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BERNSTEIN WIDTHS OF HARDY-TYPE OPERATORS IN A NON-HOMOGENEOUS CASE

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Abstract

Let $I = [a, b] \subset \mathbb{R}$, let $1 < p \leq q < \infty$, let u and v be positive functions with $u \in L_{p'}(I)$, $v \in L_q(I)$ and let $T : L_p(I) \to L_q(I)$ be the Hardy-type operator given by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \ x \in I.$$

We show that the Bernstein numbers b_n of T satisfy

$$\lim_{n\to\infty} nb_n = c_{pq} \left(\int_I (uv)^r dt \right)^{1/r}, \ r = 1/p' + 1/q,$$

where c_{pq} is an explicit constant depending only on p and q.

1. *Introduction*

Let u and v be real-valued measurable functions on an interval $I := [a, b] \subset \mathbb{R}$. In [**7**], [**10**], [**11**], [**8**] and [**12**] the Hardy-type operator T given by

$$(Tf)(x) := v(x) \int_a^x f(t)u(t)dt, \ x \in I, \tag{1.1}$$

was considered as a map from $L_p(I)$ to itself, when $1 \leq p \leq \infty$. As a consequence of this work, together with that of [**9**], it is known that under appropriate conditions on u and v the approximation numbers $a_n(T)$ of T satisfy

$$\lim_{n\to\infty} na_n(T) = \lambda_p^{-1/p} \int_I |u(t)v(t)| \, dt,$$

where λ_p is the first eigenvalue of a $p-$Laplacian eigenvalue problem on I. We recall that $a_n(T) := \inf \|T - F\|$, the infimum being taken over all bounded linear maps $F : L_p(I) \to L_p(I)$ with rank less than n. A connected account of such results concerning $a_n(T)$ is given in [**6**]. The main purpose of this paper is to study the properties of T as a map from $L_p(I)$ to $L_q(I)$ when $1 < p \leq q < \infty$, and we focus on its Bernstein widths. These are the numbers $b_n = b_n(T)$ $(n \in \mathbb{N})$ given by

$$b_n = \sup_{X_{n+1}} \inf_{Tf \in X_{n+1}\setminus\{0\}} \|Tf\|_{q,I} / \|f\|_{p,I} \,,$$

where the supremum is taken over all subspaces X_{n+1} of $T(L_p(I))$ with dimension $n + 1$. The Bernstein widths of various maps have been extensively studied: for embeddings of Sobolev spaces we refer to Pinkus [**19**], Bourgain and Gromov [**1**] and Lang [**13**]; and for the map $T : L_p(I) \to L_p(I)$, in the special case when

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$u = v = 1$, see [**3**]. Our main result is that if u and v are positive functions with $u \in L_{p'}(I)$ and $v \in L_q(I)$, then

$$\lim_{n\to\infty} nb_n = c_{pq} \left(\int_I (uv)^r dt \right)^{1/r}, \ r = 1/p' + 1/q,$$

where c_{pq} is an explicit constant.

For appropriate u and v it can be proved that when T is viewed as a map from $L_p(I)$ to $L_p(I)$ we have $a_n(T) = b_n(T)$ for all $n \in \mathbb{N}$ (for more details see [**14**]). In contrast, when T is a map from $L_p(I)$ to $L_q(I)$ with $p < q$, we know that $a_n(T) > b_n(T)$ for all $n \in \mathbb{N}$ (see Section IV, Theorem 1.1 in [**18**]). Moreover, with suitable u and v, it can be shown that (again with $p < q$)

$$\lim_{n\to\infty} na_n(T) = \infty,$$

so that the decay of $a_n(T)$ is slower than $1/n$: see [**16**]. This underlines the difference between the approximation numbers and the Bernstein numbers.

Throughout the paper we suppose that $1 < p \le q < \infty$, that u and v are positive functions $I = [a, b] \subset \mathbb{R}$ with $u \in L_{p'}(I)$ and $v \in L_q(I)$, and that T is a compact map from $L_p(I)$ to $L_q(I)$. The standard norm on $L_p(I)$ will be denoted by $\|\cdot\|_{p,I}$ or by $\|\cdot\|_p$ if no ambiguity is possible. We write $A \lesssim B$ (or $A \gtrsim B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B. By $A \approx B$ we shall mean that $A \lesssim B$ and $B \lesssim A$.

2. *Preliminaries and technical results*

We start with the definition of special generalisations of the trignometric functions, the $\sin_{pq}$ and $\cos_{pq}$ functions (see [**4**]). (Note that these functions have their origin in [**15**] and [**20**])

Definition 2.1. For $\sigma \in [0, q/2]$ we set

$$\text{arc } \sin_{pq}(\sigma) = \frac{q}{2} \int_0^{2\sigma/q} \frac{ds}{(1 - s^q)^{1/p}}.$$

We put

$$\pi_{pq} = 2\text{arc } \sin_{pq}(q/2) = B(1/q, 1/p'),$$

where $1/p' = 1 - 1/p$ and B denotes the Beta function. By $\sin_{pq}$ we mean the inverse of arc $\sin_{pq}$ and the extension of this inverse as a $2\pi_{pq}-$periodic function on $\mathbb{R}$.

More precisely, since arc $\sin_{pq} : [0, q/2] \to [0, \pi_{pq}/2]$ is increasing, $\sin_{pq}$ is well-defined on $[0, \pi_{pq}/2]$. We extend it to $[\pi_{pq}/2, \pi_{pq}]$ by defining $\sin_{pq} x = \sin_{pq}(\pi_{pq} - x)$ for $x \in [\pi_{pq}/2, \pi_{pq}]$, to $[-\pi_{pq}, \pi_{pq}]$ by oddness, and finally to all of $\mathbb{R}$ by $2\pi_{pq}-$periodicity. Now define $\cos_{pq}$ by

$$\cos_{pq}(x) = \frac{d}{dx} \sin_{pq} x;$$

this is an even, $2\pi_{pq}-$periodic function that is odd about $\pi_{pq}/2$.

Now we recall some facts concerning eigenfunctions and eigenvalues for certain non-linear problems, together with results about Bernstein widths for Sobolev embeddings. Put $I=[a,b]$, where $-\infty<a<b<\infty$, and let $(x)_{(p)}:=|x|^{p-1}\operatorname{sgn}(x)$, $x\in\mathbb{R}$. Let

$$K(x,y):=v(x)u(y)\chi_{[a,x]}(y)\text{ for }x,y\in I.$$

Then for any $m\in\mathbb{N}$ and any collection of points $x_1,x_2,...,x_m,y_1,y_2,...,y_m$ with $a\le x_1\le x_2\le...\le x_m\le b$ and $a\le y_1\le y_2\le...\le y_m\le b$, we have

$$\det\ (K(x_i,y_j))_{i,j=1}^m\ge 0.$$

This means that $K(\cdot,\cdot)$ is totally positive, in the terminology of Pinkus [**19**], Definition 3.1, p. 52. The map T given by (1.1) is represented by

$$(Tf)(x)=\int_I K(x,y)f(y)dy.$$

Let $B:=\{f\in L_p(I):\|f\|_p\le 1\}$ and consider the isoperimetric problem of determining

