. Soc. Lond.* A **441**, 501 (1993).

[5] O. Costin & S. Tanveer, On the existence and uniqueness of solutions of nonlinear evolution systems of PDEs in $\mathbb{R}^+ \times \mathbb{C}^d$, their asymptotic and Borel summability properties (submitted; available at http://www.math.rutgers.edu/~costin .

[6] O. Costin & S. Tanveer, Existence and uniqueness of solutions of nonlinear evolution systems of n-th order partial differential equations in the complex plane, Submitted.

[7] O. Costin, S. Tanveer, Existence and uniqueness for a class of nonlinear higher-order partial differential equations in the complex plane, Comm. Pure Appl. Math, Vol. LIII, 1092—1117 (2000).

[8] A. S. Fokas & S. Tanveer, A Hele-Shaw problem and the second Painleve' transcendent, *Math. Proc. Camb. Phil. Soc*, A **124**, 169 (1998).

[9] J. Écalle *Fonctions Resurgentes, Publications Mathematiques D'Orsay, 1981*

[10] J. Écalle *in Bifurcations and periodic orbits of vector fields NATO ASI Series, Vol. 408, 1993*

[11] J. Écalle *Finitude des cycles limites et accéléro-sommation de l'application de retour*, Preprint 90-36 of Universite de Paris-Sud, 1990

[12] W. Balser, B. L. J. Braaksma, J-P Ramis, Y. Sibuya *Asymptotic Anal.* ***5****(1991), 27-45*

[13] O Costin, *Exponential asymptotics, transseries, and generalized Borel summation for analytic, nonlinear, rank-one systems of ordinary differential equations*, Internat. Math. Res. Notices no. 8, 377–417 (1995).

[14] O Costin and R D Costin, *On the location and type of singularities of nonlinear differential systems* (Inventiones Mathematicae **145**, 3, pp 425-485 (2001))).

[15] O. Costin *Duke Math. J. Vol. 93, No 2: 289–344, 1998*

[16] W. Wasow *Asymptotic expansions for ordinary differential equations*, Interscience Publishers 1968.

[17] A. Bertozzi, The Mathematics of Moving Contact Lines in Thin Liquid Films, *Notices of the AMS*, June-July, 1998.

