Cohomology of Coxeter groups with group ring coefficients: II

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Abstract

For any Coxeter group W, we define a filtration of $H^*(W; \mathbf{Z}W)$ by W-submodules and then compute the associated graded terms. More generally, if $\mathcal{U}$ is a CW complex on which W acts as a reflection group we compute the associated graded terms for $H_*(\mathcal{U})$ and, in the case where the action is proper and cocompact, for $H_c^*(\mathcal{U})$.

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1   Introduction

The cohomology of a group G with coefficients in a left G-module M is denoted $H^*(G; M)$. We are primarily interested in the case where M is the group ring, $\mathbf{Z}G$. Since $\mathbf{Z}G$ is a G-bimodule, $H^*(G; \mathbf{Z}G)$ inherits the structure of a right G-module. When G is discrete and acts properly and cocompactly on a contractible CW complex Ω, there is a natural topological interpretation for this cohomology group: $H^*(G; \mathbf{Z}G) \cong H_c^*(\Omega)$. where $H_c^*(\ )$ denotes finitely supported cellular cohomology. The action of G on Ω induces a right action on cohomology and the above isomorphism is one of right G-modules. For a general group G, not much is known about the G-module structure on $H^*(G; \mathbf{Z}G)$. For example, even in the above case where G acts properly and cocompactly on a contractible Ω, we don't believe it is known whether or not $H^*(G; \mathbf{Z}G)$ is always finitely generated as a G-module.

Here we deal with the case where $G = W$, a Coxeter group. In [5] the first author computed $H^*(W; \mathbf{Z}W)$ as an abelian group but not as a W-module. We partially remedy the situation here. We do not quite determine the W-module structure on $H^*(W; \mathbf{Z}W)$. Rather, we describe a certain decreasing filtration of

$H^*(W; \mathbf{Z}W)$ by W-submodules and compute the associated graded terms. In order to describe this computation, we need some notation.

Suppose (W, S) is a Coxeter system. (W is the group and S is the distinguished set of involutions which generates W.) A subset $T \subset S$ is *spherical* if the subgroup W_T, generated by T, is finite. $\mathcal{S}$ denotes the set of spherical subsets of S, partially ordered by inclusion.

Let $A := \mathbf{Z}W$ denote the group ring. Let $\{e_w\}_{w \in W}$ be its standard basis. For each $T \in \mathcal{S}$, put

$$a_T := \sum_{u \in W_T} e_u.$$

Let A^T denote the right ideal $a_T A$. If $T \subset U \in \mathcal{S}$, then $a_U \in A^T$ (cf., formula (3.2) in Section 3); hence, $A^U \subset A^T$. Let $A^{>T}$ be the right W-submodule spanned by the A^U, with $U \supsetneq T$.

For each $w \in W$, put $\mathrm{In}'(w) := \{s \in S \mid l(sw) < l(w)\}$. It is a fact that $\mathrm{In}'(w) \in \mathcal{S}$. Set $b'_w := a_{\mathrm{In}'(w)} e_w$. We will show in Lemma 3.1 that $\{b'_w\}_{w \in W}$ is also a basis for A. Define $\widehat{A}^T$ to be the $\mathbf{Z}$-submodule of A spanned by $\{b'_w \mid w \in W, \mathrm{In}'(w) = T\}$. N.B. $\widehat{A}^T$ is not a W-submodule of A; however, $\widehat{A}^T \subset A^T$ and, as we shall see in Corollary 3.3, the natural map $\widehat{A}^T \to A^T/A^{>T}$ is an isomorphism of free abelian groups.

The Coxeter group W acts properly and cocompactly as a group generated by reflections on a certain contractible complex Σ (see [3]). A fundamental domain for the W-action on Σ is a finite simplicial complex K, which can be described as follows: K is the geometric realization of the poset $\mathcal{S}$. Since $\emptyset$ is an initial element of $\mathcal{S}$, K is contractible. For each $s \in S$, define K_s to be the geometric realization of $\mathcal{S}_{\geq \{s\}}$, where $\mathcal{S}_{\geq \{s\}} := \{T \in \mathcal{S} \mid s \in T\}$. It is a subcomplex of K. For each $U \subset S$, put $K^U := \bigcup_{s \in U} K_s$. The calculation of [5] was the following:

$$H^*(W; A) = H_c^*(\Sigma) = \sum_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{A}^T.$$

There is a decreasing filtration of right W-submodules of A:

$$F_0 \supset \cdots F_p \supset \cdots,$$

where F_p is the submodule of A spanned by the A^T with $\mathrm{Card}(T) \geq p$. This induces a filtration of $H^*(W; A)$. The main result in this paper is the following. (A more precise version of which is stated as Theorem 4.5, below.)

Theorem. *In filtration degree p, the graded right W-module associated to the above filtration of $H^*(W; A)$ is isomorphic to*

$$\bigoplus_{\substack{T \in \mathcal{S} \\ \mathrm{Card}(T)=p}} H^*(K, K^{S-T}) \otimes (A^T/A^{>T}),$$

where $A^T/A^{>T}$ is the right W-module defined above.

2

Corollary. $H^*(W; A)$ *is finitely generated as a W-module.*

In view of [5, 6, 10, 12, 13], the above computation was a natural guess for the W-module structure on $H^*(W; A) = H_c^*(\Sigma)$. In particular, in [6] we calculated the "weighted L^2-cohomology" of Σ and obtained a very similar answer (provided the "weight" q lies in a certain range).

We actually proceed in somewhat more generality than indicated above. We consider an action of W as a reflection group on a CW complex $\mathcal{U}$ with strict fundamental domain X and then compute certain equivariant homology and cohomology groups of $\mathcal{U}$ with coefficients in $\mathbf{Z}W$. The equivariant (co)homology groups we are interested in have the following well known interpretations: $H_*^W(\mathcal{U}; \mathbf{Z}W) = H_*(\mathcal{U})$ and when the action is proper and cocompact, $H_W^*(\mathcal{U}; \mathbf{Z}W) = H_c^*(\mathcal{U})$. In Theorems 3.5 and 4.5 we prove formulas similar to the ones above for $H_c^*(\mathcal{U})$ and $H_*(\mathcal{U})$. In both cases the formulas involve terms of the form $H_*(X, X^U)$ or $H^*(X, X^U)$ with $U \subset S$. The difference is that in homology only the spherical subsets $U \in \mathcal{S}$ appear, while in cohomology only cospherical U appear (i.e., $S - U \in \mathcal{S}$). In the case of homology we recover the results of [4].

In the last two sections we make some comments concerning Hecke algebras and the finitely supported cellular cohomology of a building Φ associated to a BN pair. In Section 6 we point out that the results of the previous sections hold when the group ring is replaced by the Hecke algebra $A_{\mathbf{q}}$ associated to (W, S) and a multiparameter $\mathbf{q}$. In Section 7 we state the natural conjecture (Conjecture 7.4) for the computation of $H_c^*(\Phi)$ and we prove the cochain version of it as Theorem 7.2.

2   Preliminaries

Invariants and coinvariants. Given a left W-module M and a subset $T \subset S$, we have the $\mathbf{Z}$-submodule $M^T \subset M$ of the W_T-*invariants* defined by

$$M^T := M^{W_T} := \{x \in M \mid wx = x \text{ for all } w \in W_T\}. \tag{2.1}$$

More generally, for any $\mathbf{Z}$-submodule $B \subset M$, put

$$B^T := B \cap M^T.$$

For a right W-module M, the W_T-*coinvariants* are defined as a quotient $\mathbf{Z}$-module of M:

$$M_T := M_{W_T} := M \otimes_{W_T} \mathbf{Z} \cong M/MI_T, \tag{2.2}$$

where I_T is the augmentation ideal of $\mathbf{Z}W_T$ and $\mathbf{Z}$ is the trivial W_T-module. For any $\mathbf{Z}$-submodule $B \subset M$, B_T denotes the image of B in M_T.

