Functions of trigonometric type
and bases in $L_q$

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Abstract

It is shown that, for all $p \in (1, \infty)$, the eigenfunctions of the Dirichlet problem for
the $p$-Laplacian on $[0, 1]$ form a basis of $L_q(0, 1)$ for all $q \in (1, \infty)$.

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operator, Schauder basis, $L_p$ spaces, biorthogonal system
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1 Introduction

Let $q \in (1, \infty)$. It is a standard fact that the functions $\sin n\pi x$ and $\cos n\pi x$
form a basis of $L_q(-1, 1)$: see, for example, [6], pp. 342-5. Given any $f \in
L_q(0, 1)$, it follows that its odd extension to $L_q(-1, 1)$ has a unique representa-
tion in terms of the sin $n\pi x$. This means that the sin $n\pi x$ form a basis of
$L_q(0, 1)$. In this paper we show that the same is true when the sines are re-
paced by the $p$-sine functions, for any $p \in (1, \infty)$. We recall that these may

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be defined by setting

\[ F_p(x) = \int_0^x (1 - t^p)^{-1/p} dt, \quad x \in [0, 1] \quad (1.1) \]

and

\[ \pi_p = 2 \int_0^1 (1 - t^p)^{-1/p} dt. \quad (1.2) \]

The \( p \)-sine function, \( \sin_p \), is defined on \([0, \pi_p/2]\) by

\[ \sin_p x = F_p^{-1}(x); \quad (1.3) \]

it is extended to \( \mathbb{R} \) by standard procedures. Note that \( \sin_2 \) is simply the usual sine function. These \( \sin_p \) functions have attracted a great deal of attention recently, especially in connection with the one-dimensional \( p \)-Laplacian and with the sharp estimation of the approximation numbers of embeddings. For example, the functions \( \sin_p(n \pi_p x) \) turn out to be the eigenfunctions of the \( p \)-Laplacian eigenvalue problem

\[
-\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u \quad \text{on} \ (0, 1), \quad \left\{ \begin{array}{l}
u(0) = u(1) = 0, \end{array} \right. \quad (1.4)
\]

corresponding to eigenvalues \( \lambda_n = (p - 1)(n \pi_p)^p \quad (n \in \mathbb{N}) \). We refer to [2], [3] and [4] for further information and additional references on these functions and their applications. A fascinating account of early work on generalisations of trigonometric functions, is given in [5].

The only paper of which we are aware that deals with the basis properties of the \( \sin_p \) functions is that of Binding et al [1], in which it is shown that if \( 12/11 \leq p < \infty \), they form a basis of \( L_q(0, 1) \) for all \( q \in (1, \infty) \). The proof proceeds by constructing a homeomorphism of \( L_q(0, 1) \) onto itself that maps \( \sin(n \pi x) \) onto \( \sin_p(n \pi_p x) \quad (n \in \mathbb{N}) \), and while this method allows a slightly smaller value than \( 12/11 \) to be obtained, it does not enable the basis property to be established for \( p \) arbitrarily close to \( 1 \). In contrast to this, we construct a basis of the dual of \( L_q(0, 1) \) that forms a biorthogonal system with the \( \sin_p(n \pi_p x) \) functions. This enables us to conclude, by means of a general theorem about bases in Banach spaces, that the \( \sin_p(n \pi_p x) \) functions form a basis of \( L_q(0, 1) \) for all \( p, q \in (1, \infty) \).

