Estimation asymptotique des intervalles d'instabilité d'opérateurs de Hill avec potentiels à deux termes

ASYMPTOTICS OF INSTABILITY ZONES OF HILL OPERATORS WITH TWO TERM POTENTIAL

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We give a sharp asymptotics of the instability zones of the Hill operator $Ly = -y'' + (a\cos 2x + b\cos 4x)y$ for arbitrary real $a, b \neq 0$.

Résumé

Dans cette note on donne une estimation asymptotique des intervalles d'instabilité d'opérateurs de Hill de la forme $Ly = -y'' + (a\cos 2x + b\cos 4x)y$, où a et b sont des réels non nuls arbitraires.

1. The Schrödinger operator $Ly = -y'' + v(x)y, \ -\infty < x < \infty$, with real valued periodic $L^2([0,\pi])$-potential $v(x)$, $v(x+\pi) = v(x)$, has spectral gaps, or instability zones $(\lambda_n^-, \lambda_n^+)$, $n \geq 1$, close to n^2 if n is large enough. The points λ_n^-, λ_n^+ could be determined as well as eigenvalues of the Hill equation $Ly = -y'' + v(x)y = \lambda y$, considered on $[0,\pi]$ with boundary conditions $Per^+ : y(0) = y(\pi), \ y'(0) = y'(\pi)$ for n even, and $Per^- : y(0) = -y(\pi), \ y'(0) = -y'(\pi)$ for n odd. See details and basics in [**15**, **17**, **18**, **21**].

Let $\gamma_n = \lambda_n^+ - \lambda_n^-$ be the lengths of the spectral gaps. The decay rates of (γ_n) are in a close relation with smoothness of the potential v (see [**11**, **12**, **22**, **4**, **5**, **6**]). Sometimes the Lyapunov function, or the Hill discriminant (see [**17**], Sect. 2.1-2.2) $\Delta(\lambda)$ can be written explicitly as it happens in the Krönig-Penney model, made of a periodic array of δ and δ' functions, or onionlike scatterers with several channels (see details in [**2**] and the bibliography there). Then the asymptotics of the roots of Lyapunov functions (trigonometric polynomials (7), (8) in [**2**]) and consequently the asymptotics of gaps and bands become a question about roots of elementary trigonometric functions. Without explicit Lyapunov function this task is much more difficult.

2. In 1980 E. Harrell [**10**], and then J. Avron and B. Simon [**1**] gave the asymptotics of spectral gaps of the Mathieu operator $-\dfrac{d^2}{dx^2} + 2a\cos 2x$. They showed that

$$\gamma_n = \lambda_n^+ - \lambda_n^- = 8\pi^2 \left(\frac{|a|}{4\pi^2}\right)^n \frac{1}{((n-1)!)^2} \left(1 + O(1/n^2)\right).$$

In [**1**] the question was raised about these asymptotics in the case of two term potential

$$v(x) = a\cos 2x + b\cos 4x. \qquad (1)$$

Later, A. Grigis [**9**] gave generic asymptotics of spectral gaps of the Schrödinger operator $-\dfrac{d^2}{dx^2}+v(x)$ when v is a real-valued trigonometric polynomial. For him, the two term potential

$$u(x) = c\sin 2x + d\cos 4x, \quad d>0, \tag{2}$$

was of special interest as well. (Notice that the shift $x \to x+\pi/4$ transforms $u(x) \in$ (2) into $v \in$ (1) with $a=c, b=-d$. Their Schrödinger operators are isospectral, so we can consider without loss of generality just potentials (1); however, b could be positive or negative.)

3. Recently, we found [**7**, **8**] the asymptotics of (γ_n) for a potential of the form (2) when $c^2 = 8d > 0$. Our proofs were based on the relationship of Dirac operator with potential $\begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}$ and Hill operators with potential $u = \pm p' + p^2$, the Ricatti transform of p. In terms of a, b in (1), if we introduce a parameter t by

$$a^2 + 8bt^2 = 0, \tag{3}$$

then $c^2 = 8d > 0$ is a special case of (3) with $t = \pm 1$. Generally, for real $a, b \neq 0$ we set $a^2+8bt^2=0$, $b=-2\alpha^2$, $a=-4\alpha t$, where

(i) α and t are real if $b<0$,

(ii) α and t are pure imaginary if $b>0$.