$$\sup_{g\in T(B)}\|g\|_q\,. \tag{2.1}$$

This problem is related to the following non-linear integral problem:

$$g(x)=(Tf)(x) \tag{2.2}$$

and

$$(f(x))_{(p)}=\lambda(T^*((g)_q))(x), \tag{2.3}$$

where $(g)_q$ is the function with value $(g(x))_q$ at x and T^* is the map defined by $(T^*f)(x)=u(x)\int_x^b v(y)f(y)dy$. Note that when u and v are both identically equal to 1 on I, (2.2) and (2.3) can be transformed into the $p,q-$Laplacian differential equation

$$-\left((w')_{(p)}\right)'=\lambda(w)_{(q)}, \tag{2.4}$$

with the boundary condition

$$w(a)=0. \tag{2.5}$$

A pair (g,λ) for which a function f with $\|f\|_p=1$, satisfying (2.2) and (2.3), can be found, will be called a spectral pair. The set of all spectral pairs will be denoted by $SP(T,p,q)$. The number λ occurring in a spectral pair will be called a spectral number, and the set of all such numbers denoted by $sp(T,p,q)$; the function g corresponding to λ is called a spectral function. Given any continuous function f on I we denote by $Z(f)$ the number of distinct zeros of f on $\overset{o}{I}$, and by $P(f)$ the number of sign changes on this interval. The set of all spectral pairs (g,λ) with $Z(g)=n$ $(n\in\mathbb{N}_0)$ will be denoted by $SP_n(T,p,q)$, and $sp_n(T,p,q)$ will represent the set of all corresponding numbers λ.

Theorem 2.1. *For all $n\in\mathbb{N}$, $SP_n(T,p,q)\neq\emptyset$.*

Proof. This essentially follows from [**3**] (see also [**17**]), but we give the details for the convenience of the reader. For simplicity we suppose that I is the interval $[0,1]$. A key idea in the proof is the introduction of an iterative procedure used in [**3**].

Let $n \in \mathbb{N}$ and define

$$\mathcal{O}_n = \left\{ z = (z_1, ..., z_{n+1}) \in \mathbb{R}^{n+1} : \sum\nolimits_{i=1}^{n+1} |z_i| = 1 \right\}$$

and

$$f_0(x,z) = sgn(z_j) \text{ for } \sum\nolimits_{i=0}^{j-1} |z_i| < x < \sum\nolimits_{i=1}^{j} |z_i|, \; j = 1, ..., n+1, \text{ with } z_0 = 0.$$

With $g_0(x,z) = Tf_0(x,z)$ we construct the iterative process

$$g_k(x,z) = Tf_k(x,z), \; f_{k+1}(x,z) = (\lambda_k^q(z) T^*(g_k(x,z))_{(q)})_{(p')},$$

where λ_k is a constant so chosen that

$$\|f_{k+1}\|_p = 1$$

and $1/p + 1/p' = 1$. Then, all integrals being over I,

$$\begin{aligned} 1 &= \int |f_k(x,z)|^p \, dx = \int f_k (f_k)_{(p)} dx = \int f_k \left([\lambda_{k-1}^q T^*((g_{k-1})_{(q)})]_{(p')} \right)_{(p)} dx \\ &= \int f_k \lambda_{k-1}^q T^*((g_{k-1})_{(q)}) dx \\ &= \lambda_{k-1}^q \int T(f_k)(g_{k-1})_{(q)} dx \le \lambda_{k-1}^q \|g_k\|_q \|g_{k-1}\|_q^{q-1} \end{aligned}$$

and also

$$\begin{aligned} \|g_{k-1}\|_q^q &= \int |g_{k-1}(x,z)|^q \, dx = \int (g_{k-1})_{(q)} g_{k-1} dx \\ &= \int (g_{k-1})_{(q)} T(f_{k-1}) dx = \int T^*((g_{k-1})_{(q)}) f_{k-1} dx \\ &= \lambda_{k-1}^{-q} \int \lambda_{k-1}^q T^*((g_{k-1})_{(q)}) f_{k-1} dx \\ &\le \lambda_{k-1}^{-q} \left(\int \left| (\lambda_{k-1}^q T^*((g_{k-1})_{(q)})_{(p')} \right|^{p'} dx \right)^{1/p'} \left(\int |f_{k-1}|^p \, dx \right)^{1/p} \\ &= \lambda_{k-1}^{-q} \left(\int \left| (\lambda_{k-1}^q T^*((g_{k-1})_{(q)})_{(p')} \right|^{p'} dx \right)^{1/p'} \\ &= \lambda_{k-1}^{-q} \left(\int |f_k|^p \, dx \right)^{1/p} = \lambda_{k-1}^{-q}. \end{aligned}$$

From these inequalities it follows that

$$\|g_{k-1}(\cdot,z)\|_q \le \lambda_{k-1}^{-1} \le \|g_k(\cdot,z)\|_q .$$

This shows that the sequences $\{g_k(\cdot,z)\}$ and $\{\lambda_k(z)\}$ are monotonic decreasing. Put $\lambda(z) = \lim_{k\to\infty} \lambda_k(z)$; then $\|g_k(\cdot,z)\|_q \to \lambda^{-1}(z)$.

As the sequence $\{f_k(\cdot,z)\}$ is bounded in $L_p(I)$, there is a subsequence $\{f_{k_i}(\cdot,z)\}$ that is weakly convergent, to $f(\cdot,z)$, say. Since T is compact, $g_{k_i}(\cdot,z) \to Tf(\cdot,z) := g(\cdot,z)$ and we also have $f(\cdot,z) = (\lambda^q(z)T^*(g(\cdot,z))_{(q)})_{(p')}$. It follows that for each $z \in \mathcal{O}_n$, the sequence $\{g_{k_i}(\cdot,z)\}$ converges to a spectral function.

Now set $z = (0,0,...,0,1) \in \mathcal{O}_n$. Then $f_0(\cdot,z) = 1$, and as the operators T and T^* are positive, $g_k(\cdot,z) \ge 0$ for all k, so that $g(\cdot,z) \ge 0$. Thus $g(\cdot,z) \in SP_0(T,p,q)$: $SP_0(T,p,q) \ne \emptyset$.

Next we show that for all $n \in \mathbb{N}, SP_n(T,p,q) \neq \emptyset$. Given $n, k \in \mathbb{N}$, set

$$E_k^n = \{z \in \mathcal{O}_n : Z(g_k(\cdot, z)) \leq n-1\}.$$

From the definition of T it follows that $g_k(\cdot, z)$ depends continuously on z; thus E_k^n is an open subset of $\mathcal{O}_n$ and $F_k^n := \mathcal{O}_n \backslash E_k^n$ is a closed subset of $\mathcal{O}_n$. Let $0 < t_1 < ... < t_n < 1$ and put

$$F_k(\alpha) = (g_k(t_1, \alpha), ..., g_k(t_n, \alpha)), \ \alpha \in \mathcal{O}_n.$$

