Approximation numbers and Kolmogorov widths of Hardy-type operators in a non-homogeneous case

D.E.Edmunds J.Lang

June 20, 2005

Abstract

Let $I = [a, b] \subset \mathbb{R}$, let $1 < q \leq p < \infty$, let u and v be positive functions with $u \in L_{p'}(I)$, $v \in L_q(I)$ and let $T : L_p(I) \to L_q(I)$ be the Hardy-type operator given by

$$(Tf)(x) = v(x) \int_a^x f(t)u(t)dt, \ x \in I.$$

Given any $n \in \mathbf{N}$, let s_n stand for either the n-th approximation number of T or the n-th Kolmogorov width of T. We show that

$$\lim_{n\to\infty} ns_n = c_{pq} \left(\int_I (uv)^{1/r} dt \right)^r, \ r = 1/p' + 1/q,$$

where c_{pq} is an explicit constant depending only on p and q.

1 Introduction

Let u and v be real-valued measurable functions on an interval $I := [a, b] \subset \mathbf{R}$. In [5], [10], [11], [6] and [12] the Hardy-type operator T given by

$$(Tf)(x) := v(x) \int_a^x f(t)u(t)dt, \ x \in I, \tag{1.1}$$

was considered as a map from $L_p(I)$ to itself, when $1 \leq p \leq \infty$.

The main purpose of this paper is to study the properties of the Kolmogorov widths $d_n(T)$ and the approximation numbers $a_n(T)$ of T as a map from $L_p(I)$ to $L_q(I)$ when $1 < q \leq p < \infty$. These numbers are defined by (with standard notation)

$$d_n(T) = d_n = \inf_{X_n} \sup_{0<\|f\|_{p,I}\leq 1} \inf_{g\in X_n} \|Tf - g\|_{q,I} / \|f\|_{p,I} \quad (n \in \mathbf{N}),$$

where the infimum is taken over all n-dimensional subspaces X_n of $L_q(I)$, and

$$a_n(T) = a_n = \inf_{P_n} \sup_{0<\|f\|_{p,I}\le 1} \|Tf - P_n f\|_{q,I} \,/\, \|f\|_{p,I} \quad (n \in \mathbf{N}),$$

where the infimum is taken over all continuous linear operators P_n from $L_p(I)$ into $L_q(I)$ of rank at most $n-1$. Note that the n-th approximation number is identical to the $(n-1)$-th linear width of T, see Pinkus [19].

From [5]-[12] we know that under appropriate conditions on u and v the approximation numbers $a_n(T)$ of T, viewed as a map from $L_p(I)$ to $L_p(I)$, satisfy

$$\lim_{n\to\infty} n a_n(T) = \lambda_p^{-1/p} \int_I |u(t)v(t)|\, dt,$$

where λ_p is the first eigenvalue of a $p-$Laplacian eigenvalue problem on I. A connected account of such results concerning $a_n(T)$ is given in [4].

For many years the Kolmogorov widths and approximation numbers of various maps have undergone intense scrutiny: for embedding of Sobolev spaces we refer to [21], [19], [9] and [13]; and for the Hardy-type operator T, see [4], [17] and (in the special case when u=v=1) [14]. When $p \ne q$, the previously existing results for the approximation numbers of $T : L_p(I) \to L_q(I)$ are of the form (see [15] and [17])

$$c_1 \le \liminf_{n\to\infty} n^{\lambda} a_n(T) \le \limsup_{n\to\infty} n^{\lambda} a_n(T) \le c_2,$$

for some $\lambda > 0$; c_1, c_2 are positive constants independent of n but depending on p, q, u and v. The existence of $\lim_{n\to\infty} n^{\lambda} a_n(T)$ is not established in this earlier work. Here we show that this limit does exist, when $1 < q \le p < \infty$: our main result is that if u and v are positive functions with $u \in L_{p'}(I)$ and $v \in L_q(I)$, then

$$\lim_{n\to\infty} n s_n = c_{pq} \left(\int_I (uv)^r dt \right)^{1/r}, \; r = 1/p' + 1/q,$$

where c_{pq} is an explicit constant and s_n stands for either a_n or d_n. This leaves open the situation in which $p < q$. For this case, however, we obtained in [8] by rather similar techniques a corresponding formula for the asymptotic behaviour of the Bernstein widths of T.

Throughout the paper we suppose that $1 < q \le p < \infty$ and that u and v are positive functions on $I = [a,b] \subset \mathbf{R}$ with $u \in L_{p'}(I)$ and $v \in L_q(I)$. Then we have that T is a compact map from $L_p(I)$ to $L_q(I)$. The standard norm on $L_p(I)$ will be denoted by $\|\cdot\|_{p,I}$ or by $\|\cdot\|_p$ if no ambiguity is possible. By χ_s will be meant the characteristic function of a set $S \subset \mathbf{R}$; $|S|$ will denote the Lebesgue measure of S. We write $A \preceq B$ (or $A \succeq B$) if $A \le cB$ (or $cA \ge B$) for some positive constant c independent of appropriate quantities involved in the expressions A and B. By $A \approx B$ we shall mean that $A \preceq B$ and $B \preceq A$.

2 Preliminaries and technical results

We start with the definition of special generalisations of the trigonometric functions, the $\sin_{pq}$ and $\cos_{pq}$ functions (see [2]). (Note that these functions have their origin in [16] and [20])

Definition 2.1 *For $\sigma \in [0, q/2]$ we set*

$$arc\ sin_{pq}(\sigma) = \frac{q}{2} \int_0^{2\sigma/q} \frac{ds}{(1-s^q)^{1/p}}.$$

We put

$$\pi_{pq} = 2arc\ sin_{pq}(q/2) = B(1/q, 1/p'),$$

where $1/p' = 1 - 1/p$ and B denotes the Beta function. By $\sin_{pq}$ we mean the inverse of arc sin_{pq} and the extension of this inverse as a $2\pi_{pq}-$periodic function on $\mathbf{R}$.

More precisely, since arc $\sin_{pq} : [0, q/2] \to [0, \pi_{pq}/2]$ is increasing, $\sin_{pq}$ is well-defined on $[0, \pi_{pq}/2]$. We extend it to $[\pi_{pq}/2, \pi_{pq}]$ by defining $\sin_{pq} x = \sin_{pq}(\pi_{pq} - x)$ for $x \in [\pi_{pq}/2, \pi_{pq}]$, to $[-\pi_{pq}, \pi_{pq}]$ by oddness, and finally to all of $\mathbf{R}$ by $2\pi_{pq}-$periodicity. Now define $\cos_{pq}$ by

$$\cos_{pq}(x) = \frac{d}{dx} \sin_{pq} x;$$

this is an even, $2\pi_{pq}-$periodic function that is odd about $\pi_{pq}/2$.