2 Preliminaries and technical results

Throughout the paper \( p \) will stand for a number in \((1, \infty)\), \( p' = p/(p - 1) \), \( I = [0, 1] \) and \( \| \cdot \|_p \) will denote the usual norm on the Lebesgue space \( L_p(I) \).
The function $\sin_p$ is defined on $[0, \pi_p/2]$ by (1.3): note that $F_p$ is strictly increasing, as is $\sin_p$. Extension to $[0, \pi_p]$ is achieved by setting
\[
\sin_p(t) = \sin_p(\pi_p - t) \quad \text{for} \quad t \in [\pi_p/2, \pi_p];
\]
further extension to $[-\pi_p, \pi_p]$ is made by oddness, and finally $\sin_p$ is extended to $\mathbb{R}$ by $2\pi_p$-periodicity. This extension is in $C^1(\mathbb{R})$. Note that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$. We define $\cos_p : \mathbb{R} \to \mathbb{R}$ by
\[
\cos_p t = \frac{d}{dt} \sin_p t, \quad t \in \mathbb{R};
\]
$\cos_p$ is even, $2\pi_p$-periodic and odd about $\pi_p/2$. Moreover,
\[
|\sin_p t|^p + |\cos_p t|^p = 1, \quad t \in \mathbb{R}. \tag{2.2}
\]
The number $\pi_p$ is easily shown to be given by
\[
\pi_p = 2p^{-1}\Gamma(1/p')\Gamma(1/p) = \frac{2\pi}{p\sin(\pi/p)}. \tag{2.3}
\]
For shortness we shall write
\[
e_n(t) = \sin(n\pi t), \quad x_n(t) = \sin_p(n\pi_p t) \quad (n \in \mathbb{N}, \quad t \in I). \tag{2.4}
\]
We recall that a sequence $\{y_n\}_{n \in \mathbb{N}}$ of elements of a Banach space $Y$ is called a (Schauder) basis if, for every $y \in Y$, there is a unique sequence $\{a_n\}_{n \in \mathbb{N}}$ of scalars such that
\[
y = \sum_{n=1}^{\infty} a_n y_n, \tag{2.5}
\]
the series converging in the norm of $Y$. It can be shown (see [7], Proposition II.B.6, p.37) that the partial sum projections $P_N : Y \to Y$ defined by
\[
P_N \left( \sum_{n=1}^{\infty} a_n y_n \right) = \sum_{n=1}^{N} a_n y_n \quad (N \in \mathbb{N}) \tag{2.6}
\]
have the property that
\[
\sup_{N \in \mathbb{N}} \|P_N\| < \infty. \tag{2.7}
\]
Since each $x_n$ is continuous on $I$ it is in every $L_q(I)$ ($1 < q < \infty$); and as the $e_k$ form a basis in $L_q(I)$, $x_n$ has a Fourier sine expansion, converging in each $L_q(I)$:
\[
x_n(t) = \sum_{k=1}^{\infty} \hat{x}_n(k) \sin(k\pi t), \tag{2.8}
\]
where
\[
\hat{x}_n(k) = 2 \int_{0}^{1} x_n(t) \sin(k\pi t) dt. \tag{2.9}
\]
The symmetry of $x_1$ about $t = 1/2$ means that

\[ \hat{x}_1(k) = 0 \quad \text{when } k \text{ is odd,} \quad (2.10) \]

and that

\[
\hat{x}_n(k) = 2 \int_0^1 x_1(nt) \sin(k\pi t) dt \\
= 2 \sum_{m=1}^{\infty} \hat{x}_1(m) \int_0^1 \sin(k\pi t) \sin(m\pi t) dt \\
= \begin{cases} 
\hat{x}_1(m), & \text{if } mn = k \text{ for some odd } m, \\
0, & \text{otherwise.} 
\end{cases} \quad (2.11)
\]

To estimate $\hat{x}_1(m)$, we follow [1] and note that since $x''_1(t) < 0$ for all $t \in (0, 1/2)$, integration by parts twice gives

\[
|\hat{x}_1(m)| = 4 \left| \int_0^{1/2} x_1(t) \sin(m\pi t) dt \right| \\
= \left| -\frac{4}{(m\pi)^2} \int_0^{1/2} x''_1(t) \sin(m\pi t) dt \right| \\
\leq \frac{4\pi}{(m\pi)^2} \int_0^{1/2} x''_1(t) dt \\
= \frac{4\pi}{(m\pi)^2}. \quad (2.12)
\]