Then F_k is a continuous, odd mapping from $\mathcal{O}_n$ to $\mathbb{R}^n$. By Borsuk's theorem, there is a point $\alpha_k \in \mathcal{O}_n$ such that $F_k(\alpha_k) = 0$; that is, $\alpha_k \in E_k^n$. From the definition of g_k and f_{k+1}, together with the positivity of T and T^*, we have

$$Z(g_{k+1}) \leq P(f_{k+1}) \leq Z(f_{k+1}) \leq P(g_k) \leq Z(g_k),$$

so that $E_k^n \subset E_{k+1}^n$, which implies that $F_k^n \supset F_{k+1}^n$. Hence there exists $\widetilde{\alpha} \in \cap_{k\geq 1} F_k^n$, and as above we see that $g_k(\cdot, \widetilde{\alpha})$ converges, as $k \to \infty$, to a spectral function $g(\cdot, \widetilde{\alpha}) \in SP_n(T,p,q)$. Thus $SP_n(T,p,q) \neq \emptyset$ and the proof is complete. □

We denote by $SP_n^a(p,q)$ the set of all pairs (w, λ) (again called spectral pairs, w being an eigenfunction with associated eigenvalue λ) corresponding to solutions of (2.4) and (2.5) for which $Z(u) = n$. Similarly, $SP_n^{a,b}(p,q)$ will stand for the set of all spectral pairs (w, λ) corresponding to solutions of (2.4) that satisfy the Dirichlet boundary conditions

$$w(a) = w(b) = 0 \tag{2.6}$$

and have $Z(u) = n$. It is known from [**3**], [**4**] or [**19**] that for all $n \in \mathbb{N}$, $SP_n^{a,b}(p,q)$ consists of exactly one spectral pair (up to normalisation). Moreover, from [**20**] or [**4**] we have

Lemma 2.1. *For any* $\alpha \in \mathbb{R}\backslash\{0\}$, *the set of eigenvalues of problem (2.4) under the Dirichlet boundary conditions (2.6) on* $I = [a,b]$ *is given by*

$$\lambda_n(\alpha) := \left(\frac{2n\pi_{pq}}{b-a}\right)^q \cdot \frac{|\alpha|^{p-q}}{p'q^{q-1}} \quad (n \in \mathbb{N}),$$

with corresponding eigenfunctions

$$w_{n,\alpha}(t) := \frac{\alpha(b-a)}{n\pi_{pq}} \sin_{pq}\left(\frac{n\pi_{pq}}{b-a}t\right) \quad (t \in I).$$

A simple computation enables us to modify Lemma 2.1 so as to apply to the eigenvalue problem (2.4) with initial conditions at the left-hand endpoint a of I.

Lemma 2.2. *For any* $\alpha \in \mathbb{R}\backslash\{0\}$, *the set of eigenvalues of problem (2.4) under the conditions*

$$w(a) = 0, w'(a) = \alpha \tag{2.7}$$

on $I = [a,b]$ *is given by*

$$\widetilde{\lambda}_n(\alpha) := \left(\frac{2(n-1/2)\pi_{pq}}{b-a}\right)^q \cdot \frac{|\alpha|^{p-q}}{p'q^{q-1}} \quad (n \in \mathbb{N}),$$

with corresponding eigenfunctions

$$\widetilde{w}_{n,\alpha}(t) := \frac{\alpha(b-a)}{(n-1/2)\pi_{pq}} \sin_{pq}\left(\frac{(n-1/2)\pi_{pq}}{b-a}t\right) \quad (t \in I).$$

Next we recall the definitions and basic properties of the Bernstein widths, the linear widths and the approximation numbers.

Definition 2.2. Let C be a centrally symmetric subset of a normed linear space X and let $n \in \mathbb{N}$. The n^{th} Bernstein width of C, $b_n(C, X)$, is

$$b_n(C, X) := \sup_{X_{n+1}} \sup\{\lambda \geq 0 : X_{n+1} \cap (\lambda B_X) \subset C\},$$

where B_X is the closed unit ball in X and the outer supremum is taken over all subspaces X_{n+1} of X such that dim $X_{n+1} = n+1$. The linear width of C, $\delta_n(C, X)$, is

$$\delta_n(C, X) := \inf_{P_n} \sup_{x \in C} \|x - P_n x\|,$$

where the infimum is taken over all bounded linear maps $P : X \to X$ with rank at most n.

It can be shown that for all $n \in \mathbb{N}$, $b_{n+1} \leq b_n$, $\delta_{n+1} \leq \delta_n$ and $b_n \leq \delta_n$. For this and more information about Bernstein and linear widths, see [**18**].

In this paper we study the operator T given by (1.1) as a map from $L_p(I)$ to $L_q(I)$, with $1 < p \leq q < \infty$, and we are interested in the Bernstein widths $b_n(TB, L_q(I))$, where $TB = \{Tf : \|f\|_{p,I} \leq 1\}$. For every $n \in \mathbb{N}$ we have

$$b_n(TB, L_q(I)) \leq \delta_n(TB, L_q(I)) = a_{n+1}(T).$$

Here $a_n(T)$ is the n^{th} approximation number of T, given by

$$a_n(T) = \inf_{P_n} \sup_{\|f\|_{p,I} \leq 1} \|Tf - P_n f\|,$$

where the infimum is taken over all bounded linear operators from $L_p(I)$ to $L_q(I)$ of rank less than n. More details of approximation numbers will be found in [**5**] and [**6**]. From Definition 2.2, Section V in [**18**] we have

$$b_n(T(B), L_q(I)) = \sup_{X_{n+1}} \inf_{Tf \in X_{n+1} \setminus \{0\}} \|Tf\|_{q,I} / \|f\|_{p,I}, \tag{2.8}$$

where the supremum is taken over all subspaces X_{n+1} of $T(L_p(I))$ with dimension $n+1$. Since u and v are positive functions, (2.8) can be expressed as

$$b_n(T(B), L_q(I)) = \sup_{X_{n+1}} \inf_{\alpha \in \mathbb{R}^{n+1} \setminus \{0\}} \frac{\left\|T\left(\sum_{i=1}^{n+1} \alpha_i f_i\right)\right\|_{q,I}}{\left\|\sum_{i=1}^{n+1} \alpha_i f_i\right\|_{p,I}}, \tag{2.9}$$

where the supremum is taken over all $(n+1)-$dimensional subspaces $X_{n+1} =$ span $\{f_1, ..., f_{n+1}\} \subset L_p(I)$.

Now let $W_p^1(I)$ be the Sobolev space of all functions in $L_p(I)$ with first-order distributional derivatives also in $L_p(I)$. It is a familiar fact that the elements of $W_p^1(I)$ are absolutely continuous on I (more precisely, there is a representative in each equivalence class that is absolutely continuous), and so it makes sense to speak of the values of elements of this space at the endpoints of I. Let

$$W^1_{p,a}(I) = \{f \in W^1_p(I) : f(a) = 0\}, BW^1_{p,a}(I) = \{f \in W^1_{p,a}(I) : \|f'\|_{p,I} \leq 1\}.$$

We shall need the following result from [**2**].

Theorem 2.2. *Let $1 \leq p \leq q \leq \infty$. Then for each $n \in \mathbb{N}$,*

$$b_n \left(BW^1_{p,a}(I), L_q(I)\right) = \widetilde{\lambda}_n^{-1/q}(\alpha),$$

where $\widetilde{\lambda}_n(\alpha)$ is the n^{th} eigenvalue of problem (2.4) under condition (2.7), and α is so chosen that for the corresponding eigenfunction $\widetilde{u}_{n,\alpha}$ we have $\left\|\widetilde{w}'_{n,\alpha}\right\|_p = 1$.