$\mathbf{Z}(W/W_T)$ denotes the (left) permutation module defined by the W-action on W/W_T.

Lemma 2.1. *There are isomorphisms:*

(i) $\mathrm{Hom}_W(\mathbf{Z}(W/W_T), M) \cong M^T$,

(ii) $M \otimes_W \mathbf{Z}(W/W_T) \cong M_T$,

where M is a left W-module in the first case and a right W-module in the second.

Proof. (i) $\mathrm{Hom}_W(\mathbf{Z}(W/W_T), M)$ can be identified with the set of W-equivariant functions $f : W/W_T \to M$. Because of equivariance, for any such f, $f(1W_T) \in M^T$. Conversely, given any $x_0 \in M^T$, the formula $f(wW_T) = wx_0$, gives a well-defined $f : W/W_T \to M$. So, $f \to f(1W_T)$ defines an isomorphism from $\mathrm{Hom}_W(\mathbf{Z}(W/W_T), M)$ to M^T.

(ii) We have:

$$M \otimes_W \mathbf{Z}(W/W_T) = M \otimes_W \mathbf{Z}W \otimes_{W_T} \mathbf{Z} = M \otimes_{W_T} \mathbf{Z} = M_T.$$

$\square$

Remark 2.2. If M is a bimodule, then the right W-action on M gives both $\mathrm{Hom}_W(\mathbf{Z}(W/W_T), M)$ and M^T the structure of right W-modules and the isomorphism in (i) is an isomorphism of right W-modules. Similarly, (ii) is an isomorphism of left W-modules.

The basic construction. Suppose X is a CW complex. Let $\mathcal{P}(X)$ denote the set of cells in X and $X^{(i)}$ the set of i-cells. Given cells $c \in X^{(i)}$ and $c' \in X^{(i-1)}$, let $[c : c']$ denote the incidence number. Write $c' < c$ whenever the incidence number $[c : c']$ is nonzero. Extend this to a partial order on $\mathcal{P}(X)$.

A *mirror structure* on a CW complex X is a family of subcomplexes $(X_s)_{s \in S}$ indexed by some set S (which for us will always be the fundamental set of generators for the Coxeter group W). For each $T \subset S$, define subcomplexes of X:

$$X_T := \bigcap_{s \in T} X_s \quad \text{and} \quad X^T := \bigcup_{s \in T} X_s$$

and set $X_\emptyset := X$. For each cell c in X, set

$$S(c) := \{s \in S \mid c \subset X_s\}.$$

Similarly, for each $x \in X$, $S(x) := \{s \in S \mid x \in X_s\}$.

From the above data we construct another CW complex $\mathcal{U}(W, X)$, with a cellular W-action, as follows. Give W the discrete topology. Define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', x') \iff x = x'$ and $wW_{S(x)} = w'W_{S(x')}$. $\mathcal{U}(W, X)$ is the quotient space $(W \times X)/\sim$. The W-action on it is the obvious one. X is a fundamental domain for this action in the strict sense: the natural inclusion $X \hookrightarrow \mathcal{U}(W, X)$, which takes x to the class of $(1, x)$, induces a homeomorphism $X \to \mathcal{U}(W, Z)/W$.

When X is the complex K, discussed in the Introduction, $\mathcal{U}(W, K)$ is the contractible complex Σ.

Coefficient systems. A *system of coefficients* on a CW complex X is a functor $\mathcal{F}$ from $\mathcal{P}(X)$ to the category of abelian groups. Here the poset $\mathcal{P}(X)$ is regarded

as a category with $\mathrm{Hom}_{\mathcal{P}(X)}(c, d)$ equal to a singleton whenever $c \leq d$ and empty otherwise. The functor $\mathcal{F}$ will be contravariant whenever we are dealing with chains, homology or coinvariants and covariant in the case of cochains, cohomology or invariants. Define chains and cochains with coefficients in $\mathcal{F}$ by

$$C_i(X; \mathcal{F}) := \bigoplus_{c \in X^{(i)}} \mathcal{F}(c) \quad \text{and} \quad C^i(X; \mathcal{F}) := \prod_{c \in X^{(i)}} \mathcal{F}(c).$$

We regard both i-chains and i-cochains as functions f from $X^{(i)}$ to $\bigcup \mathcal{F}(c)$ such that $f(c) \in \mathcal{F}(c)$ for each $c \in X^{(i)}$. Boundary and coboundary maps are then defined by the usual formulas:

$$\partial(f)(c) := \sum [d : c] \mathcal{F}_{dc}(f(d))$$
$$\delta(f)(c) := \sum [c : d] \mathcal{F}_{dc}(f(d)),$$

where, given an i-cell c, the first sum is over all $(i+1)$-cells d which are incident to c and the second sum is over all $(i-1)$-cells d which are incident to c and where $\mathcal{F}_{dc} : f(d) \to f(c)$ is the homomorphism corresponding to $d > c$ (in the first case) or $c > d$ (in the second).

Examples 2.3. Suppose $\{X_s\}_{s \in S}$ is a mirror structure on X.

(i) (*Invariants*). Given a left W-module M, define a (covariant) system of coefficients $\mathcal{I}(M)$ on X by

$$\mathcal{I}(M)(c) := M^{S(c)}.$$

If $B \subset M$ is any $\mathbf{Z}$-submodule of M, then we have a sub-coefficient system $\mathcal{I}(B) \subset \mathcal{I}(M)$, defined by $\mathcal{I}(B)(c) := B^{S(c)}$.

(ii) (*Coinvariants*). For a right W-module M, define a (contravariant) system of coefficients $\mathcal{C}(M)$ on X by

$$\mathcal{C}(M)(c) := M_{S(c)}.$$

Similarly, for any $\mathbf{Z}$-submodule B of M, $\mathcal{C}(B)(c) := B_{S(c)}$.

The following observation is the key to our results. Suppose M is a left W-module and that we have a direct sum decomposition (of $\mathbf{Z}$-modules), $M = B \oplus C$, satisfying the following condition:

$$M^T = B^T \oplus C^T \quad \text{for all } T \subset S. \tag{2.3}$$

Then we have a direct sum decomposition of coefficient systems: $\mathcal{I}(M) = \mathcal{I}(B) \oplus \mathcal{I}(C)$. This leads to a decomposition of cochain groups: $C^i(X; \mathcal{I}(M)) = C^i(X; \mathcal{I}(B)) \oplus C^i(X; \mathcal{I}(C)$ and a decomposition in cohomology:

$$H^*(X; \mathcal{I}(M)) = H^*(X; \mathcal{I}(B)) \oplus H^*(X; \mathcal{I}(C)). \tag{2.4}$$

Similarly, suppose M is a right W-module and $M = B \oplus C$ is a $\mathbf{Z}$-module decomposition satisfying:

$$M_T = B_T \oplus C_T \quad \text{for all } T \subset S.$$

Then we get a decomposition of coefficient systems: $\mathcal{C}(M) = \mathcal{C}(B) \oplus \mathcal{C}(C)$ and a corresponding decomposition of homology groups:

$$H_*(X; \mathcal{C}(M)) = H_*(X; \mathcal{C}(B)) \oplus H_*(X; \mathcal{C}(C)). \tag{2.5}$$

Equivariant (co)homology. Given a discrete group G acting cellularly on a CW complex Ω, we will associate a certain equivariant homology and cohomology groups. Given a left G-module M, the *G-equivariant cochains on Ω with coefficients in M* are defined by

$$C_G^i(\Omega; M) := \mathrm{Hom}_G(C_i(\Omega), M).$$

Similarly, if M is a right G-module, we have the *G-equivariant chains*:

$$C_i^G(\Omega; M) := M \otimes_W C_i(\Omega),$$

where $C_i(\Omega)$ denotes the group of cellular i-chains on Ω. (Some people think that "equivariant (co)homology" refers to the (co)homology of $\Omega \times_G EG$ with local coefficients in M. However, there are other equivariant theories, for example, the one described above.)