Let $B := \{f \in L_p(I) : \|f\|_p \le 1\}$ and consider the isoperimetric problem of determining

$$\sup_{g \in T(B)} \|g\|_q . \tag{2.1}$$

This problem is related to the following non-linear integral problem:

$$g(x) = (Tf)(x) \tag{2.2}$$

and

$$(f(x))_{(p)} = \lambda(T^*((g)_q))(x), \tag{2.3}$$

where $(g)_q$ is the function with value $(g(x))_q$ at x and T^* is the map defined by $(T^* f)(x) = u(x) \int_x^b v(y) f(y) dy$. Note that when u and v are both identically equal to 1 on I, (2.2) and (2.3) can be transformed into the $p, q-$Laplacian differential equation

$$-\left((w')_{(p)}\right)' = \lambda(w)_{(q)}, \tag{2.4}$$

with the boundary condition

$$w(a) = 0. \tag{2.5}$$

A pair (g, λ) for which a function f with $\|f\|_p = 1$, satisfying (2.2) and (2.3), can be found, will be called a spectral pair. The set of all spectral pairs will be denoted by $SP(T,p,q)$. The number λ occurring in a spectral pair will be called a spectral number, and the set of all such numbers denoted by $sp(T,p,q)$; the function g corresponding to λ is called a spectral function. Let g, f and λ satisfy (2.2) and (2.3); then

$$\begin{aligned}\int |g(x)|^q dx &= \int g(g)_q dx = \int Tf(x)(g)_q dx \\ &= \int f(x)T^*(g)_q dx = \int f(x)(f)_p \lambda^{-1} \\ &= \lambda^{-1} \int |f(x)|^p dx.\end{aligned}$$

From this it follows that $\lambda^{-1} = \frac{\|g\|_q^q}{\|f\|_p^p}$ and then for $(g_1, \lambda) \in SP(T,p,q)$ we have $\lambda^{-1/q} = \|g_1\|_q$.

Given any continuous function f on I we denote by $Z(f)$ the number of distinct zeros of f on $\overset{o}{I}$, and by $P(f)$ the number of sign changes on this interval. The set of all spectral pairs (g, λ) with $Z(g) = n$ ($n \in \mathbf{N}_0$) will be denoted by $SP_n(T,p,q)$, and $sp_n(T,p,q)$ will represent the set of all corresponding numbers λ.

We denote by $SP_n^a(p,q)$ the set of all pairs (w, λ) (again called spectral pairs, w being an eigenfunction with associated eigenvalue λ) corresponding to solutions of (2.4) and (2.5) for which $Z(u) = n$. Similarly, $SP_n^{a,b}(p,q)$ will stand for the set of all spectral pairs (w, λ) corresponding to solutions of (2.4) that satisfy the Dirichlet boundary conditions

$$w(a) = w(b) = 0 \tag{2.6}$$

and have $Z(u) = n$. It is known from [1], [2] or [19] that for all $n \in \mathbf{N}$, $SP_n^{a,b}(p,q)$ consists of exactly one spectral pair (up to normalisation). Moreover, from [20] or [2] we have

Lemma 2.2 *For any $\alpha \in \mathbf{R}\backslash\{0\}$, the set of eigenvalues of problem (2.4) under the Dirichlet boundary conditions (2.6) on $I = [a,b]$ is given by*

$$\lambda_n(\alpha) := \left(\frac{2n\pi_{pq}}{b-a}\right)^q \cdot \frac{|\alpha|^{p-q}}{p'q^{q-1}} \quad (n \in \mathbf{N}),$$

with corresponding eigenfunctions

$$w_{n,\alpha}(t) := \frac{\alpha(b-a)}{n\pi_{pq}} \sin_{pq}\left(\frac{n\pi_{pq}}{b-a}t\right) \quad (t \in I).$$

It is easy to modify Lemma 2.2 so as to apply to the eigenvalue problem (2.4) with initial conditions at the left-hand endpoint a of I.

Lemma 2.3 *For any $\alpha \in \mathbf{R}\backslash\{0\}$, the set of eigenvalues of problem (2.4) under the conditions*

$$w(a) = 0, w'(a) = \alpha \tag{2.7}$$

on $I = [a, b]$ is given by

$$\widetilde{\lambda}_n(\alpha) := \left(\frac{2(n-1/2)\pi_{pq}}{b-a}\right)^q \cdot \frac{|\alpha|^{p-q}}{p' q^{q-1}} \quad (n \in \mathbf{N}),$$

with corresponding eigenfunctions

$$\widetilde{w}_{n,\alpha}(t) := \frac{\alpha(b-a)}{(n-1/2)\pi_{pq}} \sin_{pq}\left(\frac{(n-1/2)\pi_{pq}}{b-a} t\right) \quad (t \in I).$$

Next we recall that a basic property of the Kolmogorov widths and the approximation numbers is that for all $n \in \mathbf{N}$, $a_n \leq d_{n+1}$. For this and more information about Kolmogorov widths, see [18]. Now let $W_p^1(I)$ be the Sobolev space of all functions in $L_p(I)$ with first-order distributional derivatives also in $L_p(I)$. It is a familiar fact that the elements of $W_p^1(I)$ are absolutely continuous on I (more precisely, there is a representative in each equivalence class that is absolutely continuous), and so it makes sense to speak of the values of elements of this space at the endpoints of I. Let

$$W_{p,a}^1(I) = \{f \in W_p^1(I) : f(a) = 0\}, BW_{p,a}^1(I) = \{f \in W_{p,a}^1(I) : \|f'\|_{p,I} \leq 1\}$$

and denote the embedding from $W_{p,a}^1(I)$ into $L_q(I)$ by E_a. We shall need the following result from [1].

Theorem 2.4 *Let $1 \leq q \leq p \leq \infty$. Then for each $n \in \mathbf{N}$,*

$$d_n(E_a) := \inf_{X_n} \sup_{f \in BW_{p,a}^1} \inf_{g \in X_n} \|E_a f - g\|_{q,I} = \widetilde{\lambda}_n^{-1/q}(\alpha),$$

where the outer infimum is taken over all n-dimensional subspaces X_n of $L^q(I)$, $\widetilde{\lambda}_n(\alpha)$ is the n^{th} eigenvalue of problem (2.4) under condition (2.7), and α is so chosen that for the corresponding eigenfunction $\widetilde{u}_{n,\alpha}$ we have $\left\|\widetilde{u}'_{n,\alpha}\right\|_p = 1$.

It is clear that $BW_{p,a}^1(I) = \{Tf; \|f\|_{p,I} \leq 1\}$, where T is the special case of (1.1) with $u = v = 1$, so that $(Tf)(x) = \int_a^x f(t)dt$. Together with Lemma 2.3 this enables us to make the following observation.

Remark 2.5 *Let $1 \leq q \leq p \leq \infty$ and suppose that T is given by $(Tf)(x) = \int_a^x f(t)dt$. Then*

$$d_n(E_a) = \frac{b-a}{2(n-1/2)\pi_{pq}} \left(\frac{p' q^{q-1}}{|\alpha|^{p-q}}\right)^{1/q},$$

where α is chosen so that

$$\left\| \frac{\alpha(b-a)}{(n-1/2)\pi_{pq}} \left(\sin_{pq}\left(\frac{(n-1/2)\pi_{pq}}{b-a}\cdot\right)\right)' \right\|_p = 1.$$

3 Technical Lemmas

Here we introduce various techniques that will be used to establish the main theorem. Although there are similarities between these and the procedures used in [8], we give details here for the convenience of the reader. We suppose throughout this section that $u \in L_{p'}(I)$ and $v \in L_q(I)$: these assumptions are sufficient to ensure the compactness of T. We begin with an elementary lemma that is a simple consequence of Hölder's inequality.

Lemma 3.1 *Let $1 < q \leq p < \infty$ and $n \in \mathbf{N}$. Then*

$$\sup_{\alpha \in \mathbf{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = n^{1/q-1/p},$$

and the supremum is attained when $|\alpha_i| = 1$, $i = 1, ..., n$; and

$$\inf_{\alpha \in \mathbf{R}^n} \frac{\left(\sum_{i=1}^n |\alpha_i|^q\right)^{1/q}}{\left(\sum_{i=1}^n |\alpha_i|^p\right)^{1/p}} = 1,$$

where the infimum is attained when $|\alpha_i| = 1$ for only one i and $\alpha_j = 0$ for each $j \neq i$.