It is clear that $BW^1_{p,a}(I) = T(B)$, where T is the special case of (1.1) with $u = v = 1$, so that $(Tf)(x) = \int_a^x f(t)dt$. Together with Lemma 2.2 this enables us to make the following observation.

Remark 2.1. Let $1 \leq p \leq q \leq \infty$ and suppose that T is given by $(Tf)(x) = \int_a^x f(t)dt$. Then

$$b_n(T(B), L_q(I)) = \frac{b-a}{2(n-1/2)\pi_{pq}} \left(\frac{p' q^{q-1}}{|\alpha|^{p-q}}\right)^{1/q},$$

where α is chosen so that

$$\left\| \frac{\alpha(b-a)}{(n-1/2)\pi_{pq}} \left(\sin_{pq}\left(\frac{(n-1/2)\pi_{pq}}{b-a}\cdot\right)\right)' \right\|_p = 1.$$

3. *Technical Lemmas*

Here we introduce various techniques that will be used to establish the main theorem. We suppose throughout this section that $u \in L_{p'}(I)$ and $v \in L_q(I)$, and remark that these assumptions are sufficient to ensure the compactness of T. We begin with an elementary lemma that is a simple consequence of Hölder's inequality.

Lemma 3.1. *Let $1 < p \leq q < \infty$ and $n \in \mathbb{N}$. Then*

$$\inf_{\alpha \in \mathbb{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = n^{1/q-1/p},$$

and the infimum is attained when $|\alpha_i| = 1$, $i = 1, ..., n$.

Definition 3.1. Let $J = [c,d] \subset I$. Then

$$C_{v,u,0}(J) := C_0(J) := \sup\left\{\frac{\|Tf\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J)\backslash\{0\}, (Tf)(c) = (Tf)(d) = 0\right\}$$

and

$$C_{v,u,+}(J) := C_+(J) := \sup\left\{\frac{\|Tf\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J)\backslash\{0\}, (Tf)(c) = 0\right\},$$

where T is defined in (1.1).

From this definition we immediately have

Lemma 3.2. *Let I_1 and I_2 be intervals with $I_1 \subset I_2 \subset I$. Then*

$$C_0(I_1) \le C_0(I_2),\ C_+(I_1) \le C_+(I_2) \text{ and } C_0(I_1) \le C_+(I_1).$$

A characterisation of $C_0(J)$ and $C_+(J)$ is given in the next lemma.

Lemma 3.3. *Let $J = [c,d] \subset I$. Then*

$$C_0(J) = \|g_1\|_{q,J} = \lambda_1^{-q},$$

where

$$(g_1,\lambda_1) \in SP_0(T,p,q) \text{ on } J,\ g_1(c) = g_1(d) = 0;$$

and

$$C_+(J) = \|g_0\|_{q,J} = \lambda_0^{-q},$$

where

$$(g_0,\lambda_0) \in SP_0(T,p,q) \text{ on } J.$$

Proof. Since T is a compact map from $L_p(J)$ to $L_q(J)$, there exist $h_0, h_1 \in L_p(J)$ such that
(a) $C_0(J) = \|Th_1\|_{q,J}\,, \|h_1\|_{p,J} = 1$ and $(Th_1)(c) = (Th_1)(d) = 0$;
(b) $C_+(J) = \|Th_0\|_{q,J}\,, \|h_0\|_{p,J} = 1$ and $(Th_0)(c) = 0$.
Put

$$G(f) = \|Tf\|_{q,J} \,/\, \|f\|_{p,J}\,,\ f \ne 0.$$

Then $G'(f) = 0$ if, and only if, $f \in SP(T,p,q)$ on J. From (a) and (b) it follows that $G'(h_0) = G'(h_1) = 0$, and the result is now clear. □

Next we give a monotonicity result.

Lemma 3.4. *Let I_1, I_2 be intervals contained in I, with $I_1 \subsetneq I_2$ and $|I_2 \backslash I_1| > 0$. Then*
(a) $C_0(I_1) < C_0(I_2)$
and
(b) $C_+(I_1) < C_+(I_2)$.

Proof. We prove (b) and consider the following cases:
(i) $I_1 = [c,d] \subset I_2 = [c,b]$, $d < b$;
(ii) $I_1 = [c,d] \subset I_2 = [a,d]$, $a < c$;
(iii) $I_1 = [c,d] \subset I_2 = [a,b]$, $a < c < d < b$.

Clearly (b) will be established if we can handle these three cases. First suppose that (i) holds. Since T is a compact map, there exists $f_1 \ge 0$ such that

$$C_+(I_1) = \|Tf_1\|_{q,I_1} \,/\, \|f_1\|_{p,I_1} > 0.$$

Define f_2 on I_2 by $f_2(x) = f_1(x)$ if $x \in I_1$, $f_2(x) = 0$ if $x \in I_2 \backslash I_1$. Then $\|f_1\|_{p,I_1} = \|f_2\|_{p,I_2}\,$, $(Tf_1)(x) = (Tf_2)(x)$ $(x \in I_1)$, $(Tf_2)(x) > 0$ $(x \in I_2 \backslash I_1)$ and

$$C_+(I_1) = \|Tf_1\|_{q,I_1} \,/\, \|f_1\|_{p,I_1} < \|Tf_2\|_{q,I_1} \,/\, \|f_2\|_{p,I_1} \le C_+(I_2).$$

For case (ii), note that there exists $f_1 > 0$, with supp $f_1 \subset I_1$, such that

$$C_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1}.$$

Since u is locally integrable, there exists $z \in \left(a, \frac{1}{2}(a+c)\right)$ such that $u(z) = \lim_{\varepsilon \to 0+} \int_z^{z+\varepsilon} u(x)dx$. Let $\delta > 0$ and define

$$f_2(x) = \delta\chi_{(z,z+\varepsilon)}(x) + f_1(x),\ x \in I_2.$$

Then for small $\delta > 0$ and $\varepsilon > 0$, there is a positive constant C_1 such that

$$\|f_2\|_{p,I_2} \le C_1\varepsilon\delta^p + \|f_1\|_{p,I_2}.$$

For Tf_2 we have, with $S(z) \approx \delta\varepsilon u(z)$,

$$(Tf_2)(x) \begin{cases} = 0, & a \le x \le z, \\ > 0, & z < x \le z+\varepsilon, \\ = S(z)v(x), & z+\varepsilon < x \le c, \\ = S(z)v(x) + (Tf_1)(x), & c < x \le d. \end{cases}$$

From this it follows that for small positive δ and ε, there is a positive constant C_2 such that

$$\|Tf_2\|_{q,I_2} = \left\{ (S(z))^q \int_{z+\varepsilon}^{c} v^q(x)dx + \int_c^d |S(z)v(x) + (Tf_1)(x)|^q\, dx \right\}^{1/q}$$

$$\ge C_2\{(\delta\varepsilon)^q + \delta\varepsilon\} + \|Tf_1\|_{q,I_1}.$$

Hence for small positive δ and ε, t

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} \ge \frac{C_2\delta\varepsilon + \|Tf_1\|_{q,I_1}}{C_1\varepsilon\delta^p + \|f_1\|_{p,I_2}},$$

which implies that there exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for $\varepsilon = \varepsilon_1$ and $0 < \delta < \delta_1$,

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} > \frac{\|Tf_1\|_{q,I_1}}{\|f_1\|_{p,I_1}}.$$

This gives the proof of (b) in case (ii). Case (iii) follows from (i) and (ii).