If the G-action is free and the projection to the orbit space is a covering projection, then equivariant (co)chains on Ω are equal to the (co)chains on the orbit space with local coefficients in M. This is a useful viewpoint even when the action is not free. In general, M does not induce a locally constant coefficient system on the orbit space. Rather, it induces a coefficient system on the orbit space thought of as an "orbihedron" or "complex of groups." The theory of such coefficient systems can be found in [11]. These general coefficient systems on orbihedra are more general then the type considered above. (They correspond to "lax functors" rather than to functors.) However, as we shall see in Lemma 2.6, when $\Omega = \mathcal{U}(W, X)$, the induced coefficient system on X coincides with one of the coefficient systems described in Examples 2.3.

In the case of coefficients in the group ring, we have the following well-known interpretation of equivariant (co)homology.

Proposition 2.4. *Suppose G acts cellularly on a CW complex Ω. Then*

(i) $C_*^G(\Omega; \mathbf{Z}G) \cong C_*(\Omega)$.

(ii) *If the G-action is proper and there are only finitely many orbits of cells, then*
$$C_G^*(\Omega; \mathbf{Z}G) \cong C_c^*(\Omega).$$

Proof. (i) $C_i^G(\Omega; \mathbf{Z}G) = \mathbf{Z}G \otimes_{\mathbf{Z}G} C_i(\Omega) \cong C_i(\Omega)$.

(ii) For any G-module M, by [2, Lemma 7.4, p.208], $\mathrm{Hom}_G(M, \mathbf{Z}G) \cong \mathrm{Hom}_c(M, \mathbf{Z})$, where $\mathrm{Hom}_c(M, \mathbf{Z})$ denotes the set of $\mathbf{Z}$-module maps $f : M \to \mathbf{Z}$ such that for each $m \in M$, $f(gm) = 0$ for all but finitely many $g \in G$. Hence, $C_G^i(\Omega; \mathbf{Z}G) = \mathrm{Hom}_G(C_i(\Omega), \mathbf{Z}G) = \mathrm{Hom}_c(C_i(\Omega), \mathbf{Z}) = C_c^i(\Omega)$. $\square$

Now let $\mathcal{U} = \mathcal{U}(W, X)$. W acts properly on $\mathcal{U}$ with compact quotient if and only if X is compact and $X_U = \emptyset$ whenever $U \notin \mathcal{S}$. In view of Proposition 2.4, *when dealing with the cohomology of $\mathcal{U}$, we shall always assume that these conditions hold* (i.e., X is compact and $X_U = \emptyset$ for all $U \notin \mathcal{S}$). However, in the formulas for the homology of $\mathcal{U}$, we need no extra assumptions on X. In this special case $\Omega = \mathcal{U}$, Proposition 2.4 becomes the following.

Corollary 2.5.

(i) $C_*^W(\mathcal{U}; \mathbf{Z}W) \cong C_*(\mathcal{U})$.

(ii) $C_W^*(\mathcal{U}; \mathbf{Z}W) \cong C_c^*(\mathcal{U})$.

Lemma 2.6.

(i) *For any right W-module M, $C_*^W(\mathcal{U}; M) \cong C_*(X; \mathcal{C}(M))$.*

(ii) *For any left W-module M, $C_W^*(\mathcal{U}; M) \cong C^*(X; \mathcal{I}(M))$.*

Proof. Any orbit of cells in $\mathcal{U}$ has the form Wc for some unique cell c in X. As a W-set, this orbit is isomorphic to $W/W_{S(c)}$. Hence, using Lemma 2.1, we get

$$C_W^i(\mathcal{U}; M) \cong \bigoplus_{c \in X^{(i)}} M^{S(c)} = C^i(X; \mathcal{I}(M))$$

$$C_i^W(\mathcal{U}; M) \cong \bigoplus_{c \in X^{(i)}} M_{S(c)} = C_i(X; \mathcal{C}(M)).$$

$\square$

Remark 2.7. Combining the isomorphisms in Corollary 2.5 (ii) and Lemma 2.6 (ii), we get

$$C_c^i(\mathcal{U}) \cong C_W^i(\mathcal{U}; \mathbf{Z}W) \cong C^i(X; \mathcal{I}(\mathbf{Z}W)) \cong \bigoplus_{c \in X^{(i)}} (\mathbf{Z}W)^{S(c)}.$$

The composition of these gives an isomorphism $C_c^i(\mathcal{U}) \to \bigoplus_{c \in X^{(i)}} (\mathbf{Z}W)^{S(c)}$, such that its component corresponding to $c \in X^{(i)}$ is given by

$$f \to \sum_{w \in W} f(w^{-1}c)e_w$$

where (e_w) is the standard basis for $\mathbf{Z}W$. Similarly, the composition of the isomorphisms in Corollary 2.5 (i) and Lemma 2.6 (i) gives the obvious identification

$$C_i(\mathcal{U}) \cong \bigoplus_{c \in X^{(i)}} (\mathbf{Z}W)_{S(c)}.$$

3 Group ring coefficients

Subsets of W. For any $U \subset S$, put

$$X_U := \{w \in W \mid l(sw) > l(w) \text{ for all } s \in U\}$$
$$Y_U := \{w \in W \mid l(ws) > l(w) \text{ for all } s \in U\} = (X_U)^{-1}$$

X_U (resp., Y_U) is the set of elements in W which are $(U, \emptyset)$-*reduced* (resp., $(\emptyset, U)$-*reduced*). X_U (resp., Y_U) is a set of representatives for $W_U \backslash W$ (resp., W/W_U).

Given $w \in W$, set

$$\text{In}(w) := \{s \in S \mid l(ws) < l(w)\},$$
$$\text{In}'(w) := \{s \in S \mid l(sw) < l(w)\} = \text{In}(w^{-1})$$

$\text{In}(w)$ (resp., $\text{In}'(w)$) is the set of letters of S with which a reduced word for w can end (resp., begin). By [3, Lemma 7.12], for any $w \in W$, $\text{In}(w)$ is a spherical subset. We note that, for any $T \in \mathcal{S}$,

$$w_T X_T = \{w \in W \mid T \subset \text{In}'(w)\},$$
$$Y_{S-T} = \{w \in W \mid T \supset \text{In}(w)\},$$

where $w_T \in W_T$ is the element of longest length. Thus, $w_T X_T$ is also a set of representatives for $W_T \backslash W$.

Symmetrization and alternation. From now on, A denotes the group ring $\mathbf{Z}W$. For each spherical subset T of S, define elements a_T and h_T in A by

$$a_T := \sum_{w \in W_T} e_w \quad \text{and} \quad h_T := \sum_{w \in W_T} (-1)^{l(w)} e_w \tag{3.1}$$

called, respectively, *symmetrization* and *alternation* with respect to T. If $T \subset U \in \mathcal{S}$, define

$$c_{(U,T)} := \sum_{u \in X_T \cap W_U} e_u \quad \text{and} \quad d_{(U,T)} := \sum_{u \in Y_T \cap W_U} (-1)^{l(w)} e_u.$$

It is easily checked that

$$a_U = a_T c_{(U,T)} \quad \text{and} \quad h_U = d_{(U,T)} h_T. \tag{3.2}$$

For any subset T of S, let A^T denote the W_T-invariants in A, defined as in (2.1). Notice that A^T is 0 if $T \notin \mathcal{S}$ and is equal to the right ideal $a_T A$ if $T \in \mathcal{S}$. Similarly, for $T \in \mathcal{S}$, define H^T to be the left ideal Ah_T and to be 0 otherwise. By (3.2), $A^U \subset A^T$ and $H^U \subset H^T$ whenever $T \subset U$. Let A_T denote the W_T-coinvariants, defined as in (2.2) and let I_T denote the augmentation ideal of $\mathbf{Z}W_T$. For any $s \in S$, note that $AI_{\{s\}} = H^{\{s\}}$. Hence, $A_{\{s\}} = A/H^{\{s\}}$. More generally, for any $T \subset S$,

$$AI_T = \sum_{s \in T} H^{\{s\}} \quad \text{so,} \quad A_T = A/\sum_{s \in T} H^{\{s\}}.$$