Definition 3.2 *Let $J = [c, d] \subset I$ and $x \in I$. Then*

$$T_{x,J} f(.) := v(.)\chi_J(.) \int_x^{.} f(t)u(t)\chi_J(t)dt,$$

$$A_{v,u}(J) := A(J) = \inf_{x \in J} \|T_{x,J}|L_p(J) \to L_q(J)\|,$$

$$C_{v,u,+}(J) := C_+(J) := \sup\left\{ \frac{\|T_{c,J}f\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J)\backslash\{0\} \right\}.$$

$$C_{v,u,0}(J) := C_0(J) := \sup\left\{ \frac{\|Tf\|_{q,J}}{\|f\|_{p,J}} : f \in L_p(J)\backslash\{0\}, (Tf)(c) = (Tf)(d) = 0 \right\}.$$

From this definition we have (see section 2.4.2 in [4] or [11] for details of similar arguments)

Lemma 3.3 *Let I_1 and I_2 be intervals with $I_1 \subset I_2 \subset I$. Then*

$$A(I_1) \leq A(I_2), C_+(I_1) \leq C_+(I_2).$$

and

$$C_0(I_1) \leq C_0(I_2), C_0(I_1) \leq C_+(I_1).$$

The quantities $A(J)$, $C_0(J)$ and $C_+(J)$ are characterised in the next lemma.

Lemma 3.4 *Let $J = [c, d] \subset I$. Then*

$$A(J) = \|T_{e,J}|L^p(J) \to L^q(J)\| \tag{3.1}$$

for some $e \in \overset{o}{J}$ and

$$A(J) = \frac{\|g_1\|_{q,J}}{\|f_1\|_{p,J}} = \lambda_1^{-1/q},$$

where

$$(g_1, \lambda_1) \in SP(T, p, q) \text{ on } J \text{ and } g_1(e) = 0;$$

and

$$C_+(J) = \frac{\|g_0\|_{q,J}}{\|f_0\|_{p,J}} = \lambda_0^{-1/q},$$

where

$$(g_0, \lambda_0) \in SP(T, p, q) \text{ on } J, \text{ and } g_0(c) = 0;$$

also

$$C_0(J) = \|g_1\|_{q,J} = \lambda_1^{-q},$$

where

$$(g_1, \lambda_1) \in SP(T, p, q) \text{ on } J, \ g_1(c) = g_1(d) = 0.$$

Proof. Since T is a compact map from $L_p(J)$ to $L_q(J)$, there exist $h_0, h_1, h_2 \in L_p(J)$ and $x \in J$ such that
(a) $A(J) = \|T_{x,J} h_1\|_{q,J}$, $\|h_1\|_{p,J} = 1$;
(b) $C_+(J) = \|T_{c,J} h_0\|_{q,J}$, $\|h_0\|_{p,J} = 1$;
(c) $C_0(J) = \|T h_0\|_{q,J}$, $\|h_0\|_{p,J} = 1$.

Put

$$G(f) = \|Tf\|_{q,J} / \|f\|_{p,J} , \ f \neq 0.$$

Then $G'(f) = 0$ if, and only if, $Tf \in SP(T, p, q)$ on J. From (b) it follows that $G'(h_0) = 0$. By a simple modification of this argument, with the help of (a), the statement concerning A follows. The rest is proved in a similar manner. ■

Next we give a monotonicity result.

Lemma 3.5 *Let I_1, I_2 be intervals contained in I, with $I_1 \subset I_2$ and $|I_2 \backslash I_1| > 0$. Then*
(a) $C_+(I_1) < C_+(I_2)$,
(b) $C_0(I_1) < C_0(I_2)$,
(c) $A(I_1) < A(I_2)$.

Proof. First we prove (a) and consider the following cases:
(i) $I_1 = [c, d] \subset I_2 = [c, b]$, $d < b$;
(ii) $I_1 = [c, d] \subset I_2 = [a, d]$, $a < c$;
(iii) $I_1 = [c, d] \subset I_2 = [a, b]$, $a < c < d < b$.

Clearly (a) will be established if we can handle these three cases. First suppose that (i) holds. Since T is a compact map, there exists $f_1 \geq 0$ such that

$$C_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} > 0.$$

Define f_2 on I_2 by $f_2(x) = f_1(x)$ if $x \in I_1$, $f_2(x) = 0$ if $x \in I_2 \backslash I_1$. Then $\|f_1\|_{p,I_1} = \|f_2\|_{p,I_2}$, $(Tf_1)(x) = (Tf_2)(x)$ $(x \in I_1)$, $(Tf_2)(x) > 0$ $(x \in I_2 \backslash I_1)$ and

$$C_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} < \|Tf_2\|_{q,I_1} / \|f_2\|_{p,I_1} \leq C_+(I_2).$$

For case (ii), note that there exists $f_1 > 0$, with supp $f_1 \subset I_1$, such that

$$C_+(I_1) = \|Tf_1\|_{q,I_1} / \|f_1\|_{p,I_1} .$$

Since u is locally integrable, there exists $z \in \left(a, \frac{1}{2}(a+c)\right)$ such that $u(z) = \lim_{\varepsilon \to 0+} \int_z^{z+\varepsilon} u(x)dx$. Let $\delta > 0$ and define

$$f_2(x) = \delta \chi_{(z,z+\varepsilon)}(x) + f_1(x),\ x \in I_2.$$

Then for small $\delta > 0$ and $\varepsilon > 0$, there is a positive constant C_1 such that

$$\|f_2\|_{p,I_2} \leq C_1 \varepsilon^{1/p} \delta + \|f_1\|_{p,I_2} .$$

For Tf_2 we have, with $S(z) \approx \delta\varepsilon u(z)$,

$$(Tf_2)(x) \begin{cases} = 0, & a \leq x \leq z, \\ > 0, & z < x \leq z+\varepsilon, \\ = S(z)v(x), & z+\varepsilon < x \leq c, \\ = S(z)v(x) + (Tf_1)(x), & c < x \leq d. \end{cases}$$

From this it follows that for small positive δ and ε, there is a positive constant C_2 such that

$$\|Tf_2\|_{q,I_2} \geq \left\{ (S(z))^q \int_{z+\varepsilon}^c v^q(x)dx + \int_c^d |S(z)v(x) + (Tf_1)(x)|^q\, dx \right\}^{1/q}$$

$$\geq C_2\{(\delta\varepsilon)^q + \delta\varepsilon\} + \|Tf_1\|_{q,I_1} .$$

Hence for small positive δ and ε,

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} \geq \frac{C_2\delta\varepsilon + \|Tf_1\|_{q,I_1}}{C_1\varepsilon^{1/p}\delta + \|f_1\|_{p,I_2}},$$

which implies that there exist $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that for $\varepsilon = \varepsilon_1$ and $0 < \delta < \delta_1$,

$$\frac{\|Tf_2\|_{q,I_2}}{\|f_2\|_{p,I_2}} > \frac{\|Tf_1\|_{q,I_1}}{\|f_1\|_{p,I_1}}.$$

This gives the proof of (a) in case (ii). Case (iii) follows from (i) and (ii).