The proof of (a) can be carried out by the use of the techniques used in the proofs just given, and is left to the reader. □

Lemma 3.5. *Both $C_0([x,y])$ and $C_+([x,y])$ are continuous as functions of x and of y.*

Proof. Suppose that $C_0([x,y])$ is not right-continuous as a function of the right-hand endpoint. Then there exist x and y, with $x < y$, and $t > 0$, such that

$$C_0([x,y]) < t < C_0([x,y+\varepsilon]) \text{ for all small enough } \varepsilon > 0. \tag{3.1}$$

Given each small enough $\varepsilon > 0$, there is a function f_ε such that

$$C_0([x,y+\varepsilon]) = \frac{\|Tf_\varepsilon\|_q}{\|f_\varepsilon\|_p},\ \text{supp } f_\varepsilon \subset [x,y+\varepsilon],\ \text{supp } Tf_\varepsilon \subset [x,y+\varepsilon] \text{ and } \|f_\varepsilon\|_p = 1.$$

Since T is bounded, there exists $C > 0$ such that $\|Tf_\varepsilon\|_q \le C$. As T is compact, there are a sequence (ε_k) of positive numbers converging to zero and an element g of $L_q(I)$, with supp $g \subset \cap_k [x, y+\varepsilon_k] = [x,y]$, such that $Tf_{\varepsilon_k} \to g$ in $L_q(I)$. From (3.1) we see that

$$\inf \|g - Tf\|_q > 0, \tag{3.2}$$

where the infimum is taken over all f with supp $f \subset [x,y]$ and supp $(Tf) \subset [x,y]$. However, since T has closed range, there exists $h \in L_p(I)$, with $\|h\|_p = 1$ and supp $h \subset [x,y]$, such that $Th = g$. This contradiction with (3.2) establishes the right-continuity of C_0 in its dependence on the right-hand endpoint. Left continuity is proved in much the same way, as are the remaining claims of the Lemma. □

Now we introduce a function that is going to play an important rôle in our proofs.

Definition 3.2. Suppose that $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ and let $\mathcal{P}$ be the family of all partitions $P = \{a_0, a_1, ..., a_n\}$ of $[a,b]$, $a = a_1 < a_2 < ... < a_{n-1} < a_n = b$. Let

$$\begin{aligned} S(\varepsilon) : \quad = \quad & \{n \in \mathbb{N} : \text{ for some } P \in \mathcal{P} \text{ , } C_0(a_{i-1}, a_i) \le \varepsilon \ (1 \le i \le n-1), \\ & C_+(a_{n-1}, a_n) \le \varepsilon\}, \end{aligned}$$

and define

$$B(\varepsilon) = \min S(\varepsilon) \text{ if } S(\varepsilon) \neq \emptyset, \ B(\varepsilon) = \infty \text{ otherwise.} \tag{3.3}$$

As an obvious consequence of this definition we have

Lemma 3.6. *If* $0 < \varepsilon_1 < \varepsilon_2 < \|T : L_p(I) \to L_q(I)\|$ *, then* $B(\varepsilon_1) \ge B(\varepsilon_2)$.

We also have

Lemma 3.7. *Let* $0 < \varepsilon < \|T : L_p(I) \to L_q(I)\|$ *and suppose that* $B(\varepsilon) \ge 1$. *Then there is a partition* $P = \{a = a_0, a_1, ..., a_{B(\varepsilon)} = b\}$ *of* $[a,b]$ *such that* $C_0([a_{i-1}, a_i]) = \varepsilon$ *(*$1 \le i \le B(\varepsilon) - 1$), $C_+([a_{B(\varepsilon)-1}, a_{B(\varepsilon)}]) \le \varepsilon$.

Proof. This follows from Lemmas 3.4 and 3.5, together with the techniques used for the construction of $N(\varepsilon)$ in [**11**]. □

Lemma 3.8. *For all* $\varepsilon \in (0, \|T : L_p(I) \to L_q(I)\|)$, $B(\varepsilon) < \infty$.

Proof. Suppose that $B(\varepsilon) = \infty$ for some $\varepsilon > 0$. Then by Lemma 3.7, there is a strictly increasing sequence $\{a_i\}_{i=0}^{\infty}$ with $C_0([a_{i-1}, a_i]) = \varepsilon$ for all $i \in \mathbb{N}$. Let B be the closed unit ball in $L_p(I)$. Since T is compact, $T(B)$ is a compact subset of $L_q(I)$. For each $i \in \mathbb{N}$ let f_i be an extremal function from the definition of $C_0([a_{i-1}, a_i])$. Then supp $f_i \subset [a_{i-1}, a_i]$, supp $Tf_i \subset [a_{i-1}, a_i]$, $\|f_i\|_p = 1$ and $\|Tf_i\|_q = C_0([a_{i-1}, a_i]) = \varepsilon$. This gives an infinite sequence of functions $\{Tf_i\}_{i=1}^{\infty}$ with disjoint supports and L_q norms equal to ε. Hence $T(B)$ cannot be a compact subset of $L_q(I)$ and we have a contradiction. □

Lemma 3.9. *Let* $n = B(\varepsilon_0)$ *for some* $\varepsilon_0 > 0$. *Then there exist* ε_1 *and* ε_2,

$0 < \varepsilon_2 < \varepsilon_1 \leq \varepsilon_0$, *such that* $B(\varepsilon_2) = n+1$ *and* $B(\varepsilon_1) = n$; *and there is a partition* $\{a = a_0, a_1, ..., a_{B(\varepsilon_1)} = b\}$ *of* $[a,b]$ *such that* $C_0([a_{i-1}, a_i]) = \varepsilon_1$ *whenever* $1 \leq i \leq n-1$ *and* $C_+([a_{n-1}, a_n]) = \varepsilon_1$.

Proof. This is based on the continuity of $C_0([x,y])$ and $C_+([x,y])$ as functions of the endpoints x and y, together with the fact that $B(\varepsilon) < \infty$ for all $\varepsilon \in (0, \|T : L_p(I) \to L_q(I)\|)$. Suppose that whenever $0 < \varepsilon \leq \varepsilon_0$, either $B(\varepsilon) > n+1$ or $B(\varepsilon) = n$. Put $\varepsilon_3 = \inf\{\varepsilon > 0 : \varepsilon \leq \varepsilon_0, B(\varepsilon) = n\}$. In view of the continuity properties of C_0 and C_+, if $\varepsilon_3 < \varepsilon \leq \varepsilon_0$, there is a sequence $a_0 = a, a_1, ..., a_n$ such that $C_0([a_{i-1}, a_i]) = \varepsilon$ if $1 \leq i \leq n-1$, and $C_+([a_{n-1}, a_n]) < \varepsilon$. Then there is a sequence $\{b_i\}_{i=1}^{n=B(\varepsilon_3)}$ such that $C_0([b_{i-1}, b_i]) = \varepsilon$ if $1 \leq i \leq n-1$, and $C_+([b_{n-1}, b_n]) < \varepsilon$. Hence by the continuity of C_+ and C_0 there exists $\varepsilon < \varepsilon_3$ with $B(\varepsilon) = n+1$. The proof is complete. □

The final lemmas in this section deal with the properties of $C_0(I)$ and $C_+(I)$, beginning with their explicit computation when the functions u and v are constant. In these we shall use the following notation:

$$(T_{v,u}f)(x) := v(x) \int_a^x u(t) f(t) dt,$$

and as before, $C_{v,u,0}(I)$ and $C_{v,u,+}(I)$ will stand for $C_0(I)$ and $C_+(I)$ respectively for the operator $T_{v,u}$.