Now let $A$ be the infinite matrix with $(i, j)^{th}$-entry $a_{ij}$ ($i, j, \in \mathbb{N}$), where

\[
a_{ij} = \begin{cases} 
\hat{x}_1(m), & \text{if } i = jm \text{ for some odd integer } m, \\
0, & \text{otherwise.} 
\end{cases} \quad (2.13)
\]

Evidently $A$ is lower triangular, with each diagonal element equal to $\hat{x}_1(1)$. In view of the structure of $A$, there exists a matrix $B = (b_{ij})_{i,j \in \mathbb{N}}$ such that

\[ BA = (\delta_{ij}), \quad (2.14) \]

the identity matrix; $B$ is lower triangular, with each diagonal element equal to $1/\hat{x}_1(1)$. We use $B$ to define elements of the dual $L_q(I)^*$ of $L_q(I)$ as follows. Let $f \in L_q(I)$, so that

\[ f = \sum_{j=1}^{\infty} \hat{f}(j) e_j. \]
For each $i \in \mathbb{N}$, define a functional $f_i^*$ on $L_q(I)$ by

$$f_i^*(f) = \sum_{j=1}^{i} b_{ij} \hat{f}(j). \quad (2.15)$$

We claim that $f_i^* \in L_q(I)^*$. As the linearity of $f_i^*$ is clear, it remains to show that $f_i^*$ is bounded. Since

$$|f_i^*(f)| \leq \max_{1 \leq j \leq i} |b_{ij}| \max_{j \in \mathbb{N}} |\hat{f}(j)|$$

and

$$|\hat{f}(j)| \leq 2\|f\|_1 \leq 2\|f\|_q \quad (j \in \mathbb{N}),$$

it follows that $f_i^* \in L_q(I)^*$.

Next we claim that the systems $\{x_i\}$ and $\{f_i^*\}$ are biorthogonal, by which we mean that

$$f_i^*(x_j) = \delta_{ij} \quad (i, j \in \mathbb{N}). \quad (2.16)$$

To verify this, note that

$$f_i^*(x_j) = \sum_{k=1}^{j} b_{jk} \hat{x}_j(k) = \sum_{k=1}^{\infty} b_{ij} a_{kj} = \delta_{ij}.$$

Since $f_i^* \in L_q(I)^*$, there exists $f_i \in L_{q'}(I)$ such that for all $f \in L_q(I)$,

$$f_i^*(f) = \int_0^1 f_i(t)f(t)dt.$$

In fact, $f_i$ is given by

$$f_i = \sum_{j=1}^{i} b_{ij} e_j. \quad (2.17)$$

To check this, write

$$f = \sum_{k=1}^{\infty} \hat{f}(k)e_k$$

and observe that, with $f_i$ given by (2.17),

$$\int_0^1 f_i(t)f(t)dt = \int_0^1 \left(\sum_{j=1}^{i} b_{ij} e_j(t)\right) \left(\sum_{k=1}^{\infty} \hat{f}(k)e_k(t)\right) dt$$

$$= \sum_{j=1}^{i} b_{ij} \hat{f}(j) = f_i^*(f).$$

Finally, we note that the $L_q(I)$-norm of each $x_i$ can be calculated. In fact, with
\[ I = \int_0^1 |\sin_p(i\pi x)|^q dx = i \int_0^{1/i} (\sin_p(i\pi x))^q dx \]
\[ = \int_0^1 (\sin_p(\pi t))^q dt = 2 \int_0^{1/2} (\sin_p(\pi t))^q dt, \]

the substitutions \( u = \sin_p(\pi t) \) and then \( u^p = w \) give

\[ I = \frac{2}{\pi p} \int_0^1 u^q (1 - u^p)^{-1/p} du = \frac{2}{p\pi p} \int_0^1 u^{(q+1)/p-1} (1 - u)^{-1/p} dw \]
\[ = \frac{2}{p\pi p} B((q + 1)/p, 1/p'), \]

where \( B \) is the beta function. Hence for all \( i \in \mathbb{N}, \)

\[ \|x_i\|_q = \left\{ \frac{2}{p\pi p} B((q + 1)/p, 1/p') \right\}^{1/q}. \] (2.18)

3 The main result

After the technical preparation of §2, we can now obtain the desired result fairly quickly. The strategy is to show that the \( f^*_i \) defined by (2.15) form a basis of \( L_q(I)^* \), and then to use the biorthogonality of the systems \( \{x_i\} \) and \( \{f^*_i\} \) to conclude, via general theorems, that the \( x_i \) form a basis of \( L_q(I) \).