Two bases for A. For each $w \in W$, define elements $b_w', b_w \in A$ by

$$b_w' := a_{\text{In}'(w)} e_w \quad \text{and} \quad b_w := e_w h_{\text{In}(w)}.$$

Lemma 3.1. $\{b_w' \mid w \in W\}$ *is a basis for A (as a $\mathbf{Z}$-module). More generally, for any $T \in \mathcal{S}$, $\{b_w' \mid T \subset \text{In}'(w)\}$ is a basis for A^T.*

Proof. We first show $\{b_w' \mid w \in W\}$ is a basis. The point is that the matrix which expresses the b_w' in terms of the e_w has 1's on the diagonal and is "upper triangular with respect to word length." In detail: first note that b_v' is the sum of e_v with various e_w having $l(w) < l(v)$. Suppose $\sum \beta_w b_w' = 0$ is a nontrivial linear relation. Let $v \in W$ be an element with $\beta_v \neq 0$ and $l(v)$ maximum. Since the coefficient of e_v in the linear relation must be 0, we have $\beta_v = 0$, a contradiction. Similarly, one shows, by induction on word length, that each e_v is a linear combination of b_w' with $l(w) \leq l(v)$. Hence, $\{b_w'\}$ spans A.

To prove the second sentence, we must first show that $b_w' \in A^T$ whenever $T \subset \text{In}'(w)$. If this condition holds, then, by (3.2),

$$b_w' = a_{\text{In}'(w)} e_w = a_T c_{(\text{In}'(w),T)} e_w \in A^T.$$

Note that $T \subset \text{In}'(w)$ if and only if $w \in w_T X_T$. Since, by the previous paragraph, $\{b_w' \mid w \in w_T X_T\}$ is linearly independent, it remains to show that it spans A^T. Since $w_T X_T$ is a set of coset representatives for $W_T \backslash W$, a basis for A^T is $\{a_T e_w \mid w \in w_T X_T\}$. Let $\bar{e}_w := c_{(\text{In}'(w),T)} e_w$. For $w \in w_T X_T$, the matrix which expresses $\{\bar{e}_w \mid w \in w_T X_T\}$ in terms of $\{e_w \mid w \in w_T X_T\}$ has 1's on the diagonal and is upper triangular with respect to word length. So,

$$\{a_T \bar{e}_w \mid w \in w_T X_T\} = \{b_w' \mid T \subset \text{In}'(w)\}$$

is also a basis for A^T. $\qquad\square$

Lemma 3.2. $\{b_w \mid w \in W\}$ *is a basis for A. More generally, for any subset U of S, the projection $A \to A_{S-U}$ maps $\{b_w \mid U \supset \text{In}(w)\}$ injectively to a basis for A_{S-U}.*

Proof. The proof of the first sentence is omitted since it is similar to that of the first sentence of the previous lemma.

Fix a subset $U \subset S$ and let $p : A \to A_{S-U}$ denote the projection. Since $A_{S-U} = \mathbf{Z}(W/W_{S-U})$, $\{p(e_w) \mid w \in Y_{S-U}\}$ is the obvious basis for A_{S-U} (as a $\mathbf{Z}$-module). Any element $y \in A$ can be written in the form

$$y = \sum_{w \in Y_{S-U}} \sum_{u \in W_{S-U}} \alpha_{wu} e_{wu}. \tag{3.3}$$

Moreover, $y \in AI_{S-U} = \text{Ker}(p)$ if and only if $\sum_{u \in W_{S-U}} \alpha_{wu} = 0$ for each $w \in Y_{S-U}$. Let y be an element in the submodule spanned by $\{b_w \mid U \supset \text{In}(w)\}$ $(= \{b_w \mid w \in Y_{S-U}\})$, i.e., let

$$y = \sum_{w \in Y_{S-U}} y_w b_w.$$

Suppose $p(y) = 0$. Let $v \in Y_{S-U}$ be such that $y_v \neq 0$ and $l(v)$ is maximum with respect to this property. Since b_v is the sum of e_v and ± 1 times various e_w with $l(w) < l(v)$, the coefficients α_{vu} in (3.3) are 0 for all $u \neq 1$ in W_{S-U}. So, $\sum \alpha_{vu} = 0$ forces $\alpha_{vu} = 0$, a contradiction. Thus, $\{p(b_w) \mid w \in Y_{S-U}\}$ is linearly independent in A_{S-U}. The usual argument, using induction on word length, shows that $\{p(b_w) \mid w \in Y_{S-U}\}$ spans A_{S-U}. $\qquad\square$

In view of Lemmas 3.1 and 3.2, for each $T \in \mathcal{S}$, we define **Z**-submodules of A:

$$\widehat{A}^T := \mathrm{Span}\{b'_w \mid \mathrm{In}'(w) = T\},$$
$$\widehat{H}^T := \mathrm{Span}\{b_w \mid \mathrm{In}(w) = T\}$$

A corollary to Lemma 3.1 is the following.

Corollary 3.3. *For any $U \in \mathcal{S}$,*

$$A^U = \bigoplus_{T \in \mathcal{S}_{\geq U}} \widehat{A}^T.$$

Consequently, given $T \in \mathcal{S}$, for any $U \subset S$ we have:

$$(\widehat{A}^T)^U = \begin{cases} \widehat{A}^T, & \text{if } U \subset T; \\ 0, & \text{if } U \cap (S - T) \neq \emptyset. \end{cases} \qquad (3.4)$$

It follows that the direct sum decomposition in Corollary 3.3 satisfies (2.3) and hence, gives a decomposition of coefficient systems:

$$\mathcal{I}(A) = \bigoplus_{T \in \mathcal{S}} \mathcal{I}(\widehat{A}^T). \qquad (3.5)$$

In terms of the $\widehat{H}^T$, the version of this we are interested in is the following:

$$(\widehat{H}^T)_U \cong \begin{cases} \widehat{H}^T, & \text{if } U \subset S - T; \\ 0, & \text{if } U \cap T \neq \emptyset. \end{cases} \qquad (3.6)$$

In the above formula, by writing $(\widehat{H}^T)_U \cong \widehat{H}^T$, we mean that the projection $A \to A_U$ maps $\widehat{H}^T$ isomorphically onto $(\widehat{H}^T)_U$. To see that $(\widehat{H}^T)_U = 0$ when $U \cap T \neq \emptyset$, note that if $s \in T \cap U$, then $\widehat{H}^T \subset H^s \subset AI_U$.

The $\widehat{H}^T$ version of Corollary 3.3 is the following.

Corollary 3.4. *For any $U \subset S$,*

$$A_{S-U} = \bigoplus_{T \in \mathcal{S}_{\leq U}} (\widehat{H}^T)_{S-U}.$$

10

So, the decomposition in Corollary 3.4 gives a decomposition of coefficient systems:

$$\mathcal{C}(A) = \bigoplus_{T \in \mathcal{S}} \mathcal{C}(\widehat{H}^T). \tag{3.7}$$

Hence, (2.4) and (2.5) apply to give the following calculation of (co)homology with group ring coefficients.

Theorem 3.5. *Let* $\mathcal{U} = \mathcal{U}(W, X)$.

$$H_c^i(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^i(X, X^{S-T}) \otimes \widehat{A}^T,$$

$$H_i(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H_i(X, X^T) \otimes \widehat{H}^T.$$

Proof. To prove the first formula, note that by Proposition 2.4 and observation (2.4),

$$C_c^i(\mathcal{U}) = C^i(X; \mathcal{I}(A)) = \bigoplus_{T \in \mathcal{S}} C^i(X; \mathcal{I}(\widehat{A}^T)).$$

Given a cell $c \in X^{(i)}$, by (3.4),

$$(\widehat{A}^T)^{S(c)} = \begin{cases} 0, & \text{if } c \subset X^{S-T}; \\ \widehat{A}^T, & \text{otherwise.} \end{cases}$$

Hence,

$$C^i(X; \mathcal{I}(\widehat{A}^T)) = \{f : X^{(i)} \to \widehat{A}^T \mid f(c) = 0 \text{ if } c \subset X^{S-T}\} = C^i(X, X^{S-T}) \otimes \widehat{A}^T.$$

Combining these formulas and taking cohomology, we get the first formula.