Lemma 3.10. *Let u and v be constant on the interval I. Then*
(i) $C_{v,u,0}(I) = uv\,|I|^{1/p'+1/q}\, C_{1,1,0}([0,1])$,
(ii) $C_{v,u,+}(I) = uv\,|I|^{1/p'+1/q}\, C_{1,1,+}([0,1])$,
(iii) $C_{v,u,+}(I) = 2C_{v,u,0}(I)$.

Proof. For (i) we observe that

$$C_{v,u,0}(I) = \sup_{\text{supp } f,\ \text{supp } T_{v,u}f \subset I} \frac{\|T_{v,u}f\|_{q,I}}{\|f\|_{p,I}} = \sup_{\text{supp } f,\ \text{supp } T_{v,u}f \subset I} \frac{\left\|v\int_a^{\cdot} uf(t)dt\right\|_{q,I}}{\|f\|_{p,I}}$$

$$= uv\,|I|^{1/p'+1/q} \sup_{\text{supp } f,\ \text{supp } T_{v,u}f \subset I} \frac{\left\|\int_a^{\cdot} f(t)dt\right\|_{q,I}}{\|f\|_{p,I}}$$

$$= uv\,|I|^{1/p'+1/q} \sup_{\text{supp } f,\ \text{supp } T_{1,1}f \subset [0,1]} \frac{\left\|\int_0^{\cdot} f(t)dt\right\|_{q,[0,1]}}{\|f\|_{p,[0,1]}}$$

$$= uv\,|I|^{1/p'+1/q}\, C_{1,1,0}([0,1]).$$

In the same way we can prove (ii). Finally, (iii) follows from (i) and (ii), together with Lemmas 2.1, 2.2 and 3.3. □

From [**4**] and [**20**] (see also [**15**] for $p = q$) we have

Lemma 3.11. *Let* $f(t) = c(Sf)'(t)$, *where* $(Sf)(t) = c sin_{pq}(\pi_{pq} t)$ *and c is an* *arbitrary non-zero constant. Then*

$$C_{1,1,0}([0,1]) = \frac{\|Sf\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} = \frac{(p')^{1/q} q^{1/p'} (p'+q)^{1/p-1/q}}{2\pi_{pq}}.$$

Now we establish the continuous dependence of $C_{v,u,0}(I)$ and $C_{v,u,+}(I)$ on u and v.

Lemma 3.12. *Let u_1, u_2 and v be positive weights on I with $u_1, u_2 \in L_{p'}(I)$ and $v \in L_q(I)$. Then*
(i) $|C_{v,u_1,0}(I) - C_{v,u_2,0}(I)| \le 2\|v\|_q \|u_1 - u_2\|_{p'}$
and
(ii) $|C_{v,u_1,+}(I) - C_{v,u_2,+}(I)| \le \|v\|_q \|u_1 - u_2\|_{p'}$.

Proof. For $i = 0, 1$ we set

$$U_i = \left\{ f : \int_a^b u_i(t) f(t) dt = 0, \|f\|_p = 1 \right\},$$
$$V_i = \left\{ f : \left| \int_a^b u_i(t) f(t) dt \right| \le \|u_2 - u_1\|_{p'}, \|f\|_p = 1 \right\}.$$

Since

$$\left| \int_a^b u_1(t) f(t) dt \right| \le \|u_2 - u_1\|_{p'} \|f\|_p + \left| \int_a^b u_2(t) f(t) dt \right|,$$

we have $U_1 \subset V_2$. Correspondingly, $U_2 \subset V_1$. Either $C_{v,u_1,0}(I) \le C_{v,u_2,0}(I)$ or $C_{v,u_1,0}(I) \ge C_{v,u_2,0}(I)$. Suppose that the first case holds. Then

$$C_{v,u_2,0}(I) = \sup_{f \in U_2} \left\| v(\cdot) \int_a^{\cdot} f(u_2 - u_1 + u_1) dt \right\|_q$$

$$\le \sup_{f \in U_2} \left\{ \|v\|_q \|u_2 - u_1\|_{p'} \|f\|_p + \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q \right\}$$

$$\le \|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1 \cup (V_1 \setminus U_1)} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q$$

$$\le 2\|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q.$$

Hence

$$C_{v,u_2,0}(I) \le 2\|v\|_q \|u_2 - u_1\|_{p'} + C_{v,u_1,0}(I).$$

The other case is handled similarly, and the proof of (i) is complete.

For (ii) the argument is simpler. In what follows all the suprema are taken over all functions f such that supp $f \subset I$ and $\|f\|_p \leq 1$. Then

$$C_{v,u_1,+}(I) = \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_1(t) dt \right\|_q$$

$$\leq \sup \left\{ \left\| v(\cdot) \int_a^{\cdot} f(t) \left| u_1(t) - u_2(t) \right| dt \right\|_q + \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q \right\}$$

$$\leq \|v\|_q \|f\|_p \|u_1 - u_2\|_{p'} + \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q$$

$$\leq \|v\|_q \|u_1 - u_2\|_{p'} + C_{v,u2,+}(I).$$

The proof is complete. □

Lemma 3.13. *Let u, v_1 and v_2 be weights on I with $u \in L_{p'}(I)$ and $v_1, v_2 \in L_q(I)$. Then*
(i) $|C_{v_2,u,0}(I) - C_{v_1,u,0}(I)| \leq \|v_1 - v_2\|_q \|u\|_{p'}$
and
(ii) $|C_{v_2,u,+}(I) - C_{v_1,u,+}(I)| \leq \|v_1 - v_2\|_q \|u\|_{p'}$.

Proof. The suprema in what follows are taken over all functions f such that supp f, supp $T_{v_1,u} f \subset I$ and $\|f\|_p \leq 1$. Note that supp $T_{v_1,u} f =$ supp $T_{v_2,u} f$. Then

$$C_{v_1,u,0}(I) = \sup \left\| v_1(\cdot) \int_a^{\cdot} f(t) u(t) dt \right\|_q$$

$$\leq \sup \left\{ \left\| (v_1 - v_2) \int_a^{\cdot} f(t) u(t) dt \right\|_q + \left\| v_2 \int_a^{\cdot} f(t) u(t) dt \right\|_q \right\}$$

$$\leq \sup \left\{ \|v_1 - v_2\|_q \|f\|_p \|u\|_{p'} + \left\| v_2 \int_a^{\cdot} f(t) u(t) dt \right\|_q \right\}$$

$$\leq \|v_1 - v_2\|_q \|u\|_{p'} + \sup \left\| v_2 \int_a^{\cdot} f(t) u(t) dt \right\|_q$$

$$\leq \|v_1 - v_2\|_q \|u\|_{p'} + C_{v_2,u,0}(I).$$

The rest is now clear. □

4. *The main theorem*

First we clarify the relation between $B(\varepsilon)$ and ε. As in the previous section we suppose that $u \in L_{p'}(I)$ and $v \in L_q(I)$.