The proof of (b) and (c) can be accomplished by modification of this argument with use of (3.1). ■

Lemma 3.6 *The functions $C_+([x,y])$, $C_0([x,y])$ and $A([x,y])$ are continuous in their dependence on x and y.*

Proof. Suppose that $C_+([x,y])$ is not right-continuous as a function of the right-hand endpoint. Then there exist x and y, with $x<y$, and $t>0$, such that

$$C_+([x,y]) < t < C_+([x,y+\varepsilon]) \text{ for all small enough } \varepsilon > 0. \tag{3.2}$$

Given each small enough $\varepsilon>0$, there is a function f_ε such that

$$C_+([x,y+\varepsilon]) = \frac{\|T_x f_\varepsilon\|_q}{\|f_\varepsilon\|_p}, \text{ supp } f_\varepsilon \subset [x,y+\varepsilon], \text{ supp } T_x f_\varepsilon \subset [x,y+\varepsilon] \text{ and } \|f_\varepsilon\|_p = 1.$$

Since T_x is bounded, there exists $C>0$ such that $\|T_x f_\varepsilon\|_q \le C$. As T_x is compact, there is a sequence (ε_k) of positive numbers converging to zero and an element g of $L_q(I)$, with supp $g \subset \cap_k [x, y+\varepsilon_k] = [x,y]$, such that $T_x f_{\varepsilon_k} \to g$ in $L_q(I)$. From (3.2) we see that

$$\inf \|g - T_x f\|_{q,[x,y]} > 0, \tag{3.3}$$

where the infimum is taken over all f with supp $f \subset [x,y]$. However, since T_x has closed range, there exists $h \in L_p(I)$, with $\|h\|_p = 1$ and supp $h \subset [x,y]$, such that $Th = g$. This contradiction with (3.3) establishes the right-continuity of C_+ in its dependence on the right-hand endpoint. Left continuity is proved in much the same way. Continuity of A and C_0 can be proved by modification of the previous arguments (for more details see the proof of Lemma 2.2 in [12].
■

After this preparation we introduce a function that will be of crucial importance in our proofs.

Definition 3.7 *Suppose that $0<\varepsilon<\|T : L_p(I) \to L_q(I)\|$ and let $\mathcal{P}$ be the family of all partitions $P = \{a_1, ..., a_n\}$ of $[a,b]$, $a = a_1 < a_2 < ... < a_{n-1} < a_n = b$. Let*

$$\begin{aligned} S(\varepsilon): \quad = \quad & \{n \in \mathbf{N} : \text{ for some } P = \{a_1,...,a_n\} \in \mathcal{P}, C_+([a_1,a_2]) \le \varepsilon, \\ & A([a_2,a_3]) \le \varepsilon, \ ... \ , A([a_{n-1},a_n]) \le \varepsilon\} \end{aligned}$$

and define

$$B(\varepsilon) = \min S(\varepsilon) \text{ if } S(\varepsilon) \neq \emptyset, \ B(\varepsilon) = \infty \text{ otherwise.} \tag{3.4}$$

Monotonicity of B is clear:

Lemma 3.8 *If $0<\varepsilon_1<\varepsilon_2<\|T : L_p(I) \to L_q(I)\|$, then $B(\varepsilon_1) \ge B(\varepsilon_2)$.*

We also have

Lemma 3.9 *Let $0<\varepsilon<\|T : L_p(I) \to L_q(I)\|$ and suppose that $B(\varepsilon) \ge 1$. Put $B(\varepsilon) = n$. Then there is a partition $P = \{a = a_1, a_2, ..., a_{B(\varepsilon)} = b\}$ of $[a,b]$ such that $C_+([a_1,a_2]) = \varepsilon$, $A([a_2,a_3]) = \varepsilon$, ..., $A([a_{n-2},a_{n-1}]) = \varepsilon$, $A([a_{n-1},a_n]) \le \varepsilon$.*

Proof. This follows from Lemmas 3.5 and 3.6, together with the techniques used for the construction of $N(\varepsilon)$ in [11]. ■

Lemma 3.10 *Let T be a compact map from $L^p(I)$ into $L^q(I)$. Then for all $\varepsilon \in (0, \|T : L_p(I) \to L_q(I)\|)$, $B(\varepsilon) < \infty$.*

Proof. This follows from the definition of compactness of T and a simple modification of the proof of Remark 2.4 of [12]. See also Lemma 3.8 of [8]. ■

Lemma 3.11 *Let $n = B(\varepsilon_0)$ for some $\varepsilon_0 > 0$. Then there exist ε_1 and ε_2, $0 < \varepsilon_2 < \varepsilon_1 \leq \varepsilon_0$, such that $B(\varepsilon_2) = n+1$ and $B(\varepsilon_1) = n$; and there is a partition $\{a = a_1, a_2, ..., a_{B(\varepsilon_1)} = b\}$ of $[a, b]$ such that the conclusion of Lemma 3.9 is satisfied with $A([a_{n-1}, a_n]) = \varepsilon_1$.*

Proof. We use the continuity of $C_+([x, y])$ and $A([x, y])$ as functions of the end-points x and y, together with the fact that $B(\varepsilon) < \infty$ for all $\varepsilon \in (0, \|T : L_p(I) \to L_q(I)\|)$. Suppose that whenever $0 < \varepsilon \leq \varepsilon_0$, either $B(\varepsilon) > n+1$ or $B(\varepsilon) = n$. Put $\varepsilon_3 = \inf\{\varepsilon > 0 : \varepsilon \leq \varepsilon_0, B(\varepsilon) = n\}$. In view of the continuity properties of A and C_+, if $\varepsilon_3 < \varepsilon \leq \varepsilon_0$, there is a sequence $a_1 = a, a_2, ..., a_n$ such that the conclusion of Lemma 3.9 is satisfied for the sequence with $C_+([a_1, a_2]) = \varepsilon$, $A([a_{i-1}, a_i]) = \varepsilon$ if $1 \leq i \leq n-1$, and $A([a_{n-1}, a_n]) \leq \varepsilon$. Then there is a sequence $\{b_i\}_{i=1}^{n=B(\varepsilon_3)}$ such that $C_+([a_1, a_2]) = \varepsilon$, $A([b_{i-1}, b_i]) = \varepsilon$ if $1 \leq i \leq n-1$, and $A([b_{n-1}, b_n]) = \varepsilon$. Hence by the continuity of C_+ and A there exists $\varepsilon < \varepsilon_3$ with $B(\varepsilon) = n+1$. The proof is complete. ■

The final lemmas in this section deal with additional properties of $A(I)$ and $C_+(I)$. In these we shall use the following notation:

$$(T_{v,u}f)(x) := v(x) \int_a^x u(t)f(t)dt,$$

and as before, $A_{v,u}(I)$, $C_{v,u,0}(I)$ and $C_{v,u,+}(I)$ will stand for $A(I)$, $C_0(I)$ and $C_+(I)$ respectively for the operator $T_{v,u}$. Of crucial importance is the next Lemma, which gives the values of these functions when u and v are constant.

Lemma 3.12 *Let u and v be constant on the interval I. Then*
(i) $A_{v,u}(I) = C_{v,u,0}(I) = uv\,|I|^{1/p'+1/q}\,C_{1,1,0}([0,1])$,
(ii) $C_{v,u,+}(I) = uv\,|I|^{1/p'+1/q}\,C_{1,1,+}([0,1])$,
(iii) $C_{v,u,+}(I) = 2A_{v,u}(I) = 2C_{v,u,0}(I)$.