**Lemma 3.1** For any \( q \in (1, \infty) \), the sequence \( \{f^*_i\}_{i \in \mathbb{N}} \) is complete in \( L_q(I)^* \) in the sense that its closed linear span is \( L_q(I)^* \).

**Proof.** For each \( n \in \mathbb{N} \) set \( P_n = \text{sp}\{f^*_1, f^*_2, \ldots, f^*_n\} \), the span of \( f^*_1, f^*_2, \ldots, f^*_n \). Then \( s^* \in P_n \) if and only if there exist \( d_1, \ldots, d_n \in \mathbb{R} \) such that for all \( f = \sum_{j=1}^{\infty} \hat{f}(j)e_j \in L_q(I), \)

\[ s^*(f) = \sum_{j=1}^{n} d_j \hat{f}(j). \] (3.1)

Moreover,

\[ s^*(f) = \int_0^1 s(t)f(t)dt, \] (3.2)

where

\[ s = \sum_{j=1}^{n} d_j e_j. \] (3.3)
Now let $g^* \in L_q(I)^*$ and let $g \in L_{q'}(I)$ be such that

$$g^*(f) = \int_0^1 g(t)f(t)dt, \quad f \in L_q(I).$$

Since $g \in L_{q'}(I)$, the basis property of the $e_i$ in $L_{q'}(I)$ means that

$$g = \sum_{i=1}^{\infty} \hat{g}(i)e_i,$$

and for each $N \in \mathbb{N}$,

$$\|g - \sum_{i=1}^{N} \hat{g}(i)e_i\|_{q'} = \|\sum_{i=N+1}^{\infty} \hat{g}(i)e_i\|_{q'} \to 0 \text{ as } N \to \infty. \quad (3.4)$$

For each $n \in \mathbb{N}$ put

$$g_n = \sum_{i=1}^{n} \hat{g}(i)e_i \quad (\in L_{q'}(I))$$

and

$$g_n^*(f) = \int_0^1 g_n(t)f(t)dt, \quad f \in L_q(I).$$

From (3.2) we see that $g_n^* \in P_n$. For every $f \in L_q(I)$ we have, with the help of Hölder’s inequality,

$$|g^*(f) - g_n^*(f)| = \left|\int_0^1 (g(t) - g_n(t))f(t)dt\right|$$

$$\leq \|g - g_n\|_{q'}\|f\|_q.$$

Hence by (3.4),

$$\sup_{\|f\|_q \leq 1} \|g^*(f) - g_n^*(f)\| \leq \|g - g_n\|_{q'} \to 0$$

as $n \to \infty$, so that $g_n^* \to g^*$ in $L_q(I)^*$, as required. ■

**Lemma 3.2** Let $q \in (1, \infty)$. There is a sequence $\{u_n^*\}$ of bounded linear maps of $L_q(I)^*$ to itself (i.e. endomorphisms of $L_q(I)^*$ ) such that

(i) $u_n^*(x^*) = x^*$ for all $x^* \in P_n$ $(n \in \mathbb{N})$,

(ii) $u_n^*(x^*) = 0$ for all $x^* \in P^{(n)}$ $(n \in \mathbb{N})$

and

(iii) $1 \leq C := \sup_{n \in \mathbb{N}} \|u_n^*\| < \infty$.

Here $P_n = \text{sp}\{f_1^*, \ldots, f_n^*\}$, $P^{(n)} = \text{sp}\{f_{n+1}^*, f_{n+2}^*, \ldots\}$ (the closed linear span of $f_{n+1}^*, f_{n+2}^*, \ldots$) and the $f_i^*$ are as defined in (2.15).