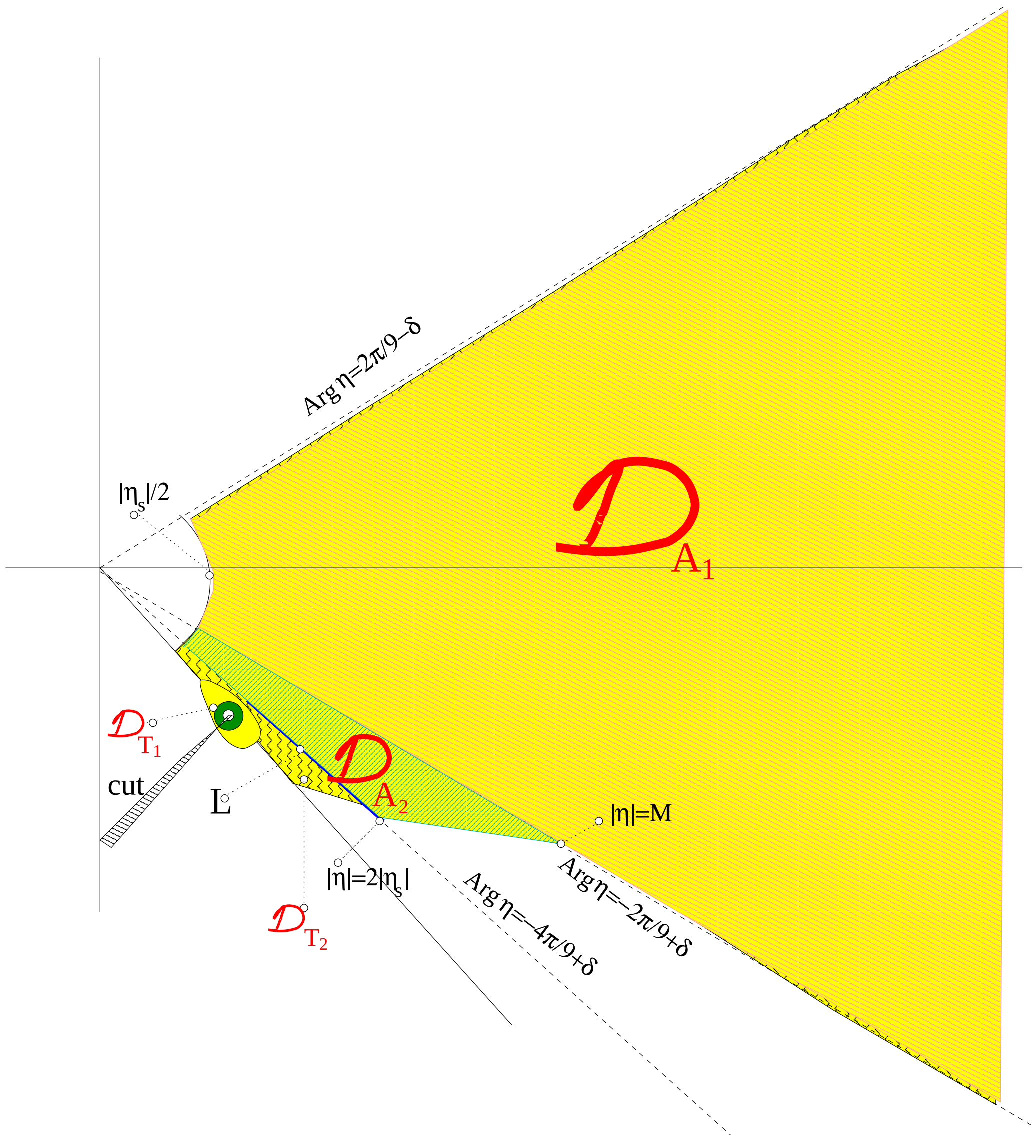

Figure 1: Subdomains of $\mathcal{D}$

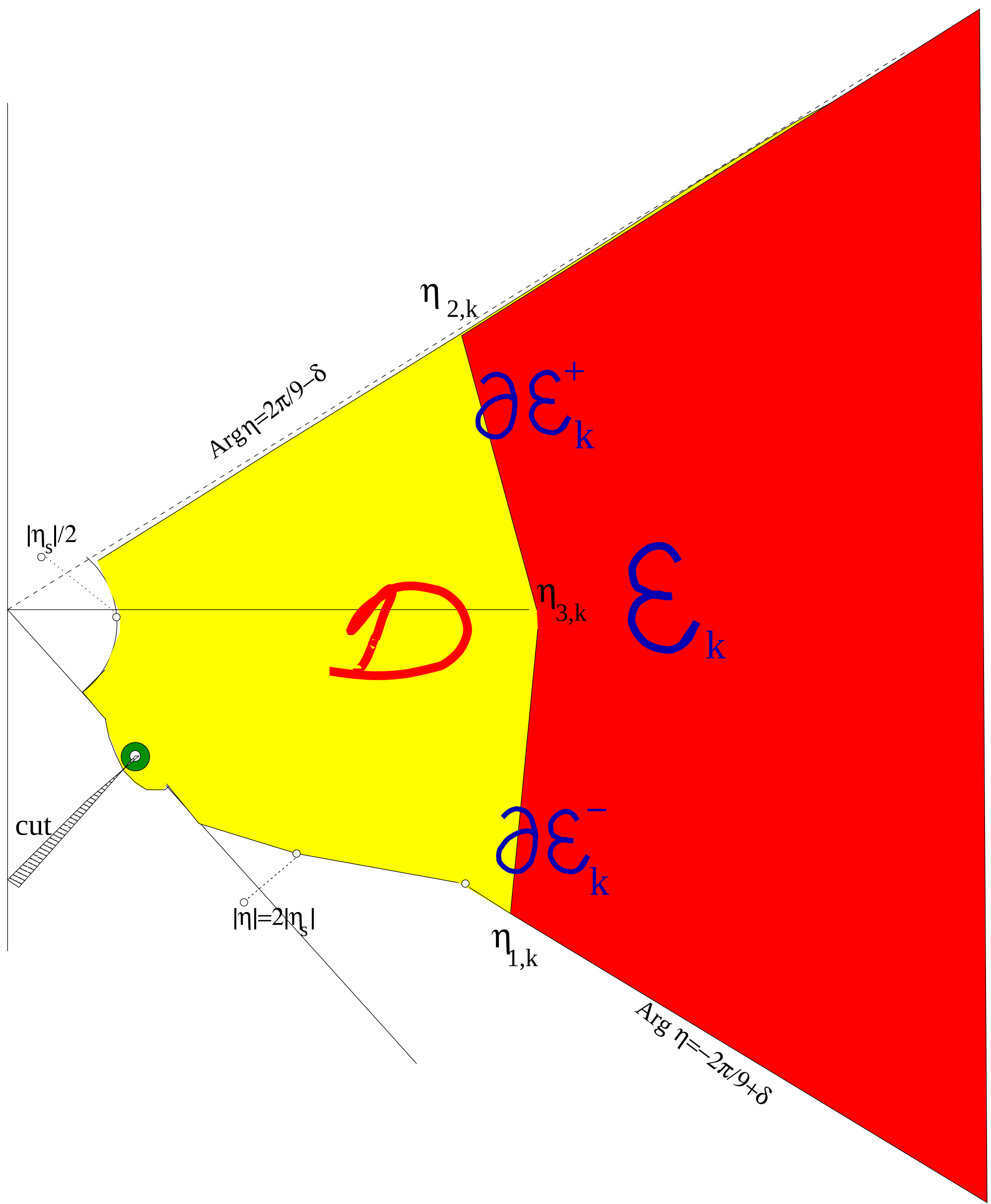