Proof. For (ii) note that

$$C_{v,u,+}(I) = \sup_{\text{supp } f\subset I} \frac{\|T_{v,u}f\|_{q,I}}{\|f\|_{p,I}} = \sup_{\text{supp } f\subset I} \frac{\left\|v\int_a^{\cdot} uf(t)dt\right\|_{q,I}}{\|f\|_{p,I}}$$

$$= uv \sup_{\text{supp } f\subset I} \frac{\left\|\int_a^{\cdot} f(t)dt\right\|_{q,I}}{\|f\|_{p,I}}$$

$$= uv\,|I|^{1/p'+1/q} \sup_{\text{supp } f\subset[0,1]} \frac{\left\|\int_0^{\cdot} f(t)dt\right\|_{q,[0,1]}}{\|f\|_{p,[0,1]}}$$

$$= uv\,|I|^{1/p'+1/q}\, C_{1,1,+}([0,1]).$$

In the same way we can prove (i). Finally, (iii) follows from (i) and (ii), together with Lemmas 2.2, 2.3 and 3.4. ■

From [2] and [20] (see also [16] for $p=q$) we have

Lemma 3.13 *Let $f(t) = c(Sf)'(t)$, where $(Sf)(t) = c\sin_{pq}(\pi_{pq}t)$, $(T_0f)(t) = c\sin_{pq}(\pi_{pq}t)$ and c is an arbitrary non-zero constant. Then*

$$A_{1,1}([-1/2,1/2]) = \frac{\|T_0 f\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} = \frac{\|Sf\|_{q,[-1/2,1/2]}}{\|f\|_{p,[-1/2,1/2]}} = \frac{(p')^{1/q}q^{1/p'}(p'+q)^{1/p-1/q}}{2\pi_{pq}}$$

and

$$C_{1,1,0}([0,1]) = \frac{\|Sf\|_{q,[0,1]}}{\|f\|_{p,[0,1]}} = \frac{(p')^{1/q}q^{1/p'}(p'+q)^{1/p-1/q}}{2\pi_{pq}}.$$

Note that $A_{1,1}([0,1]) = A_{1,1}([-1/2,1/2])$ and the extremal functions for $A_{1,1}([0,1])$ can be obtained by translation of the extremal function for $A_{1,1}([-1/2,1/2])$. Now we establish the continuous dependence of $A_{v,u}(I)$, $C_{v,u,0}(I)$ and $C_{v,u,+}(I)$ on u and v.

Lemma 3.14 *Let u_1, u_2 and v be positive weights on I with $u_1, u_2 \in L_{p'}(I)$ and $v \in L_q(I)$. Then*
(i) $|A_{v,u_1}(I) - A_{v,u_2}(I)| \le \|v\|_q \|u_1 - u_2\|_{p'}$
(ii) $|C_{v,u_1,+}(I) - C_{v,u_2,+}(I)| \le \|v\|_q \|u_1 - u_2\|_{p'}$,
(iii) $|C_{v,u_1,0}(I) - C_{v,u_2,0}(I)| \le 2\|v\|_q \|u_1 - u_2\|_{p'}$

Proof. Let us start with (i). Without loss of generality we may suppose that $A_{v,u_1}(I) \geq A_{v,u_2}(I)$. Then

$$\begin{aligned}
A_{v,u_1}(I) &= \sup_{\|f\|_{p,I}\leq 1} \inf_{c\in I} \|v \left[\int_c^{\cdot} (u_1 - u_2 + u_2) f dt\right] \|_{q,I} \\
&\leq \sup_{\|f\|_{p,I}\leq 1} \inf_{c\in I} \Big[\|v \int_c^{\cdot} (u_1 - u_2) f dt\|_{q,I} \\
&\qquad\qquad + \|v \int_c^{\cdot} u_2 f dt\|_{q,I} \Big] \\
&\leq \sup_{\|f\|_{p,I}\leq 1} \inf_{c\in I} [\|v\|_{q,I} \|u_1 - u_2\|_{p',I} \\
&\qquad + \|v \int_c^{\cdot} u_2 f dt\|_{q,I} \Big] \\
&\leq \|v\|_{q,I} \|u_1 - u_2\|_{p',I} + A_{v,u_2}(I).
\end{aligned}$$

For (ii) the argument is simpler. In what follows all the suprema are taken over all functions f such that supp $f \subset I$ and $\|f\|_p \leq 1$. Then

$$\begin{aligned}
C_{v,u_1,+}(I) &= \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_1(t) dt \right\|_q \\
&\leq \sup \left\{ \left\| v(\cdot) \int_a^{\cdot} f(t) \left| u_1(t) - u_2(t) \right| dt \right\|_q + \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q \right\} \\
&\leq \sup \|v\|_q \|f\|_p \|u_1 - u_2\|_{p'} + \sup \left\| v(\cdot) \int_a^{\cdot} f(t) u_2(t) dt \right\|_q \\
&\leq \|v\|_q \|u_1 - u_2\|_{p'} + C_{v,u2,+}(I).
\end{aligned}$$

Finally we prove (iii). For $i = 0, 1$ we set

$$\begin{aligned}
U_i &= \left\{ f : \int_a^b u_i(t) f(t) dt = 0, \|f\|_p = 1 \right\}, \\
V_i &= \left\{ f : \left| \int_a^b u_i(t) f(t) dt \right| \leq \|u_2 - u_1\|_{p'}, \|f\|_p = 1 \right\}.
\end{aligned}$$

Since

$$\left| \int_a^b u_1(t) f(t) dt \right| \leq \|u_2 - u_1\|_{p'} \|f\|_p + \left| \int_a^b u_2(t) f(t) dt \right|,$$

we have $U_2 \subset V_1$. Correspondingly, $U_1 \subset V_2$. Either $C_{v,u_1,0}(I) \leq C_{v,u_2,0}(I)$ or $C_{v,u_1,0}(I) \geq C_{v,u_2,0}(I)$. Suppose that the first case holds. Then

$$\begin{aligned} C_{v,u_2,0}(I) &= \sup_{f \in U_2} \left\| v(\cdot) \int_a^{\cdot} f(u_2 - u_1 + u_1) dt \right\|_q \\ &\leq \sup_{f \in U_2} \left\{ \|v\|_q \|u_2 - u_1\|_{p'} \|f\|_p + \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q \right\} \\ &\leq \|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1 \cup (V_1 \setminus U_1)} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q \\ &\leq 2 \|v\|_q \|u_2 - u_1\|_{p'} + \sup_{f \in U_1} \left\| v(\cdot) \int_a^{\cdot} f u_1 dt \right\|_q . \end{aligned}$$

Hence

$$C_{v,u_2,0}(I) \leq 2 \|v\|_q \|u_2 - u_1\|_{p'} + C_{v,u_1,0}(I).$$

The other case is handled similarly, and the proof of (iii) is complete. ■

Lemma 3.15 *Let u, v_1 and v_2 be weights on I with $u \in L_{p'}(I)$ and $v_1, v_2 \in L_q(I)$. Then*
(i) $|A_{v_2,u}(I) - A_{v_1,u}(I)| \leq \|v_1 - v_2\|_q \|u\|_{p'}$
(ii) $|C_{v_2,u,+}(I) - C_{v_1,u,+}(I)| \leq \|v_1 - v_2\|_q \|u\|_{p'}$,
(iii) $|C_{v_2,u,0}(I) - C_{v_1,u,0}(I)| \leq \|v_1 - v_2\|_q \|u\|_{p'}$.