To prove the second formula, note that by Proposition 2.4 and observation (2.5),

$$C_i(\mathcal{U}) = C_i(X; \mathcal{C}(A)) = \bigoplus_{T \in \mathcal{S}} C_i(X; \mathcal{C}(\widehat{H}^T)).$$

Given a cell $c \in X^{(i)}$, by (3.6),

$$(\widehat{H}^T)_{S(c)} \cong \begin{cases} 0, & \text{if } c \subset X^T; \\ \widehat{H}^T, & \text{otherwise.} \end{cases}$$

Hence,

$$C_i(X; \mathcal{C}(\widehat{H}^T)) = \bigoplus_{\substack{c \in X^{(i)} \\ c \not\subset X^T}} \widehat{H}^T \cong C_i(X, X^T) \otimes \widehat{H}^T.$$

Taking homology, we get the second formula. $\qquad \square$

Remark. The first formula in Theorem 3.5 is one of the main results of [5]. (Actually, only a special case is stated in [5]; however, the general result is stated in [7].) The second formula is the main result of [4].

11

4 The W-module structure of $H_c^*(\mathcal{U})$ and $H_*(\mathcal{U})$.

A is a W-bimodule. So, $\mathcal{I}(A)$ is a system of right W-modules and $H^*(X; \mathcal{I}(A))$ $(= H_c^*(\mathcal{U}))$ is a right W-module. Similarly, $\mathcal{C}(A)$ is a system of left W-modules and $H_*(X; \mathcal{C}(A))$ $(= H_*(\mathcal{U}))$ is a left W-module.

For each nonnegative integer p, define

$$F_p := \sum_{|T| \geq p} A^T, \qquad\qquad E_p := \bigoplus_{|T| < p} \widehat{A}^T, \qquad\qquad (4.1)$$

$$F_p' := \sum_{|T| \geq p} H^T, \qquad\qquad E_p' := \bigoplus_{|T| < p} \widehat{H}^T, \qquad\qquad (4.2)$$

where $|T| := \mathrm{Card}(T)$. As in Section 2, these give coefficient systems $\mathcal{I}(F_p)$ and $\mathcal{C}(F_p')$ on X. Note that F_p is a right W-module and $\mathcal{I}(F_p)$ is a coefficient system of right W-submodules of $\mathcal{I}(A)$. Similarly, $\mathcal{C}(F_p')$ is a system of left W-modules. (However, E_p and E_p' only have the structure of $\mathbf{Z}$ submodules of A.)

Lemma 4.1. *We have decompositions (as $\mathbf{Z}$-modules):*

(i) $A = F_p \oplus E_p$ *and this induces a decomposition of coefficient systems,* $\mathcal{I}(A) = \mathcal{I}(F_p) \oplus \mathcal{I}(E_p)$.

(ii) $A = F_p' \oplus E_p'$ *and this induces a decomposition of coefficient systems,* $\mathcal{C}(A) = \mathcal{C}(F_p') \oplus \mathcal{C}(E_p')$.

Proof. (i) By the second formula in Corollary 3.3, $F_p = \bigoplus_{|T| \geq p} \widehat{A}^T$; hence, by the first formula in the same corollary, $A = F_p \oplus E_p$. To get the decomposition of coefficient systems, we must show that $A^U = (F_p)^U \oplus (E_p)^U$ for all $U \subset S$. Since $A^U = \bigoplus_{T \supset U} \widehat{A}^T$,

$$A^U = \bigoplus_{\substack{T \supset U \\ |T| \geq p}} \widehat{A}^T \oplus \bigoplus_{\substack{T \supset U \\ |T| < p}} \widehat{A}^T. \qquad\qquad (4.3)$$

Denote the first summation in (4.3) by B and the second one by C.

Claim. $B = (F_p)^U$.

Proof of Claim. Obviously, $B \subset (F_p)^U$. Let $x \in (F_p)^U$. Since $x \in F_p$, we have

$$x = \sum_{|T| \geq p} \alpha^T,$$

where $\alpha^T \in \widehat{A}^T$. Since $x \in A^U$,

$$x = \sum_{T \supset U} \beta^T,$$

12

where $\beta^T \in \widehat{A}^T$. But $A = \bigoplus_{T \subset S} \widehat{A}^T$, so the two decompositions of x coincide. Therefore, $\alpha^T = 0$ unless $T \supset U$ and

$$x = \sum_{\substack{T \supset U \\ |T| \geq p}} \alpha^T \in B,$$

which proves that $(F_p)^U \subset B$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Continuing with the proof of Lemma 4.1, note that a similar argument shows $(E_p)^U = C$. Hence, $A^U = (F_p)^U \oplus (E_p)^U$ and (i) is proved.

(ii) As before, by Corollary 3.4, $A = F'_p \oplus E'_p$. To get the decomposition of coefficient systems, we must show that $A_{S-U} = (F'_p)_{S-U} \oplus (E'_p)_{S-U}$ for all $U \subset S$. Since $A_{S-U} = \bigoplus_{T \subset U}(\widehat{H}^T)_{S-U}$,

$$A_{S-U} = \bigoplus_{\substack{T \subset U \\ |T| \geq p}} (\widehat{H}^T)_{S-U} \oplus \bigoplus_{\substack{T \subset U \\ |T| < p}} (\widehat{H}^T)_{S-U}. \qquad (4.4)$$

Denote the first summation in (4.4) by B' and the second one by C'. We claim that $(F'_p)_{S-U} = B'$. Obviously, $B' \subset (F'_p)_{S-U}$. Any $x \in F_p$ can be written in the form

$$x = \sum_{|T| \geq p} \gamma^T,$$

where $\gamma^T \in \widehat{H}^T$. Since $\gamma^T \in I_{S-U}$ whenever $T \cap (S - U) \neq \emptyset$, if $T \not\subset U$, we can set $\gamma^T = 0$ without changing the congruence class of x modulo I_{S-U}. So, putting

$$y = \sum_{\substack{T \subset U \\ |T| \geq p}} \gamma^T,$$

we have $y \equiv x \mod I_{S-U}$ and $y \in B'$. So, $(F'_p)_{S-U} \subset B'$. A similar argument shows $(E'_p)_{S-U} = C'$. Hence, $A_{S-U} = (F'_p)_{S-U} \oplus (E'_p)_{S-U}$ and (ii) is proved. $\quad\square$

Corollary 4.2.

(i) $F_p \hookrightarrow A$ induces a W-equivariant embedding $H^i(X; \mathcal{I}(F_p)) \hookrightarrow H^i(X; \mathcal{I}(A))$ with image a $\mathbf{Z}$-module direct summand.

(ii) $F'_p \hookrightarrow A$ induces a W-equivariant embedding $H_i(X; \mathcal{C}(F'_p)) \hookrightarrow H_i(X; \mathcal{C}(A))$ with image a $\mathbf{Z}$-module direct summand.

It follows that $F_{p+1} \hookrightarrow F_p$ induces an embedding of right W-modules, $H^*(X; \mathcal{I}(F_{p+1})) \hookrightarrow H^*(X; \mathcal{I}(F_p))$. This gives an associated graded group of right W-modules:

$$H^*(X; \mathcal{I}(F_p))/H^*(X; \mathcal{I}(F_{p+1})).$$

Similarly, we have an embedding $H_*(X; \mathcal{C}(F'_{p+1})) \hookrightarrow H_*(X; \mathcal{C}(F'_p))$ of left W-modules and an associated graded group of left W-modules. Our goal in this

section is to prove Theorem 4.5 below, which gives a complete computation of these graded W-modules.