Lemma 4.1. *Let $r = 1/q + 1/p'$. Then*

$$\lim_{\varepsilon \to 0+} \varepsilon B(\varepsilon)^{1/r} = C_{1,1,0}([0,1]) \left(\int_I (uv)^r dt \right)^{1/r}.$$

Proof. Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\|u_\beta - u\|_{p',I} \le \beta$, $\|v_\beta - v\|_{q,I} \le \beta$ and

$$\left| \int_I (uv)^r dt - \int_I (u_\beta v_\beta)^r dt \right| \le \beta.$$

Let $N(\beta)$ be the number of steps in the functions u_β, v_β and let $\varepsilon > 0$ be so chosen that $B(\varepsilon) \gg N(\beta)$. Let $\{J_i\}_{i=1}^{N(\beta)}$ be the set of all intervals on which u_β and v_β are constant, let $\{a_i\}_{i=1}^{N(\beta)}$ be the sequence from Lemma 3.7 and put $I_i = [a_{i-1}, a_i]$ for $i = 1, ..., B(\varepsilon)$. Plainly

$$I = \cup_{i=1}^{N(\beta)} J_i = \cup_{i=1}^{B(\varepsilon)} I_i.$$

Now define sets B, B_1 and B_2 by

$$B = \{1, ..., B(\varepsilon)\} = B_1 \cup B_2,$$

where

$$B_1 := \{i \in B : I_i \subset J_j \text{ for some } j, 1 \le j \le N(\beta)\}, \; B_2 = B \backslash B_1.$$

Put

$$I_{B_1} = \cup_{i \in B_1} I_i, \; I_{B_2} = \cup_{i \in B_2}.$$

Then for I_i $(i \in B_1 \backslash \{B(\varepsilon)\})$ we have, using Lemmas 3.10, 3.12 and 3.13,

$$\left| C_{v,u,0}(I_i) - u_\beta v_\beta |I_i|^{1/p'+1/q} C_{1,1,0}([0,1]) \right| \le 2 \|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i}$$

$$+ \|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i}.$$

Thus for $i \in B_1$, $i \ne B(\varepsilon)$, we have

$$\varepsilon^r = C_{v,u,0}(I_i)^r$$
$$\ge \left\{ C_{1,1,0}([0,1]) u_\beta v_\beta |I_i|^{1/p'+1/q} - 2 \|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i} - \|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i} \right\}^r$$

and hence, with the understanding that the summations are over all $i \in B_1 \backslash \{B(\varepsilon)\}$,

$$\{(\#B_1 - 1)\varepsilon^r\}^{1/r} = \left(\sum_{i \in B_1 \backslash \{B(\varepsilon)\}} C_{v,u,0}(I_i)^r \right)^{1/r}$$

$$\geq \left\{ \sum \left(u_\beta v_\beta \left| I_i \right|^{1/p' + 1/q} \right)^r \right\}^{1/r} C_{1,1,0}([0,1])$$

$$- 2 \left\{ \sum \left\| u_\beta - u \right\|_{p', I_i}^r \left\| v \right\|_{q, I_i}^r \right\}^{1/r}$$

$$- \left\{ \sum \left\| u \right\|_{p', I_i}^r \left\| v_\beta - v \right\|_{q, I_i}^r \right\}^{1/r}$$

$$\geq C_0([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} - 2 \left\| u_\beta - u \right\|_{p', I} \left\| v \right\|_{q, I}$$

$$- \left\| u \right\|_{p', I} \left\| v_\beta - v \right\|_{q, I}$$

$$\geq C_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} - 3\beta.$$

Now we look at the upper bound for $\varepsilon^r \#B_1$. We have, as in the previous case,

$$\{(\#B_1 - 1)\varepsilon^r\}^{1/r} = \left(\sum\nolimits_{i \in B_1 \backslash \{B(\varepsilon)\}} C_{v,u,0}(I_i)^r \right)^{1/r}$$

$$\leq C_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} + 2\beta.$$

Thus

$$\left| (\#B_1 - 1)^{1/r} \varepsilon - C_{1,1,0}([0,1]) \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^r \right)^{1/r} \right| \leq 3\beta.$$

When $\varepsilon \downarrow 0$, $I_{B_1 \backslash \{B(\varepsilon)}} \uparrow I$ and $\#B_1 / \#B \uparrow 1$. Hence

$$\lim_{\varepsilon \to 0+} \left| \varepsilon (\#B)^{1/r} - C_{1,1,0}([0,1]) \left(\int_I (u_\beta v_\beta)^r \right)^{1/r} \right| \leq 3\beta$$

and the result follows. □

Next we obtain information about the Bernstein widths for $T_{v,u}$.

Lemma 4.2. *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$\varepsilon (B(\varepsilon) - 1)^{1/q - 1/p} \leq b_{B(\varepsilon) - 2}.$$

Proof. Since T is compact, $B(\varepsilon) < \infty$. By Lemma 3.7, there are a sequence $\{a_i\}_{i=1}^{B(\varepsilon)}$ and intervals $I_i = [a_{i-1}, a_i]$ such that $C_{v,u,0}(I_i) = \varepsilon$ for $i = 1, ..., B(\varepsilon) - 1$ and $C_{v,u,0}(I_{B(\varepsilon)}) \leq \varepsilon$. For each i with $1 \leq i \leq B(\varepsilon)-1$, denote by f_i a function such that supp f_i, supp $Tf_i \subset I_i$, $\|f_i\|_{p,I} = 1$ and $C_{v,u,0}(I_i) = \|Tf_i\|_{q,I} / \|f_i\|_{p,I} = \varepsilon$. Put

$$X_{B(\varepsilon)} = \text{ span } \{f_i : i = 1, ..., B(\varepsilon) - 1\};$$

this is a $(B(\varepsilon) - 1)-$dimensional subspace of $L_p(I)$. From (2.9) we see that

$$b_{B(\varepsilon)-2} \geq \inf_{\alpha \in \mathbb{R}^{B(\varepsilon)-1}} \frac{\left\| T(\sum_{i=1}^{B(\varepsilon)-1} \alpha_i f_i) \right\|_q}{\left\| \sum_{i=1}^{B(\varepsilon)-1} \alpha_i f_i \right\|_p}.$$

Now use Lemma 3.1. □

Lemma 4.3. *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$b_{B(\varepsilon)} \leq (B(\varepsilon) - 2)^{1/q-1/p}\varepsilon.$$