Proof. The proof of (i) and (ii) is just a simple modification of the previous proof. Let us prove (iii). The suprema in what follows are taken over all functions f such that supp f, supp $T_{v_1,u} f \subset I$ and $\|f\|_p \leq 1$. Note that supp

$T_{v_1,u}f = \text{supp } T_{v_2,u}f$. Then

$$C_{v_1,u,0}(I) = \sup \left\| v_1(\cdot) \int_a^{\cdot} f(t)u(t)dt \right\|_q$$

$$\leq \sup \left\{ \left\| (v_1 - v_2) \int_a^{\cdot} f(t)u(t)dt \right\|_q + \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q \right\}$$

$$\leq \sup \left\{ \|v_1 - v_2\|_q \|f\|_p \|u\|_{p'} + \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q \right\}$$

$$\leq \|v_1 - v_2\|_q \|u\|_{p'} + \sup \left\| v_2 \int_a^{\cdot} f(t)u(t)dt \right\|_q$$

$$\leq \|v_1 - v_2\|_q \|u\|_{p'} + C_{v_2,u,0}(I).$$

The rest is now clear. ■

4 The main theorem

Our first objective is to make more precise the relationship between $B(\varepsilon)$ and ε. As before we suppose that $u \in L_{p'}(I)$ and $v \in L_q(I)$.

Lemma 4.1 *Let $1 < q \leq p < \infty$ and $r = 1/q + 1/p'$. Then*

$$\lim_{\varepsilon \to 0+} \varepsilon B(\varepsilon)^r = A_{1,1}([0,1]) \left(\int_I (uv)^{1/r} dt \right)^r.$$

Proof. Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\|u_\beta - u\|_{p',I} \leq \beta$, $\|v_\beta - v\|_{q,I} \leq \beta$ and

$$\left| \int_I (uv)^{1/r} dt - \int_I (u_\beta v_\beta)^{1/r} dt \right| \leq \beta.$$

Let $N(\beta)$ be the number of steps in the functions u_β, v_β and let $\varepsilon > 0$ be so chosen that $B(\varepsilon) \gg N(\beta)$. Let $\{J_i\}_{i=1}^{N(\beta)}$ be the set of all intervals on which u_β and v_β are constant, let $\{a_i\}_{i=1}^{N(\beta)}$ be the sequence from Lemma 3.9 and put $I_i = [a_{i-1}, a_i]$ for $i = 2, ..., B(\varepsilon)$. Plainly

$$I = \cup_{i=1}^{N(\beta)} J_i = \cup_{i=1}^{B(\varepsilon)} I_i.$$

Now define sets B, B_1 and B_2 by

$$B = \{1, ..., B(\varepsilon)\} = B_1 \cup B_2,$$

where

$$B_1 := \{i \in B : I_i \subset J_j \text{ for some } j, 1 \le j \le N(\beta)\},\ B_2 = B \backslash B_1.$$

Put

$$I_{B_1} = \cup_{i \in B_1} I_i,\ I_{B_2} = \cup_{i \in B_2} I_i.$$

Then for I_i $(i \in B_1 \backslash \{B(\varepsilon), 2\})$ we have, using Lemmas 3.12, 3.14 and 3.15,

$$\left| A_{v,u}(I_i) - u_\beta v_\beta \left| I_i \right|^{1/p'+1/q} A_{1,1}([0,1]) \right| \le \left\| u_\beta - u \right\|_{p',I_i} \left\| v \right\|_{q,I_i}$$

$$+ \left\| u \right\|_{p',I_i} \left\| v_\beta - v \right\|_{q,I_i}$$

We recall that for $0 < s < 1$, $\|.\|_s$ is a quasi-norm which satisfies the inequality $\|f+g\|_s^s \le \|f\|_s^s + \|g\|_s^s$ and we have also $\sum |f_i + g_i|^s \le \sum |f_i|^s + \sum |g_i|^s$.

Thus with the understanding that the summations are over all $i \in B_1 \backslash \{B(\varepsilon), 2\}$, together with help from this and the Hölder inequality,

$$\begin{aligned} &\sum_i \left| A_{v,u}(I_i) - u_\beta v_\beta |I_i|^{1/p'+1/q} A_{1,1}([0,1]) \right|^{1/r} \\ &\qquad \le \sum_i \left(\|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i} + \|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i} \right)^{1/r} \\ &\qquad \le \sum_i (\|u_\beta - u\|_{p',I_i} \|v\|_{q,I_i})^{1/r} + \sum_i (\|u\|_{p',I_i} \|v_\beta - v\|_{q,I_i})^{1/r} \\ &\qquad \le \|u_\beta - u\|_{p',I}^{1/r} \|v\|_{q,I}^{1/r} + \|u\|_{p',I}^{1/r} \|v_\beta - v\|_{q,I}^{1/r}, \end{aligned}$$

and

$$\begin{aligned} &\sum_i \left| A_{v,u}(I_i) - u_\beta v_\beta |I_i|^{1/p'+1/q} A_{1,1}([0,1]) \right|^{1/r} \\ &\qquad \ge \left| \sum_i \left| A_{v,u}(I_i) \right|^{1/r} - \sum_i \left| u_\beta v_\beta |I_i|^{1/p'+1/q} A_{1,1}([0,1]) \right|^{1/r} \right| \\ &\qquad \ge \left| \left\{ (\#B_1 - 1)\varepsilon^{1/r} \right\} - (A_{1,1}([0,1]))^{1/r} \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^{1/r} \right) \right|. \end{aligned}$$

Thus

$$\left| (\#B_1 - 1)\varepsilon^{1/r} - (A_{1,1}([0,1]))^{1/r} \left(\int_{I_{B_1 \backslash \{B(\varepsilon)\}}} (u_\beta v_\beta)^{1/r} \right) \right| \le \beta^{1/r} (\|v\|_{q,I}^{1/r} + \|u\|_{p,I}^{1/r}).$$

When $\varepsilon \downarrow 0$, $I_{B_1 \backslash \{B(\varepsilon)} \uparrow I$ and $\#B_1 / \#B \uparrow 1$. Hence

$$\lim_{\varepsilon \to 0+} \left| \varepsilon (\#B)^r - A_{1,1}([0,1]) \left(\int_I (u_\beta v_\beta)^{1/r} \right)^r \right| \le 2\beta (\|v\|_{q,I} + \|u\|_{p,I})$$

and the result follows. ■

Next we establish a connection with the Kolmogorov widths for $T_{v,u}$.

Lemma 4.2 *Let $\varepsilon > 0$ be such that $B(\varepsilon) > 2$. Then*

$$a_{B(\varepsilon)}(T) \le \varepsilon B(\varepsilon)^{1/q-1/p}.$$

Proof. Since T is compact, $B(\varepsilon) < \infty$. By Lemma 3.9, there are a sequence $\{a_i\}_{i=1}^{B(\varepsilon)}$ and intervals $I_i = [a_{i-1}, a_i]$ such that $C_{v,u,+}(I_1) = \varepsilon$, $A_{v,u}(I_i) = \varepsilon$ for $i = 2, ..., B(\varepsilon) - 1$ and $A_{v,u}(I_{B(\varepsilon)}) \le \varepsilon$. For each i with $1 < i \le B(\varepsilon) - 1$, denote by $c_i \in I_i$ a point such that

$$A_{v,u}(I_i) = \sup_{f \in L^p(I_i)} \frac{\|T_{c_i,I_i} f\|_q}{\|f\|_p},$$

Put

$$P_{B(\varepsilon)} f(x) = \left[\sum_{i=2}^{B(\varepsilon)} (Tf)(c_i) \chi_{I_i}(x)\right] + 0 \cdot \chi_{I_1}(x);$$

this is a linear map $L_p \to L_q$ with rank $B(\varepsilon) - 1$.