For each $T \in \mathcal{S}$, put

$$A^{>T} := \sum_{U \supsetneq T} A^U \quad \text{and} \quad H^{>T} := \sum_{U \supsetneq T} H^U.$$

$A^T/A^{>T}$ is a right W-module and $H^T/H^{>T}$ is a left W-module.

Example 4.3. (*The sign representation*). $A^\emptyset/A^{>\emptyset}$ is isomorphic to $\mathbf{Z}$ as an abelian group. We can take the image $\bar{b}_1$ of the basis element b_1 ($= e_1$) as the generator. Since $a_s b_1 \in A^{>\emptyset}$, $\bar{b}_1 \cdot a_s = 0$ for all $s \in S$. Hence, $\bar{b}_1 \cdot s = -\bar{b}_1$. It follows that W acts on $A^\emptyset/A^{>\emptyset}$ via the sign representation:

$$\bar{b}_1 \cdot w = (-1)^{l(w)}\bar{b}_1.$$

Example 4.4. (*The case of a finite Coxeter group*). If W is finite, then for any $T \subset S$, $A^T/A^{>T} \otimes \mathbf{Q}$ can be identified with a (right) W-submodule of the rational group algebra $\mathbf{Q}W$. Similarly, $H^T/H^{>T} \otimes \mathbf{Q}$ is a (left) W-submodule of $\mathbf{Q}W$. L. Solomon proved in [13] that we have direct sum decompositions:

$$\mathbf{Q}W = \bigoplus_{T \subset S} A^T/A^{>T} \otimes \mathbf{Q},$$

$$\mathbf{Q}W = \bigoplus_{T \subset S} H^T/H^{>T} \otimes \mathbf{Q}.$$

Of course, there is no such decomposition over $\mathbf{Z}$. For an arbitrary Coxeter group W, a similar result for $L^2_\mathbf{q}(W)$ (the "$\mathbf{q}$-weighted L^2-completion" of the regular representation) is proved in [6, Theorem 9.11].

Theorem 4.5. *For each nonnegative integer p,*

(i) *there is an isomorphism of right W-modules:*

$$H^*(X;\mathcal{I}(F_p))/H^*(X;\mathcal{I}(F_{p+1})) \cong \bigoplus_{|T|=p} H^*(X, X^{S-T}) \otimes (A^T/A^{>T}).$$

(ii) *there is an isomorphism of left W-modules:*

$$H_*(X;\mathcal{C}(F'_p))/H_*(X;\mathcal{C}(F'_{p+1})) \cong \bigoplus_{|T|=p} H_*(X, X^T) \otimes (H^T/H^{>T}).$$

In view of Corollary 2.5 and Lemma 2.6, this theorem gives a computation of the W-modules associated to the corresponding filtrations of $H^*_c(\mathcal{U})$ and $H_*(\mathcal{U})$. To prove the theorem we first need the following lemma.
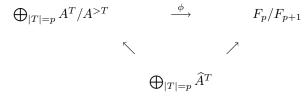
Lemma 4.6. *There are isomorphisms of W-modules:*

$$\psi : F_p/F_{p+1} \xrightarrow{\cong} \bigoplus_{|T|=p} A^T/A^{>T} \quad \text{and} \quad \psi' : F'_p/F'_{p+1} \xrightarrow{\cong} \bigoplus_{|T|=p} H^T/H^{>T}.$$

14

Proof. The inclusion $A^T \hookrightarrow F_p$ induces a map $A^T \to F_p/F_{p+1}$ and $A^{>T}$ is in the kernel; so, we get $A^T/A^{>T} \to F_p/F_{p+1}$. Therefore, we have a map of right W-modules:

$$\phi : \bigoplus_{|T|=p} A^T/A^{>T} \to F_p/F_{p+1}.$$

By Corollary 3.3, the inclusion $\widehat{A}^T \hookrightarrow A^T$ induces an isomorphism (of $\mathbf{Z}$-modules), $\widehat{A}^T \to A^T/A^{>T}$. Also, $F_p = \bigoplus_{|T|=p} \widehat{A}^T \oplus F_{p+1}$. So, we have a commutative diagram (of maps of $\mathbf{Z}$-modules):

$$\bigoplus_{|T|=p} A^T/A^{>T} \qquad \xrightarrow{\phi} \qquad F_p/F_{p+1}$$

$$\searrow \qquad\qquad \nearrow$$

$$\bigoplus_{|T|=p} \widehat{A}^T$$

Since the two slanted arrows are bijections, so is ϕ. Therefore, ϕ is an isomorphism of right W-modules. Put $\psi := \phi^{-1}$.

The definition of the second isomorphism ψ' is similar. $\qquad\qquad\square$

Here is some more notation. For any $T \subset S$, put

$$Q_{\langle T \rangle} = A^T/A^{>T}, \qquad Q'_{\langle T \rangle} = H^T/H^{>T}.$$

Since the right W-module $Q_{\langle T \rangle}$ is neither a left W-module or even a $\mathbf{Z}$-submodule of a left W-module, the definition of its (left) W_U-invariants as in (2.1) cannot be applied directly. Similarly, the definition of (right) coinvariants from (2.2) does not apply directly to $Q'_{\langle T \rangle}$. Nevertheless, for each $U \subset S$, define:

$$(Q_{\langle T \rangle})^U := (A^T \cap A^U)/(A^{>T} \cap A^U),$$
$$(Q'_{\langle T \rangle})_U := (H^T)_U/(H^{>T})_U.$$

These give coefficient systems of W-modules on X defined by

$$\mathcal{I}(Q_{\langle T \rangle})(c) := (Q_{\langle T \rangle})^{S(c)},$$
$$\mathcal{C}(Q'_{\langle T \rangle})(c) := (Q'_{\langle T \rangle})_{S(c)},$$

respectively. As in (3.4) and (3.6),

$$(Q_{\langle T \rangle})^U = \begin{cases} A^T/A^{>T}, & \text{if } U \subset T; \\ 0, & \text{otherwise} \end{cases} \qquad (4.5)$$

$$(Q'_{\langle T \rangle})_U = \begin{cases} H^T/H^{>T}, & \text{if } U \subset S - T; \\ 0, & \text{otherwise.} \end{cases} \qquad (4.6)$$

Lemma 4.7.

(i) $H^i(X; \mathcal{I}(Q_{\langle T \rangle})) = H^i(X, X^{S-T}) \otimes Q_{\langle T \rangle}$.

15

(ii) $H_i(X; \mathcal{C}(Q'_{\langle T \rangle})) = H^i(X, X^T) \otimes Q'_{\langle T \rangle}$.

Proof. (i) Using (4.5), we have

$$C^i(X; \mathcal{I}(Q_{\langle T \rangle})) = \{ f : X^{(i)} \to Q_{\langle T \rangle} \mid f(c) = 0 \text{ if } c \subset X^{S-T} \}$$
$$= C^i(X, X^{S-T}; Q_{\langle T \rangle})$$
$$\cong C^i(X, X^{S-T}) \otimes Q_{\langle T \rangle}.$$

(ii) Using (4.6),

$$C_i(X; \mathcal{C}(Q'_{\langle T \rangle})) = \bigoplus_{c \in X^{(i)}} (Q'_{\langle T \rangle})_{S(c)}$$
$$= \begin{cases} Q'_{\langle T \rangle}, & \text{if } c \not\subset X^T; \\ 0, & \text{if } c \subset X^T; \end{cases}$$
$$= C_i(X, X^T) \otimes Q'_{\langle T \rangle}.$$

$\square$

Lemma 4.8. *For any $U \subset S$,*

(i) *the following sequence of right W-modules is exact,*

$$0 \longrightarrow (F_{p+1})^U \longrightarrow (F_p)^U \xrightarrow{\tilde{\psi}} \bigoplus_{|T|=p} (Q_{\langle T \rangle})^U \longrightarrow 0,$$

where $\tilde{\psi}$ is the map induced by ψ and

(ii) *the following sequence of left W-modules is exact,*

$$0 \longrightarrow (F'_{p+1})_{S-U} \longrightarrow (F'_p)_{S-U} \xrightarrow{\tilde{\psi}'} \bigoplus_{|T|=p} (Q'_{\langle T \rangle})_{S-U} \longrightarrow 0,$$

where $\tilde{\psi}'$ is the map induced by ψ'.