Proof. Suppose that there exists $\varepsilon > 0$ such that $B(\varepsilon) > 2$ and

$$(B(\varepsilon) - 2)^{1/q-1/p}\varepsilon < b_{B(\varepsilon)}.$$

Set $B(\varepsilon) = n$. Then there exists an $(n + 1)-$dimensional subspace $X_{n+1} =$ span $\{f_1, ..., f_{n+1}\}$ of $L_p(I)$ such that $T(X_{n+1})$ is an $(n + 1)-$dimensional subspace of $L_q(I)$ and

$$b_n \geq \inf_{\alpha \in \mathbb{R}^{n+1}} \frac{\left\| T(\sum_{i=1}^{n+1} \alpha_i f_i) \right\|_q}{\left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_p} > (B(\varepsilon) - 2)^{1/q-1/p}\varepsilon.$$

Let

$$S_n := \left\{ \alpha \in \mathbb{R}^{n+1} : \left\| \sum_{i=1}^{n+1} \alpha_i f_i \right\|_p = 1 \right\}$$

and put

$$u_0(\cdot, \alpha) = \sum_{i=1}^{n+1} \alpha_i f_i(\cdot)$$

for every $\alpha \in S_n$. For each $u_0(\cdot, \alpha)$ we construct an iterative process and a sequence $\{g_j(\cdot, \alpha)\}_{j \in \mathbb{N}}$ as follows:

$$g_j(\cdot, \alpha) = Tu_j(\cdot, \alpha), \ u_{j+1}(\cdot, \alpha) = \left(\lambda_j^q(\alpha) T^*((g_j(\cdot, \alpha))_{(q)}) \right)_{(p')},$$

where the $\lambda_j(\alpha)$ are chosen so that $\|u_{j+1}(\cdot, \alpha)\|_p = 1$.

Following arguments similar to those used in the proof of Theorem 2.1 we see that as j increases, $\|g_j(\cdot, \alpha)\|_q$ is monotone non-decreasing and $g_j(\cdot, \alpha)$ converges to a spectral function of (2.2) and (2.3). Moreover, if we let

$$g(\cdot, \alpha) := \lim_{j \to \infty} g_j(\cdot, \alpha) \text{ and } \lambda^{-1/q}(\alpha) := \lim_{j \to \infty} \|g_j(\cdot, \alpha)\|_q,$$

then $(g(\cdot, \alpha), \lambda(\alpha)) \in S(T, p, q)$ for every $\alpha \in S_n$. For each $l \in \mathbb{N}$ let

$$E_l^n := \{\alpha \in S_n : Z(g_l(\cdot, \alpha)) \leq n - 1\}.$$

From the definition of T we see that $g_j(\cdot,\alpha)$ depends continuously on α, and so, by the definition of S_n, it follows that E_l^n is an open subset of S_n for each $l\in\mathbb{N}$. Then $F_l^n := S_n\backslash E_l^n$ is a closed subset of S_n and $F_l^n\supset F_{l+1}^n$.

Take $\varepsilon>0$ so that $B(\varepsilon)=n+1$ and with Lemma 3.9 in mind, let ε_1 be optimal in the sense that $B(\varepsilon_1)=n+1$ and $\varepsilon_1:=\inf\{\varepsilon>0: B(\varepsilon)=n\}$. Let $\{a_i\}_{i=1}^{n+1}$ be a sequence, forming a partition of I, such that

$$C_0([a_{i-1},a_i])=\varepsilon_1\ (i=1,...,n),\ C_+([a_n,a_{n+1}])=\varepsilon_1,$$

and put

$$F_l(\alpha):=(g_l(a_1,\alpha),...,g_l(a_n,\alpha))\,;$$

F_l is a continuous, odd mapping from S_n to $\mathbb{R}^n$, and by Borsuk's theorem, there exists $\alpha_l\in S_n$ such that $F_l(\alpha_l)=0$, that is, $\alpha_l\in F_l^n$. There is a subsequence $\{\alpha_{l_k}\}_{k=1}^\infty$ of $\{\alpha_l\}_{l=1}^\infty$ with limit $\widetilde{\alpha}=\lim_{k\to\infty}\alpha_{l_k}$. Then $(g(\cdot,\widetilde{\alpha}),\lambda(\widetilde{\alpha}))\in S_n(T,p,q)$, and from the construction of $g_j(\cdot,\widetilde{\alpha})$ we have (see the proof of Theorem 2.1, Definition 3.1 and Lemma 3.3)

$$\min_{\alpha\in\mathbb{R}^{n+1}}\frac{\left\|T(\sum_{i=1}^{n+1}\alpha_i f_i)\right\|_q}{\left\|\sum_{i=1}^{n+1}\alpha_i f_i\right\|_p}\le\|g_j(\cdot,\widetilde{\alpha})\|_q\le\|g(\cdot,\widetilde{\alpha})\|_q=\lambda^{-1/q}(\widetilde{\alpha}).$$

Also

$$C_0(I_i)=\frac{\|g(\cdot,\widetilde{\alpha})\|_{q,I_i}}{\|f(\cdot,\widetilde{\alpha})\|_{p,I_i}}=\varepsilon_1,\ I_i=[a_{i-1},a_i], i=1,...,n,$$

and

$$C_+(I_{n+1})=\frac{\|g(\cdot,\widetilde{\alpha})\|_{q,I_{n+1}}}{\|f(\cdot,\widetilde{\alpha})\|_{p,I_{n+1}}}=\varepsilon_1,\ I_{n+1}=[a_n,b].$$

Now let $G_{n+1}:=\text{span}\ \{\widetilde{f}_1,...,\widetilde{f}_{n+1}\}$, where $\widetilde{f}_i(\cdot):=f_i(\cdot,\widetilde{\alpha})$. Then

$$\inf_{\alpha\in\mathbb{R}^{n+1}}\frac{\left\|T(\sum_{i=1}^{n+1}\alpha_i\widetilde{f}_i)\right\|_q}{\left\|\sum_{i=1}^{n+1}\alpha_i\widetilde{f}_i\right\|_p}=\|g(\cdot,\widetilde{\alpha})\|_q=\lambda^{-1/q}.$$

It can be seen that the infimum is attained when $\left\|\alpha_i\widetilde{f}_i\right\|_{p,I_i}=\left\|\alpha_j\widetilde{f}_j\right\|_{p,I_j}$. Then it follows that

$$\|g(\cdot,\widetilde{\alpha})\|_q=\varepsilon_1 B(\varepsilon_1)^{1/p-1/q},$$

and the proof is complete. □

Theorem 4.1. *Suppose that $u\in L_{p'}(I)$ and $v\in L_q(I)$. Then the Bernstein numbers of the compact map $T:L_p(I)\to L_q(I)$ $(1\le p\le q\le\infty)$ satisfy*

$$\lim_{n\to\infty}nb_n=C_{1,1,0}([0,1])\left(\int_I(uv)^r\right)^{1/r},$$

where $r=1/q+1/p'$.

Proof. From the combination of Lemma 4.2, Lemma 4.1 and the strict monotonicity of $B(\varepsilon)$ given by Lemma 3.9 we have

$$\lim_{\varepsilon\to 0}\varepsilon[B(\varepsilon)]^{1/q+1/p'} = \lim_{\varepsilon\to 0}\varepsilon[B(\varepsilon)]^{1/q-1/p}B(\varepsilon) = \lim_{\varepsilon\to 0} b_{B(\varepsilon)}B(\varepsilon) = \lim_{n\to\infty} b_n.$$

Together with Lemma 4.1 this completes the proof. □

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