We see that

$$\begin{aligned}
a_{B(\varepsilon)}(T) &\le \sup_{f \in L^p(I)} \frac{\|Tf - P_{B(\varepsilon)} f\|_{q,I}}{\|f\|_{p,I}} \\
&= \sup_{f \in L^p(I)} \frac{\left(\sum_{i=2}^{B(\varepsilon)} \|Tf(.) - Tf(c_i)\|_{q,I_i}^q + \|Tf(.)\|_{q,I_1}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f \in L^p(I)} \frac{\left(\sum_{i=2}^{B(\varepsilon)} \|T_{c_i} f(.)\|_{q,I_i}^q + \|Tf(.)\|_{q,I_1}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f \in L^p(I)} \frac{\varepsilon \left(\sum_{i=1}^{B(\varepsilon)} \|f\|_{p,I_i}^q\right)^{1/q}}{\|f\|_{p,I}} \\
&\le \sup_{f \in L^p(I)} \frac{\varepsilon [B(\varepsilon)]^{1/q-1/p)} \left(\sum_{i=1}^{B(\varepsilon)} \|f\|_{p,I_i}^p\right)^{1/p}}{\|f\|_{p,I}} \\
&\le \varepsilon [B(\varepsilon)]^{1/q-1/p}.
\end{aligned}$$

■

To prove the reverse inequality with the Kolmogorov numbers we first recall the Makovoz lemma (see 3.11 in [1]).

Lemma 4.3 *Let $U_n \subset \{Tf; \|f\|_{p,I} \le 1\}$ be a continuous and odd image of the unit sphere S^n in $\mathbf{R}^{n+1}$ endowed with the l_1 norm. Then*

$$d_n(T) \ge \inf\{\|x\|_{q,I} : x \in U_n\}$$

Lemma 4.4 *Let $1 < q \le p < \infty$. Then*

$$\liminf_{n\to\infty} n d_n(T) \ge C_{1,1,0}([0,1]) \left(\int_I |uv|^{1/r}\right)^r .$$

Proof. Let $n \in \mathbf{N}$ and define

$$\mathcal{O}_n = \left\{ z = (z_1, ..., z_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} |z_i| = 1 \right\}.$$

For the sake of simplicity we suppose that $I = [a, b] = [0, 1]$. We define

$$u_{n,z}(.) = \sum_{i=1}^{n+1} \chi_{I_i}(.) T f_i(.)$$

where $z = (z_1, z_2, ..., z_{n+1}) \in \mathcal{O}_n$, $I_j = [\sum_{i=0}^{j-1} |z_i|, \sum_{i=1}^{j} |z_i|]$, for $j = 1, ..., n+1$, with $z_0 = 0$ and

$$\operatorname{supp} f_i = I_i, \quad f_i(t)\,\operatorname{sign}(z_i) \ge 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,I_i} = 1, \qquad \frac{\|T f_i\|_{q,I_i}}{\|f_i\|_{p,I_i}} = C_{v,u,0}(I_i).$$

Then we can put $U_n = \{u_{n,z}(.); z \in \mathcal{O}_n\}$ and have

$$d_n(T) \ge \inf\{\|u_{n,z}(.)\|_{q,I}, u_{n,z}(.) \in U_n\} n^{-1/p} = \inf\{\|u_{n,z}(.)\|_{q,I}, z \in \mathcal{O}_n\} n^{-1/p}.$$

Let $\beta > 0$. There are step functions u_β, v_β, with the same steps, such that $\|u_\beta - u\|_{p',I} \le \beta$, $\|v_\beta - v\|_{q,I} \le \beta$ and

$$\left| \int_I (uv)^{1/r} dt - \int_I (u_\beta v_\beta)^{1/r} dt \right| \le \beta,$$

where $r = 1/q + 1/p'$.

Let $N(\beta)$ be the number of steps in the functions u_β, v_β. Denote by $\{y_i\}_{i=1}^{N(\beta)}$ the set of points of discontinuity of u_β, v_β. We define $T_\beta f(.) = v_\beta(.) \int_a^. u_\beta(t) f(t) dt$ and

$$u_{n,z}^\beta(.) = \sum_{i=1}^{n+1} \chi_{I_i}(.) T_\beta f_i(.)$$

where $z = (z_1, z_2, ..., z_{n+1}) \in \mathcal{O}_n$, $I_j = [\sum_{i=0}^{j-1} |z_i|, \sum_{i=1}^{j} |z_i|]$, for $j = 1, ..., n+1$, with $z_0 = 0$ and

$$\operatorname{supp} f_i = I_j, \qquad f_i(t)\operatorname{sign}(z_i) \ge 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,I_i} = 1, \qquad \frac{\|T_\beta f_i\|_{q,I_i}}{\|f_i\|_{p,I_i}} = C_{v_\beta,u_\beta,0}(I_i).$$

Putting $U_n^\beta = \{u_{n,z}^\beta(.); z \in \mathcal{O}_n\}$ we have

$d_n(T_\beta) \geq \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, u_{n,z}^\beta(.) \in U_n^\beta\} n^{-1/p} = \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, z \in \mathcal{O}_n\} n^{-1/p}.$

Now we modify the set U_n^β. Put

$$\tilde{u}_{n,z}^\beta(.) = \sum_i \chi_{J_i}(.) T_\beta f_i(.)$$

where $z = (z_1, z_2, ..., z_{n+1}) \in \mathcal{O}_n$, the J_i are intervals built from consecutive pairs of points from $\mathcal{P} := \{\sum_{i=1}^j |z_i|, j = 1, .., n+1\} \cup \{y_i, i = 1, .., N(\beta)\}$ and

$$\text{supp} f_i = J_j, \qquad f_i(t)\text{sign}(z_i) \geq 0 \text{ for all } t \in I,$$

$$\|f_i\|_{p,J_i} = 1, \qquad \frac{\|T_\beta f_i\|_{q,J_i}}{\|f_i\|_{p,J_i}} = C_{v_\beta,u_\beta,0}(J_i).$$

Then with $\tilde{U}_n^\beta = \{\tilde{u}_{n,z}^\beta(.); z \in \mathcal{O}_n\}$ we have

$d_n(T_\beta) \geq \inf\{\|u_{n,z}^\beta(.)\|_{q,I}, u_{n,z}^\beta(.) \in U_n^\beta\} n_\beta^{-1/p} \geq \inf\{\|\tilde{u}_{n,z}^\beta(.)\|_{q,I}, \tilde{u}_{n,z}^\beta(.) \in \tilde{U}_n^\beta\} n_\beta^{-1/p},$

where $n \leq n_\beta := \#\mathcal{P} \leq n + N(\beta)$. It follows that

$$\|u_{n,z}(.)\|_q = (\sum_{i=1}^{n+1} (C_{v,u,0}(I_i))^q)^{1/q} \geq (\sum_{j=1}^{n_\beta} (C_{v,u,0}(J_j))^q)^{1/q}$$

and with the help of Lemma 3.12:

$$\|\tilde{u}_{n,z}(.)\|_q = (\sum_{j=1}^{n_\beta} (C_{v_\beta,u_\beta,0}(J_j))^q)^{1/q} = (\sum_{j=1}^{n_\beta} (u_\beta v_\beta |J_j|^{1/p'+1/q} C_{1,1,0}([0,1]))^q)^{1/q}.$$

By Lemma 3.14 and Lemma 3.15:

$$\begin{aligned}
&(\sum_{j=1} |C_{v_\beta,u_\beta,0}(J_j) - C_{v,u,0}(J_j)|^q)^{1/q} \leq \\
&\quad \leq (\sum_{j=1} |2 \|u_\beta - u\|_{p',J_i} \|v\|_{q,J_i} + \|u\|_{p',J_i} \|v_\beta - v\|_{q,J_i} |^q)^{1/q} \\
&\quad \leq 2(\max_j \|u_\beta - u\|_{p',J_i}) \|v\|_{q,I} + \|u\|_{p',I} \beta \\
&\quad \leq \beta(2\|v\|_{q,I} + \|u\|_{p',I}).
\end{aligned}$$