Now, this parametrization plays a special role in asymptotic behavior of gaps $\gamma_n(\alpha)$, both for $\alpha \to 0$ and $n \to \infty$.

Theorem 1. *Let γ_n, $n \in \mathbb{N}$ be the spectral gaps (lengths of instability zones) of the operator*

$$Ly = -y'' - [4\alpha t \cos 2x + 2\alpha^2 \cos 4x]y. \tag{4}$$

If t and n are fixed, then for even n

$$\gamma_n = \frac{\pm 8\alpha^n}{2^n[(n-1)!]^2} \prod_{k=1}^{n/2} \left(t^2 - (2k-1)^2\right)((1+O(\alpha))\,, \tag{5}$$

and for odd n

$$\gamma_n = \frac{\pm 8\alpha^n t}{2^n[(n-1)!]^2} \prod_{k=1}^{(n-1)/2} \left(t^2 - (2k)^2\right)((1+O(\alpha))\,. \tag{6}$$

Remark 1. In the case (ii), if we put $\alpha = i\beta$, $t = i\tau$, β, τ real, then we can rewrite, say, (6), as

$$\gamma_n = \frac{\pm 8\beta^n \tau}{2^n[(n-1)!]^2} \prod_{k=1}^{(n-1)/2} \left(\tau^2 + (2k)^2\right)((1+O(\beta))\,.$$

Of course, (5) could be rewritten in terms of β, τ in the same way.

Proof is based, on one hand, on our analytic methods [**3**, **4**, **5**, **6**], and on the other hand, on using the approach to coexistence probem of W. Magnus and S. Winkler (see [**16**], [**17**], Ch.7, in particular, Thm 7.9)) and sharpening their result about the multiplicities of eigenvalues of the operator (4) in the case where t is an integer.

4. The essential components of the asymptotics (5) and (6) are polynomials in t of degree n. The combinatorial meaning of their coefficients unearthed in the course of the proof of Theorem 1 leads to a series of algebraic identities.

Theorem 2. *The following formulae hold:*

$$\sum (m^2 - i_1^2)\cdots(m^2 - i_k^2) = \sum_{1\le j_1<\cdots<j_k\le m} (2j_1-1)^2\cdots(2j_k-1)^2, \tag{7}$$

where the first sum is over all indicies i_s such that

$$-m < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \ge 2;$$

$$\sum [(2m-1)^2 - (2i_1-1)^2]\cdots[(2m-1)^2 - (2i_k-1)^2] = \sum_{1\le j_1<\cdots<j_k\le m-1} (4j_1)^2\cdots(4j_k)^2, \tag{8}$$

where the first sum is over all indicies i_s such that

$$-m+1 < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \ge 2.$$

The terms in (7) and (8) look to be similar to the terms in the identity conjectured by V. Kac and M. Wakimoto [**13**] and proved by S. Milne [**19**], and later by D. Zagier [**23**]; see details and further bibliography in [**20**], in particular, Sect. 7 and Cor. 7.6, pp. 120-121. Our asymptotic analysis involves eigenvalues of Schrödinger operators. This occurrence of eigenvalues suggests a possible link with advanced determinant calculus developed by G. Andrews (see C . Krattenthaler [**14**] and references there) and Hankel determinants in S. Milne [**20**].

5. Asymptotics for $n \to \infty$.

Theorem 3. *Under the notations of Theorem* 1, *let $\alpha \ne 0$ and $t \ne 0$ be fixed. Then for even n*

$$\gamma_n = \pm \frac{8\alpha^n}{2^n[(n-2)!!]^2} \cos\left(\frac{\pi}{2}t\right)\left[1 + O((\log n)^3/n)\right], \tag{9}$$

and for odd n

$$\gamma_n = \pm \frac{8\alpha^n}{2^n[(n-2)!!]^2} \frac{2}{\pi} \sin\left(\frac{\pi}{2}t\right)\left[1 + O((\log n)^3/n)\right]. \tag{10}$$

[Let us recall that $(2k-1)!! = 1\cdot 3\cdots(2k-1), \quad (2k)!! = 2\cdot 4\cdots 2k$.]

Remark 2. As in Remark 1, in the case (ii), $\cos(\pi t/2) = \cosh(\pi\tau/2)$ and if n is odd, then the product $\alpha^n \sin(\pi t/2) = i^{n+1}\beta^n \sinh(\pi\tau/2)$ is real.

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