Some groups of finite homological type

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Abstract

For each $n \geq 0$ we construct a torsion-free group that satisfies K. S. Brown's FHT condition and is FP_n, but is not FP_{n+1}.

1 Introduction

While working on comparing different notions of Euler characteristic, K. S. Brown introduced a new homological finiteness condition for discrete groups [5, IX.6]. The group G is said to be of finite homological type or FHT if G has finite virtual cohomological dimension, and for every G-module M whose underlying abelian group is finitely generated, the homology groups $H_i(G; M)$ are all finitely generated. If G is FHT, then one may define a 'naïve Euler characteristic' for every finite-index subgroup H of G, as the alternating sum of the dimensions of the homology groups of H with rational coefficients.

One question that arises is the connection between FHT and the usual homological finiteness conditions FP and FP_n, which were introduced by J.-P. Serre [8]. (We shall define these conditions below.) It is easy to see that any group G of type FP is FHT, and one might conjecture that every torsion-free group that is FHT is also of type FP. The aim of this paper is to show that this is not the case. For each $n \geq 0$, we exhibit a torsion-free group G_n that is FHT and of type FP_n, but that is not of type FP_{n+1}.

Our construction is based on R. Bieri's construction of a group that is FP_n but not FP_{n+1} [2, Prop 2.14]. We also use G. Higman's group H which

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has the properties that it has no non-trivial finite quotients and that it is acyclic (i.e., has the same integral homology as the trivial group) [1, 6]. The group G_0 is just an infinite free product of copies of H, and for $n > 0$, G_n may be described as a free product of two groups of type FP, amalgamating a common subgroup isomorphic to G_{n-1}.

A construction of the groups G_n was given in the University of Southampton PhD thesis of the second named author [7].

2 Definitions

Let G be a discrete group, let $\mathbb{Z}G$ be the integral group ring of G, and let $\mathbb{Z}$ stand for the trivial $\mathbb{Z}G$-module, i.e., the module whose underlying abelian group is infinite cyclic upon which each element of G acts as the identity. Modules will be left modules unless otherwise stated.

We begin by recalling some classical homological finiteness conditions, which were introduced by J.-P. Serre [8], but see also [2, Ch I] and [5, Ch VIII]. A projective resolution P_* for $\mathbb{Z}$ over $\mathbb{Z}G$ is an exact sequence

$$\cdots \to P_i \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}G$-modules in which each P_i is projective. The group G is FP_n if there exists P_* such that P_i is finitely generated for all $i \leq n$ and is FP_∞ if there is a P_* in which P_i is finitely generated for all i. The group G is of finite cohomological dimension if there exists P_* in which some $P_n = 0$, in which case we may take $P_i = 0$ for all $i \geq n$. It can be shown that any group of finite cohomological dimension is torsion-free. The group G is FP if G is of finite cohomological dimension and of type FP_∞. If G is FP, then there exists a resolution P_* in which each P_i is finitely generated and only finitely many P_i are non-zero.

If G contains a finite-index subgroup H which is of finite cohomological dimension, G is said to be of finite virtual cohomological dimension or finite vcd. By an argument due to Serre, any group of finite vcd admits an action with finite stabilizers on a finite-dimensional contractible CW-complex [5, Theorem VIII.11.1].

Remark 1 Since there are many interesting discrete groups that are not virtually torsion-free, one might argue that the condition 'finite vcd' is an unnatural one, which should be replaced by the condition 'admits an action with finite stabilzers on a finite-dimensional contractible space' whenever possible.

Brown defines a group G to be FHT if G is of finite vcd, and for every right G-module M whose underlying abelian group is finitely-generated, each of the homology groups $H_i(G, M)$ is a finitely generated abelian group [5, IX.6]. These homology groups may be computed as the homology of the chain complex

$$M \otimes_{\mathbb{Z}G} P_*$$

for any projective resolution P_* for $\mathbb{Z}$ over $\mathbb{Z}G$. The homology groups of a chain complex of finitely generated abelian groups are themselves finitely generated. Hence each $H_i(G, M)$ is finitely generated whenever G is type FP_∞, and we see that any group G of finite vcd that is of type FP_∞ is FHT. In particular, any torsion-free group of type FP is FHT.