From the definition of J_j and with help of the Hölder inequality we have

$$\left[\int_I |u_\beta v_\beta|^{1/r}\right]^r = \left[\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^{1/r} |J_i|\right]^r \leq \left[\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^q |J_j|^{1+q/p'}\right]^{1/q} n_\beta^{1/p'}.$$

By combining all previous observations we have:

$$
\begin{aligned}
\|u_n(z)\|_q &\geq \left(\sum_{j=1}^{n_\beta} (C_{v,u,0}(J_j))^q\right)^{1/q} \\
&\geq \left(\sum_{j=1}^{n_\beta} \left(u_\beta v_\beta\, |J_j|^{1/p'+1/q}\, C_{1,1,0}([0,1])\right)^q\right)^{1/q} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&\geq C_{1,1,0}([0,1]) \left(\sum_{j=1}^{n_\beta} |u_\beta v_\beta|^{1/r}\, |J_j|\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&= C_{1,1,0}([0,1]) \left(\int_I |u_\beta v_\beta|^{1/r}\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) \\
&\geq C_{1,1,0}([0,1]) \left(\int_I (uv)^{1/r} dt\right)^r n_\beta^{-1/p'} - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) - \beta^r C_{1,1,0}([0,1]) n_\beta^{-1/p'}.
\end{aligned}
$$

Take small $\beta > 0$ and let $n \to \infty$: then $n_\beta/n \to 1$ and

$$
\liminf_{n\to\infty} d_n(T)n \geq C_{1,1,0}([0,1]) \left(\int_I |uv|^{1/r}\right)^r - \beta(2\|v\|_{q,I} + \|u\|_{p',I}) - \beta^r C_{1,1,0}([0,1]) n_\beta^{-1/p'}.
$$

Taking $\beta \to 0$ we finish the proof.

■

Theorem 4.5 *Suppose that $u \in L_{p'}(I)$ and $v \in L_q(I)$ and $1 < q \leq p < \infty$. Let s_n denote $a_n(T)$ or $d_n(T)$. Then*

$$
\lim_{n\to\infty} n s_n = A_{1,1}([0,1]) \left(\int_I (uv)^{1/r}\right)^r,
$$

where $r = 1/q + 1/p'$.

Proof. From the combination of Lemma 4.2, Lemma 4.1, Lemma 4.4, the strict monotonicity of $B(\varepsilon)$ given by Lemma 3.11 and the fact that $a_n(T) \geq d_n(T)$, we have

$$
\begin{aligned}
A_{1,1}([0,1]) \left(\int_I (uv)^{1/r}\right)^r &= \lim_{\varepsilon\to 0} \varepsilon [B(\varepsilon)]^r = \lim_{\varepsilon\to 0} \varepsilon [B(\varepsilon)]^{1/q-1/p} B(\varepsilon) \\
&\geq \limsup_{\varepsilon\to 0} a_{B(\varepsilon)} B(\varepsilon) = \limsup_{n\to\infty} a_n n \geq \limsup_{n\to\infty} n d_n \\
&\geq \liminf_{n\to\infty} n d_n \geq A_{1,1}([0,1]) \left(\int_I (uv)^{1/r}\right)^r.
\end{aligned}
$$

The result follows. ■

Acknowledgement 4.6 *The second author is indebted to the Ohio State University for an international travel grant.*

References

[1] Buslaev, A.P. and Tikhomirov, V.M., Spectra of nonlinear differential equations and widths of Sobolev classes. Mat. Sb. **181** (1990), 1587-1606; English transl. in Math. USSR Sb. **71** (1992), 427-446.

[2] Drábek, P. and Manásevich, R., On the solution to some $p-$Laplacian nonhomogeneous eigenvalue problems, Diff. and Int. Eqns. **12** (1999), 723-740.

[3] Edmunds, D.E. and Evans, W.D., *Spectral theory and differential operators,* Oxford University Press, Oxford, 1987.

[4] Edmunds, D.E. and Evans, W.D., *Hardy operators, function spaces and embeddings,* Springer, Berlin-Heidelberg-New York, 2004.

[5] Edmunds, D.E., Evans, W.D. and Harris, D.J., Two-sided estimates of the approximation numbers of certain Volterra integral operators, Studia Math. **124** (1997), 59-80.

[6] Edmunds, D.E., Kerman, R. and Lang, J., Remainder estimates for the approximation numbers of weighted Hardy operators acting on L^2, J. d'Analyse Math. **85** (2001), 225-243.

[7] Edmunds, D.E. and Lang, J., Behaviour of the approximation numbers of a Sobolev embedding in the one-dimensional case. J. Funct. Anal. **206** (2004), 149-166.

[8] Edmunds, D.E. and Lang, J.,Bernstein widths of Hardy-type operators in a non-homogeneous case. The Ohio State Mathematical Research Institute Preprints, (05-1), 2005

[9] Edmunds, D.E. and Triebel, *H., Function Spaces, Entropy Numbers, Differential Operators,* Cambridge University Press, Cambridge, 1996.

[10] Evans, W.D., Harris, D.J. and Lang, J., Two-sided estimates of the approximation numbers of Hardy-type operators in L^∞ and L^1, Studia Math. **130** (1998), 171-192.

[11] Evans, W.D., Harris, D.J. and Lang, J., The approximation numbers of Hardy-type operators on trees, Proc Lond. Math. Soc. **83** (2000), 390-418.

[12] Lang, J., Improved estimates for the approximation numbers of Hardy-type operators, J. Approx. Theory **121** (2003), 61-70.

[13] Lang, J., The best $n-$dimensional linear approximation of the Hardy operator and the Sobolev classes on unit circle and on line, The Ohio State Mathematical Research Institute Preprints,(04-3) 2004.

[14] Lang, J.,Estimates for the Approximation Numbers and n-widths of the Weighted Hardy-Type Operators, The Ohio State Mathematical Research Institute Preprints,(04-6) 2004.

[15] Lang, J., Mendez, O. and Nekvinda, A., Asymptotic behaviour of the approximation numbers of the Hardy-type operator from L^p into L^q, JIPAM. J. Inequal. Pure Appl. Math. 5 (2004), no.1., Article 18, 36 pp.

[16] Levin, V.I., Inequality II, Recueil Mathématique **4** (46), N.2 (1938), 309-324.

[17] Lifschits, M.A. and Linde, W., Approximation and entropy numbers of Volterra operators with application to Brownian motion, AMS Memoirs **157**. no. 745 (2002).

[18] Pinkus, A., *$n-$widths in approximation theory,* Springer, 1985.

[19] Pinkus, A., $n-$widths of Sobolev spaces in L^p, Constr. Approx. **1** (1985), 15-62.

[20] Schmidt, E., Über die Ungleichung, welch die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet, Math. Annalen **117** (1940), 301-326.

[21] Triebel, H., *Interpolation theory, function spaces, differential operators,* Berlin: VEB Deutsch Verl. Wissenschaften, 1978; Amsterdam: North-Holland, 1978, 2nd (revised) edition Leipzig: Barth 1995.

Address of authors:

D.E. Edmunds
Department of Mathematics
Mantell Building
University of Sussex
Brighton BN1 9RF
UK
e-mail: `d.e.edmunds@sussex.ac.uk`

J. Lang
Department of Mathematics
The Ohio State University
100 Math Tower
231 West 18th Avenue
Columbus, OH 43210-1174
USA
e-mail: `lang@math.ohio-state.edu`