Figure 2: Domains $\mathcal{D}_k$, $\mathcal{E}_k$ and common boundary ∂E_k.

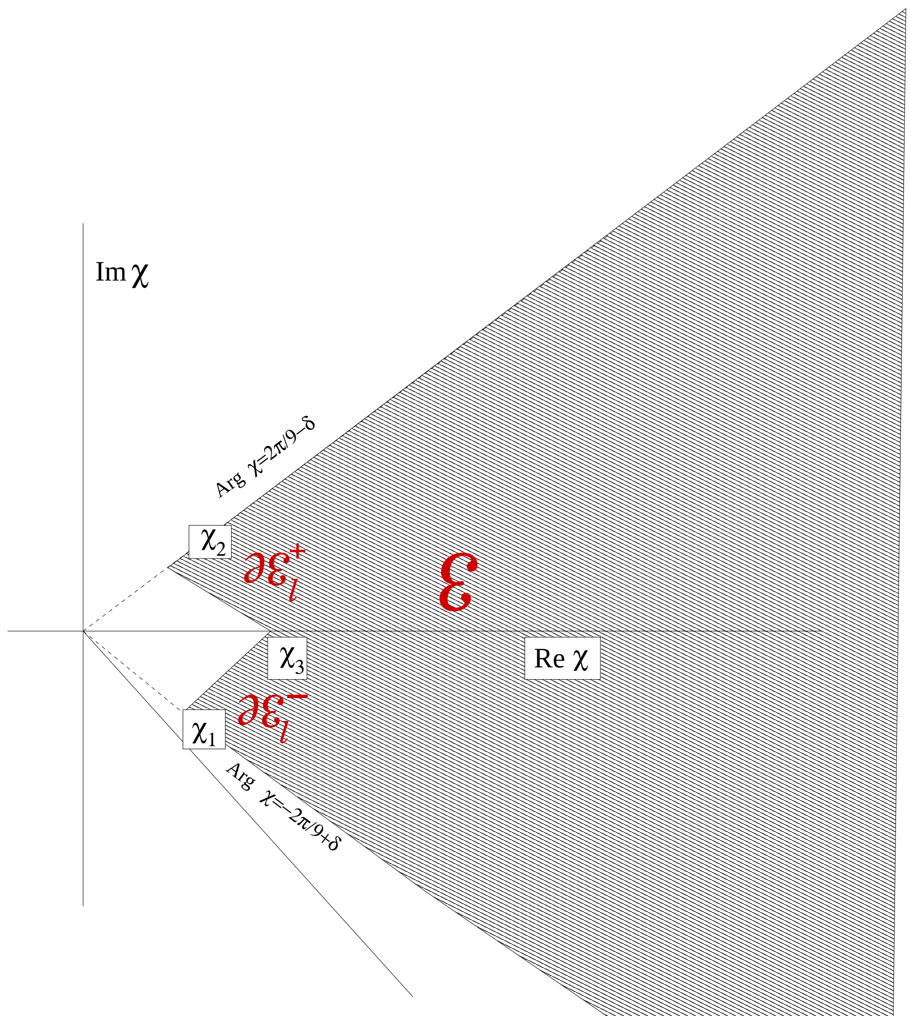

Figure 3: Domain $\mathcal{E}$ in the χ-plane