Remark 2 In his original papers on Euler characteristics, Brown gave a different definition of finite homological type, which we shall call FHT'. The group G is is FHT' if G is of finite vcd, and for each torsion-free finite-index subgroup $H \leq G$, the integral homology groups $H_i(H; \mathbb{Z})$ are finitely generated [3, 4]. By Shapiro's lemma, $H_i(H; \mathbb{Z}) \cong H_i(G; \mathbb{Z}[H\backslash G])$, where $\mathbb{Z}[H\backslash G]$ denotes the permutation module with basis the right cosets of H in G. Hence any group that is FHT is also FHT'. We do not know whether FHT is equivalent to FHT'.

3 Background

Here we collect together some known results that are used in our construction.

Proposition 3 *Let A be a finitely generated abelian group. There exists n such that $G = \mathrm{Aut}(A)$ is isomorphic to a subgroup of $GL_n(\mathbb{Z})$.*

Proof. Let $T(A)$ be the torsion subgroup of A, and let A' be a complement to $T(A)$, so that $A = T(A) \oplus A'$ and $A' \cong \mathbb{Z}^r$ for some $r \geq 0$. The subgroup $T(A)$ is characteristic in A and so there is a natural map

$$\phi : G \to \mathrm{Aut}(T(A)) \oplus \mathrm{Aut}(A/T(A)).$$

Let H be the subgroup of G consisting of those $f \in G$ such that $f(A') = A'$. Then H is a direct product

$$H = \mathrm{Aut}(T(A)) \oplus \mathrm{Aut}(A') \cong \mathrm{Aut}(T(A)) \oplus GL_r(\mathbb{Z}),$$

and ϕ induces an isomorphism from H to $\mathrm{Aut}(T(A)) \oplus \mathrm{Aut}(A/T(A))$. Elements $f \in \ker(\phi)$ act as the identity on $T(A)$, and for each $a \in A'$, $f(a) = a + b$ for some $b \in T(A)$. It follows that $\ker(\phi)$ is isomorphic to $\mathrm{Hom}(A', T(A)) \cong T(A)^r$, and so $\ker(\phi)$ is a finite group. Since ϕ restricted to H is an isomorphism, it follows that the index of H in G is equal to the order of $\ker(\phi)$, and so the index of H in G is finite.

The finite group $\mathrm{Aut}(T(A))$ is isomorphic to a subgroup of $GL_s(\mathbb{Z})$ for some s (for example, $s = |\mathrm{Aut}(T(A))|$ will suffice). Hence H is isomorphic to a subgroup of $GL_{r+s}(\mathbb{Z})$. Equivalently, there is a faithful H-module N whose underlying abelian group is free abelian of rank $r + s$. If the index of H in G is m, then the induced module $\mathbb{Z}G \otimes_{\mathbb{Z}H} N$ is a faithful G-module whose underlying abelian group is free abelian of rank $n = m(r + s)$. The action map for this module is an embedding of G in $GL_n(\mathbb{Z})$. ▪

Lemma 4 *Let X be a connected CW-complex, let Y be a connected subcomplex, and let $y_0 \in Y$ be a basepoint for both spaces. Let G be $\pi_1(X, y_0)$, the fundamental group of X, let $i : \pi_1(Y, y_0) \to G$ be the induced map of fundamental groups, and let H be a subgroup of G. Let $\widehat{X}$ be the covering space of X with fundamental group H and let $\widehat{Y}$ be the subspace of $\widehat{X}$ consisting of lifts of points of Y. There is a bijective correspondence between components of $\widehat{Y}$ and orbits in the coset space G/H for the action of $\pi_1(Y)$. The fundamental group of the component corresponding to the orbit of the coset gH is a conjugate of $i^{-1}(gHg^{-1})$ in $\pi_1(Y)$.*

Proof. Let $\widetilde{X}$ denote the universal cover of X, and let $\widetilde{Y}$ denote the subspace corresponding to Y. Pick $x_o \in \widetilde{X}$ a lift of y_0. Each component of $\widetilde{Y}$ contains some $g.x_0$. A loop γ in Y based at y_0 lifts to a path from $g.x_0$ to $g'.x_0$, for $g' = i([\gamma]).g$, where $[\gamma]$ denotes the element of $\pi_1(Y)$ represented by the loop γ. The points $g.x_0$ and $g'.x_0$ map to the same point of $\widehat{X}$ if and only if $gH = g'H$. Hence there is a path in $\widehat{Y}$ from the image of $g.x_0$ to the image of $g'.x_0$ if and only if the cosets gH and $g'H$ are in the same $\pi_1(Y)$-orbit, as claimed.