Proof. In the proof of Lemma 4.1, in formula (4.3), we showed

$$(F_p)^U = \bigoplus_{\substack{|T| \geq p \\ T \supset U}} \widehat{A}^T.$$

Put

$$B := \bigoplus_{\substack{|T|=p \\ T \supset U}} \widehat{A}^T,$$

Then B is a $\mathbf{Z}$-submodule of $(F_p)^U$ and it maps isomorphically onto $(F_p)^U/(F_{p+1})^U$. The image of B under ψ is

$$\bigoplus_{\substack{|T|=p \\ T \supset U}} A^T/A^{>T} = \bigoplus_{\substack{|T|=p \\ T \supset U}} Q_{\langle T \rangle}.$$

This proves (i).

The proof that the sequence in (ii) is short exact is similar. $\square$

Proof of Theorem 4.5. (i) By Lemma 4.8 (i), we have a short exact sequence of coefficient systems on X:

$$0 \longrightarrow \mathcal{I}(F_{p+1}) \longrightarrow \mathcal{I}(F_p) \longrightarrow \bigoplus_{|T|=p} \mathcal{I}(Q_{\langle T \rangle}) \longrightarrow 0$$

inducing a short exact sequence of cochain complexes:

$$0 \longrightarrow C^*(X; \mathcal{I}(F_{p+1})) \longrightarrow C^*(X; \mathcal{I}(F_p)) \longrightarrow \bigoplus_{|T|=p} C^*(X; \mathcal{I}(Q_{\langle T \rangle})) \longrightarrow 0.$$

By the argument for Corollary 4.2, $H^*(X; \mathcal{I}(F_{p+1})) \to H^*(X; \mathcal{I}(F_p))$ is an injection onto a ($\mathbf{Z}$-module) direct summand. Hence, the long exact sequence in cohomology decomposes into short exact sequences and we have:

$$H^i(X; \mathcal{I}(F_p))/H^i(X; \mathcal{I}(F_{p+1})) \cong \bigoplus_{|T|=p} H^i(X; \mathcal{I}(Q_{\langle T \rangle}))$$

$$\cong \bigoplus_{|T|=p} H^i(X, X^{S-T}) \otimes (A^T/A^{>T}),$$

where the second isomorphism comes from Lemma 4.7 (i).

(ii) The proof of (ii) is similar. $\qquad\square$

Remark 4.9. The decreasing filtration $\supset F_p \supset F_{p+1} \cdots$ of (4.1) gives a decreasing filtration of cochain complexes

$$\cdots \supset C^*(X; \mathcal{I}(F_p)) \supset C^*(X; \mathcal{I}(F_{p+1})) \cdots .$$

So, the quotient cochain complexes have the form $C^*(X; \mathcal{I}(F_p)/\mathcal{I}(F_{p+1}))$. Taking homology, we get a spectral sequence with E_1-term:

$$E_1^{pq} := H^{p+q}(X; \mathcal{I}(F_p)/\mathcal{I}(F_{p+1})).$$

It converges to

$$E_\infty^{pq} := \frac{H^{p+q}(X; \mathcal{I}(F_p))}{\mathrm{Im}(H^{p+q}(X; \mathcal{I}(F_{p+1})))}. \tag{4.7}$$

So, the import of Theorem 4.5 is that $E_1^{pq} = E_\infty^{pq}$.

5 $H^*(W; \mathbf{Z}W)$

Let K denote the geometric realization of the poset $\mathcal{S}$ of spherical subsets. (The simplicial complex K is contractible because it is a cone; the cone point corresponds to the minimum element $\emptyset \in \mathcal{S}$.) For each $s \in S$, define a subcomplex $K_s \subset K$ as the geometric realization of $\mathcal{S}_{\geq\{s\}}$. Put $\Sigma := \mathcal{U}(W, K)$. (Alternatively, Σ could be described as the geometric realization of the poset $W\mathcal{S}$ of all "spherical cosets," i.e., the poset of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}$.)

By construction W acts properly on Σ. It is proved in [3] that Σ is contractible. Hence,
$$H^*(W; \mathbf{Z}W) = H_c^*(\Sigma).$$

As before, $A := \mathbf{Z}W$. The filtration $A = F_0 \supset \cdots F_p \supset \cdots$ gives $H_c^*(\Sigma) = H^*(K; \mathcal{I}(A))$ the structure of a graded W-module. As in (4.7), let E_∞^{pq} be the right W-module in filtration degree p associated to the grading of $H^{p+q}(K; \mathcal{I}(A))$. Theorem 4.5 then has the following corollary.

Theorem 5.1. *The associated graded group of $H^{p+q}(W; \mathbf{Z}W)$ is given, as a right W-module, by*
$$E_\infty^{pq} = \bigoplus_{|T|=p} H^{p+q}(K, K^{S-T}) \otimes (A^T/A^{>T}).$$

It follows from Theorem 3.5 that we have a direct sum decomposition of $\mathbf{Z}$-modules:
$$H_c^*(\Sigma) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{A}^T.$$

In view of Theorem 5.1, it is natural to conjecture that $H_c^*(\Sigma)$ decomposes as above into a direct sum of right W-modules. However, in general, there is no such decomposition, as we can see by considering the following example.

Example 5.2. Suppose W is the free product of 3 copies of $\mathbf{Z}/2$. Then K is the cone on 3 points. So, it has 3 edges. By Theorem 4.5, $H^1(K, K^S) \otimes A/A^{>\emptyset}$ is a quotient of $H_c^1(\Sigma)$. Let $x \in C^1(K)$ be a cochain (= cocycle) which evaluates to 1 on one of the edges, call it c, and to 0 on the other two edges. Choose $s \in S$ which is not a vertex of c. Let y denote the image of $x \otimes 1$ in $H^1(K, K^S) \otimes A/A^{>\emptyset}$. By Example 4.3, $A/A^{>\emptyset}$ has rank 1 as an abelian group and the W-action on it is given by the sign representation. Hence, $y \cdot s = -y$ in $H^1(K, K^S) \otimes A/A^{>\emptyset}$. Suppose we had a W-equivariant splitting $\varphi : H^1(K, K^S) \otimes A/A^{>\emptyset} \to H_c^1(\Sigma)$. When regarded as an element of $C_c^1(\Sigma)$, $x + x \cdot s$ represents $\varphi(y + y \cdot s)$ in $H_c^1(\Sigma)$, i.e., it represents 0. But x and $-x \cdot s$ are not cohomologous cocycles in $C_c^1(\Sigma)$. (One can see this by noting that there is a line (= infinite 1-cycle) on which x evaluates to 1 and $x \cdot s$ to 0.) Hence, there can be no such splitting φ.

6 Hecke algebra coefficients

In this section and the next we work over the rational numbers $\mathbf{Q}$ rather than $\mathbf{Z}$.

Let $i : S \to I$ be a function to some index set I such that $i(s) = i(s')$ whenever s and s' are conjugate in W. Let $\mathbf{q} = (q_i)_{i \in I}$ be a fixed I-tuple of rational numbers. Write q_s instead of $q_{i(s)}$. If $s_1 \cdots s_l$ is a reduced expression for an element $w \in W$, then the number $q_{s_1} \cdots q_{s_l}$ is independent of the choice of reduced expression. We write it as q_w. The *Hecke algebra* $A_{\mathbf{q}}$ of W is a deformation of the group algebra $\mathbf{Q}W$ which is equal to $\mathbf{Q}W$ when each $q_s = 1$.