Each component of $\widehat{Y}$ is a covering space of Y, and so once we have chosen a basepoint we may identify its fundamental group with a subgroup of $\pi_1(Y)$. Taking different basepoints changes this subgroup by conjugation. The image of the point $g.x_0$ in $\widehat{Y}$ depends only on the coset gH. If we take as basepoint for a component of $\widehat{Y}$ the image of $g.x_0$, then a loop γ in Y lifts to a loop in $\widehat{Y}$ based at $g.x_0$ if and only if the cosets gH and $i([\gamma])gH$ are equal, or equivalently if and only if $i([\gamma]) \in gHg^{-1}$. ▪

Corollary 5 *Suppose that a group G is expressed as a free product with amalgamation, $G = H *_L K$, and that $\phi : G \to Q$ is such that $\phi : L \to Q$ is surjective. Then $\ker(\phi)$ is equal to the free product with amalgamation $H' *_{L'} K'$, where $H' = \ker(\phi) \cap H$, $L' = \ker(\phi) \cap L$ and $K' = \ker(\phi) \cap K$.*

Proof. A model for the classifying space BG can be made by joining copies of BH and BK by a cylinder $BL \times I$, where I denotes the unit interval. Take this space to be the space X in Lemma 4, and for $\widehat{X}$ take the regular cover with fundamental group $\ker(\phi)$, so that $\widehat{X}$ is the classifying space for $\ker(\phi)$. Lemma 4 can be applied in the cases $Y = BH$, $Y = BK$ and $Y = BL \times I$. In each case, it follows that $\widehat{Y}$ is connected, and the fundamental group of $\widehat{Y}$ is H', K' or L' respectively. Hence $\widehat{X}$ is built by joining a copy of BH' and a copy of BK' via a cylinder $BL' \times I$, and so $\ker(\phi) \cong H' *_{L'} K'$. ▪

Corollary 6 *Suppose that $G = H * K$, and define a homomorphism $\phi : G \to K$ as the identity homomorphism on K and the trivial map on H. The kernel of ϕ is isomorphic to a free product of copies of H indexed by the elements of K.*

Proof. Build a classifying space BG as the one-point union $BH \vee BK$, and apply Lemma 4 with $\widehat{X}$ being the regular cover corresponding to $\ker(\phi)$. In the case when $Y = BK$, we see that $\widehat{Y}$ is the universal covering space EK of BK, and in the case when $Y = BH$, we see that $\widehat{Y}$ is a disjoint union of copies of BH indexed by the elements of K. Hence $B(\ker(\phi))$ can be constructed by attaching copies of BH indexed by the elements of K to the contractible space EK. ▪

The following theorem is a special case of [2, prop. 2.13(a)].

Theorem 7 (R. Bieri) *Let $G = H *_L K$ be a free product with amalgamation, and suppose that both H and K are FP. Then for any $n \geq 1$, G is FP_n if and only if L is FP_{n-1}.*

The group H below was introduced by Higman, who proved that H has the 'group theoretic' properties given in the following theorem [6]. The proof that H has the stated 'homological properties' was given by Baumslag, Dyer and Heller in [1], where the group H played an important role in their strengthened version of the Kan-Thurston theorem.

Theorem 8 (G. Higman, G. Baumslag, E. Dyer, A. Heller) *Let H be the group defined by the presentation*

$$H = \langle\ a, b, c, d : a^b = a^2,\ b^c = b^2,\ c^d = c^2,\ d^a = d^2\ \rangle.$$

Then H is an infinite torsion-free group, the presentation 2-complex for the above presentation is a classifying space for H, and H admits no non-trivial quotient in which the images of the generators have finite order.

Corollary 9 *H as above is a non-trivial torsion-free acyclic group with no proper finite-index subgroups.*

4 The groups

Lemma 10 *Let M be a module for Higman's group H whose underlying abelian group is finitely generated. Then H acts trivially on M.*

Proof. Let $G = \mathrm{Aut}(M)$, the group of abelian group automorphisms of M. An H-module structure on M is a homomomorphism $H \to G$. By Proposition 3, G is isomorphic to a subgroup of $GL_n(\mathbb{Z})$ for some n. Thus it suffices to show that there are no non-trivial homomorphisms $\phi : H \to GL_n(\mathbb{Z})$. For each $m > 1$, let $\pi_m : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/m\mathbb{Z})$ denote the homomorphism 'reduction modulo m'. By Corollary 9, H has no proper finite-index subgroups, and so the homomorphism

$$\pi_m \circ \phi : H \to GL_n(\mathbb{Z}/m\mathbb{Z})$$

must be trivial for each $m > 1$. However, the only matrix in the kernel of all of the π_m is the identity matrix, and so ϕ must be the trivial homomorphism. ∎

Remark 11 For any group G, and any right G-module A, a left G-action on A may be defined by $g * a = ag^{-1}$. This gives a bijection between the left and right G-module structures on any fixed abelian group A.

Proposition 12 *Let* G_0 *be an infinite free product of copies of Higman's group* H*, and let* M *be a right* G_0*-module whose underlying abelian group is finitely generated. Then* G_0 *acts trivially on* M*, and*

$$H_0(G_0; M) \cong M, \qquad H_i(G_0; M) = 0 \quad \textit{for } i > 0.$$

Proof. Let M be as in the statement. By Lemma 10 and the remark following it, each copy of H inside G_0 must act trivially on M. It follows that G_0 acts trivially on M. Since H is acyclic, it follows that G_0 is also acyclic, and so the homology of G_0 with integer coefficients is isomorphic to the integral homology of the trivial group. The universal coefficient theorem allows one to compute the homology of G_0 with coefficients in any trivial module. Since each $H_i(G_0; \mathbb{Z})$ is free, the tor-term in the universal coefficient theorem vanishes, and so $H_i(G_0; M) \cong M \otimes_{\mathbb{Z}} H_i(G_0; \mathbb{Z})$ for all i, giving the result claimed above. ∎

Corollary 13 *The group* G_0 *as described above is* FHT*, is* FP_0 *and is not* FP_1*.*

Proof. There is a 2-dimensional BG_0 (consisting of the one point union of infinitely many copies of a 2-dimensional BH), so G_0 has cohomological dimension at most 2. (In fact, since G_0 is not free its cohomological dimension is exactly 2, but we do not need this fact.) By Proposition 12, the homology groups $H_i(G_0; M)$ are all finitely generated whenever M is a right G_0-module whose underlying abelian group is finitely generated. A group is FP_1 if and only if it is finitely generated, and so G_0 is not FP_1. Every group is FP_0. ∎

To construct the rest of our examples, we will start by embedding G_0 in a group of type FP. Let J_0 be the free product $H * \mathbb{Z}$, and define $\phi_0 : J_0 \to \mathbb{Z}$ by the identity map on $\mathbb{Z}$ and the trivial map from H to $\mathbb{Z}$. Applying Corollary 6, we see that $\ker(\phi_0)$ is isomorphic to a free product of infinitely many copies of the Higman group H. From now on, we shall identify G_0 with $\ker(\phi_0) \leq J_0$.

Let $\mathbb{F}_2$ denote the free group on two generators, and let $\psi : \mathbb{F}_2 \to \mathbb{Z}$ be the homomorphism that sends each of the two generators to $1 \in \mathbb{Z}$.

Now suppose that we have already defined a group J_n and a homomorphism $\phi_n : J_n \to \mathbb{Z}$. Define a new group J_{n+1} containing J_n as a direct factor, and a new homomorphism $\phi_{n+1} : J_{n+1} \to \mathbb{Z}$ extending ϕ_n by

$$J_{n+1} = J_n \times \mathbb{F}_2, \qquad \phi_{n+1}(g, h) = \phi_n(g) + \psi(h) \quad \text{for all } g \in J_n \text{ and } h \in \mathbb{F}_2.$$

For each n, we shall identify J_n with $J_n \times \{0\} \leq J_{n+1}$. For $n > 0$, define $G_n = \ker(\phi_n : J_n \to \mathbb{Z})$.