As a a rational vector space, it has the same basis $\{e_w\}_{w \in W}$ as does $\mathbf{Q}W$. Multiplication is determined by the rules:

$$e_w e_{w'} = e_{ww'}, \quad \text{if } l(ww') = l(w) + l(w')$$
$$e_s^2 = (q_s - 1)e_s + q_s.$$

Given a special subgroup W_T, $A_{\mathbf{q}}(W_T)$ denotes the Hecke algebra of W_T. It is a subalgebra of $A_{\mathbf{q}}$. There are ring homomorphisms $\alpha : A_{\mathbf{q}}(W_T) \to \mathbf{Q}$ and $\beta : A_{\mathbf{q}}(W_T) \to \mathbf{Q}$, defined by $\alpha(e_w) := q_w$ and $\beta(e_w) := (-1)^{l(w)}$, respectively. Given a left $A_{\mathbf{q}}$-module M and a subset T of S, put

$$M^T := \{x \in M \mid ax = \alpha(a)x \text{ for all } a \in A_{\mathbf{q}}(W_T)\}.$$

This gives a coefficient system $\mathcal{I}(M)$ on X the same way as in Examples 2.3.

As in [6], for each $T \in \mathcal{S}$, we modify the formulas in (3.1) to define elements a_T and h_T in $A_{\mathbf{q}}$ by

$$a_T := \frac{1}{W_T(\mathbf{q})} \sum_{w \in W_T} e_w \quad \text{and} \quad h_T := \frac{1}{W_T(\mathbf{q}^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q_w^{-1} e_w$$

where
$$W_T(\mathbf{q}) := \sum_{w \in W_T} q_w, \quad \text{and} \quad W_T(\mathbf{q}^{-1}) := \sum_{w \in W_T} q_w^{-1}.$$

Put $A_{\mathbf{q}}^T := a_T A_{\mathbf{q}}$, $H_{\mathbf{q}}^T := A_{\mathbf{q}} h_T$. (If $T \notin \mathcal{S}$, put $A_{\mathbf{q}}^T := 0$, $H_{\mathbf{q}}^T := 0$.)

For each subset U of S, put

$$(A_{\mathbf{q}})_U := A_{\mathbf{q}} \otimes_{A_{\mathbf{q}}(W_U)} \mathbf{Q} = A_{\mathbf{q}}/A_{\mathbf{q}} I_U,$$

where $A_{\mathbf{q}}(W_U)$ acts on $\mathbf{Q}$ via the symmetric character α_U and $I_U := \sum_{s \in U} H_{\mathbf{q}}^s$ is the augmentation ideal of $A_{\mathbf{q}}(W_U)$. $A_{\mathbf{q}}^U$ is a right $A_{\mathbf{q}}$-module and $(A_{\mathbf{q}})_U$ is a left $A_{\mathbf{q}}$-module.

We have decreasing filtrations (F_p) and (F_p') of $A_{\mathbf{q}}$, defined exactly as in (4.1) and (4.2).

If X and $\mathcal{U}$ are as before, then the proof of Theorem 4.5 gives the following.

Theorem 6.1. *With notation as above, for each nonnegative integer p,*

(i) *there is an isomorphism of right $A_{\mathbf{q}}$-modules:*

$$H^*(X; \mathcal{I}(F_p))/H^*(X; \mathcal{I}(F_{p+1})) \cong \bigoplus_{|T|=p} H^*(X, X^{S-T}) \otimes (A_{\mathbf{q}}^T/A_{\mathbf{q}}^{>T}).$$

(ii) *there is an isomorphism of left $A_{\mathbf{q}}$-modules:*

$$H_*(X; \mathcal{C}(F_p'))/H_*(X; \mathcal{C}(F_{p+1}')) \cong \bigoplus_{|T|=p} H_*(X, X^T) \otimes (H_{\mathbf{q}}^T/H_{\mathbf{q}}^{>T}).$$

19

7 Buildings

The importance of Hecke algebras lies in their relationship to buildings and BN pairs (e.g., see [1, Exercises 22 and 24, pp. 56–58]). Suppose that (G, B) is a BN pair. Associated to (G, B) we have a Coxeter system (W, S) such that for each $s \in S$, there is a subgroup G_s of G which is equal to the double coset BsB in the Bruhat decomposition. For each subset T of S, put $G_T := \bigcap_{s \in T} G_s$. The *associated building* is the CW complex $\Phi := (G \times K)/\sim$, where the equivalence relation $\sim$ is defined as in the definition of $\mathcal{U}(W, K)$ in Section 2. The building has *finite thickness* if $(G_s : B) < \infty$ for all $s \in S$. If this is the case, put $q_s = (G_s : B) - 1$ and regard $\mathbf{q} = (q_s)$ as an I-tuple, where I is the set of conjugacy classes of elements in S.

Let $F(G/B)$ denote the $\mathbf{Q}$ vector space of finitely supported, $\mathbf{Q}$-valued functions on G/B. The left G-action on G/B gives $F(G/B)$ the structure of a right G-module. For any $T \subset S$, we have a projection $p_T : G/B \to G/G_T$. Pulling back via p_T identifies $F(G/G_T)$ with a G-submodule of $F(G/B)$.

Regard $F(G/B)$ as a subset of all $\mathbf{Q}$-valued functions on G. The Hecke algebra can be identified with the subspace of $F(G/B)$ consisting of those functions which are invariant under the B-action on $F(G/B)$ (induced from the left B-action on G/B). Under this identification, the basis element $e_w \in A_{\mathbf{q}}$ is identified with the characteristic function of the double coset, $C(w^{-1}) := Bw^{-1}B$ and the idempotent a_T with the characteristic function of G_T ($= BW_T B$).

$A_{\mathbf{q}}$ acts naturally on the left on $F(G/B)$ by convolution, i.e., given $f \in F(G/B)$, $a \in A_{\mathbf{q}}$, convolution is defined by

$$(a * f)(h) := \int_G a(g^{-1}h) f(g) dg,$$

where we are integrating with respect to Haar measure normalized so that the measure of B is 1. In fact, $A_{\mathbf{q}}$ is the intertwining algebra ($=$ the commutant) of G on $F(G/B)$.

Lemma 7.1. *For any $T \in \mathcal{S}$,*

$$A_{\mathbf{q}}^T \otimes_{A_{\mathbf{q}}} F(G/B) = F(G/G_T)$$

Proof. Let $\sum_i a_T \alpha_i \otimes f_i$ be a typical element of the left-hand side. It can be rewritten as $1 \otimes \sum a_T * \alpha_i * f_i$. The universal map to $F(G/B)$ consists of taking the second factor. Since a_T is the characteristic function of G_T, $a_T * \alpha_i * f_i$ lies in $F(G/G_T)$; so, the image of this map is the right-hand side. $\qquad\square$

Theorem 7.2.

$$C_c^i(\Phi; \mathbf{Q}) = C^i(K; \mathcal{I}(A_{\mathbf{q}})) \otimes_{A_{\mathbf{q}}} F(G/B).$$

Proof. This is a direct consequence of Lemma 7.1 since

$$C_c^i(\Phi; \mathbf{Q}) = \bigoplus_{c \in K^{(i)}} F(G/G_{S(c)}).$$

$\qquad\square$

The natural conjecture is the following.

Conjecture 7.3.

$$H_c^i(\Phi) = H^i(K; \mathcal{I}(A_{\mathbf{q}})) \otimes_{A_{\mathbf{q}}} F(G/B).$$

The filtration (F_p) induces a filtration of $H^*(K; \mathcal{I}(A_{\mathbf{q}}))$ and hence, of $H_c^*(\Phi)$. So, Lemma 7.1 leads us to the following.

Conjecture 7.4. *In filtration degree p, the associated graded group of $H_c^*(\Phi)$ is given, as a right G-module, by*

$$\bigoplus_{|T|=p} H^*(K, K^{S-T}) \otimes F^T / F^{>T},$$

where $F^T := F(G/G_T)$ and $F^{>T}$ denotes the submodule spanned by the F^U with $U \supsetneq T$.

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