Proposition 14 *For each $n \geq 0$, G_0 is a normal subgroup of G_n, and $G_n/G_0 \cong (\mathbb{F}_2)^n$. For each $n \geq 0$, there is an isomorphism $G_{n+1} \cong J_n *_{G_n} J_n$.*

Proof. J_0 is a direct factor of J_n, and G_0 is a normal subgroup of J_0. It follows that G_0 is normal in J_n and that

$$J_n/G_0 \cong (J_0/G_0) \times \mathbb{F}_2{}^n \cong \mathbb{Z} \times \mathbb{F}_2{}^n.$$

G_0 is contained in $G_n = \ker(\phi_n)$, and so there is an induced homomorphism $\bar{\phi}_n : J_n/G_0 \to \mathbb{Z}$. Under the above isomorphism $J_n/G_0 \cong \mathbb{Z} \times \mathbb{F}_2{}^n$, the homomorphism $\bar{\phi}_n$ corresponds to the homomorphism which sends $(r, s_1, \ldots, s_n) \in \mathbb{Z} \times \mathbb{F}_2{}^n$ to $r + \psi(s_1) + \cdots + \psi(s_n)$. Since this map restricts to $\mathbb{Z} \times \{e\}^n \leq \mathbb{Z} \times \mathbb{F}_2{}^n$ as the identity map of $\mathbb{Z}$, it follows that $\ker(\bar{\phi}_n)$ is isomorphic to $\mathbb{F}_2{}^n$. Hence

$$G_n/G_0 = \ker(\bar{\phi}_n) \cong \mathbb{F}_2{}^n,$$

as claimed.

We may write $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$, and thus we may write

$$J_{n+1} = J_n \times (\mathbb{Z} * \mathbb{Z}) = (J_n \times \mathbb{Z}) *_{J_n} (J_n \times \mathbb{Z}).$$

Let ϕ' be the restriction of ϕ_{n+1} to one of the two copies of $J_n \times \mathbb{Z}$. The map ϕ' is given by the formula $\phi'(g, r) = \phi_n(g) + r$. In particular, the restriction of ϕ' to the $\mathbb{Z}$ direct factor is the identity, and it follows that $\ker(\phi')$ is isomorphic to J_n. The isomorphism between G_{n+1} and $J_n *_{G_n} J_n$ follows by applying Corollary 5 to ϕ_{n+1}. ∎

Theorem 15 *For each $n \geq 0$, the group G_n is torsion-free, is FHT and is FP_n, but is not FP_{n+1}.*

Proof. The group J_n has a finite $n+2$-dimensional classifying space, so J_n is FP. Also G_n (as a subgroup of J_n) must have finite cohomological dimension, and so must be torsion-free. (In fact the cohomological dimensions of J_n and G_n are both equal to $n+2$.) The group G_0 is FP_0 but not FP_1. The assertion that G_n is FP_n but not FP_{n+1} follows by induction, using Bieri's theorem (Theorem 7) and the description $G_{n+1} \cong J_n *_{G_n} J_n$.

It remains to check that whenever M is a right G_n-module whose underlying abelian group is finitely generated, then each $H_i(G_n; M)$ is finitely generated. For this we use the Lyndon-Hochschild-Serre (or LHS) spectral sequence for the group extension $G_0 \to G_n \to G_n/G_0$. Let M be a right G_n-module whose underlying abelian group is finitely generated. The E^2-page of the LHS-spectral sequence has

$$E^2_{i,j} = H_i(G_n/G_0; H_j(G_0; M)).$$

By Lemma 12, the subgroup G_0 acts trivially on M, and $H_0(G_0; M) = M$, $H_j(G_0; M) = 0$ for $j > 0$. Also $G_n/G_0 \cong \mathbb{F}_2{}^n$ is a group of type FP. Since the spectral sequence has $E^2_{i,j} = 0$ for $j \neq 0$, it must collapse, giving isomorphisms

$$H_i(G_n; M) \cong E^2_{i,j} \cong H_i(G_n/G_0; M) \cong H_i(\mathbb{F}_2{}^n; M).$$

Since $\mathbb{F}_2{}^n$ is of type FP, it follows that each $H_i(G_n; M)$ is finitely generated as claimed. ∎

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