**Proof.** Let $s^* \in P_n$ and let $s = \sum_{i=1}^{n} d_i e_i \in L_{q'}(I)$ be such that

$$s^*(f) = \int_0^1 s(t)f(t)dt, \quad f \in L_q(I).$$
Given \( r^* \in P^{(n)} \), there exists \( r = \sum_{i=n+1}^{\infty} c_i e_i \in L_q(I) \) such that
\[
r^*(f) = \int_0^1 r(t)f(t)dt, \quad f \in L_q(I).
\]

For each \( n \in \mathbb{N} \), define \( u_n \) on \( L_q(I) \) by
\[
u_n(x) = \sum_{i=1}^{n} \hat{x}(i)e_i, \quad x = \sum_{i=1}^{\infty} \hat{x}(i)e_i \in L_q(I).
\]

We claim that each \( u_n \) is an endomorphism of \( L_q(I) \) and that we may define an endomorphism \( u^*_n \) of \( L_q(I)^* \) by
\[
u^*_n(x^*)(f) = \int_0^1 u_n(x)(t)f(t)dt, \quad f \in L_q(I), \quad x^* \in L_q(I)^*,
\]
where \( x \in L_q(I) \) is such that
\[
x^*(f) = \int_0^1 x(t)f(t)dt, \quad f \in L_q(I).
\]

Plainly \( u_n \) is a linear map from \( L_q(I) \); and it is bounded, for as the \( e_i \) form a basis of \( L_q(I) \), we see from (2.7) that there is a constant \( C \) such that for all \( n \in \mathbb{N} \),
\[
\|u_n(x)\|_q \leq C\|x\|_q, \quad x \in L_q(I). \tag{3.5}
\]

Moreover, for every \( f = \sum_{i=1}^{n} c_i e_i, \quad g = \sum_{i=n+1}^{\infty} c_i e_i \in L_q(I) \) we have
\[
u_n(f) = f, \quad u_n(g) = 0.
\]

Next we justify the claim that each \( u^*_n \) is an endomorphism on \( L_q(I)^* \). Linearity is obvious; and for each \( x^* \in L_q(I)^* \setminus \{0\} \),
\[
\|u^*_n(x^*)\|_{L_q(I)^*}/\|x^*\|_{L_q(I)^*} = \left\{ \sup_{\|f\| \leq 1} |u^*_n(x^*)(f)| \right\} / \sup_{\|f\| \leq 1} |x^*(f)|
\]
\[
= \|u_n(x)\|_{q}/\|x\|_q \leq C \tag{3.6}
\]
the inequality following from (3.5). Hence
\[
\|u^*_n\| \leq C \quad (n \in \mathbb{N}).
\]

To establish a lower bound for the \( \|u^*_n\| \), let \( x = e_1 \). Then by (3.6),
\[
\|u^*_n(x^*)\|_{L_q(I)^*}/\|x^*\|_{L_q(I)^*} = \|u_n(x)\|_{q}/\|x\|_{q} = 1.
\]

The proof is complete. \( \blacksquare \)
Theorem 3.3 For each \( q \in (1, \infty) \), the system \( \{f^*_i\}_{i \in \mathbb{N}} \) defined by (2.15) is a basis of \( L_q(I)^* \).

**Proof.** By Lemma 3.1, \( \{f^*_i\}_{i \in \mathbb{N}} \) is complete in \( L_q(I)^* \). Now apply Theorem 7.1 of [6], which shows that such a system \( \{f^*_i\} \) is a basis if there is a sequence of endomorphisms of \( L_q(I)^* \) having the properties ensured by Lemma 3.2. ■

The main result of the paper is now virtually immediate.

Theorem 3.4 The functions \( x_i \ (i \in \mathbb{N}) \) form a basis in \( L_q(I) \) for every \( q \in (1, \infty) \).

**Proof.** By (2.16), \( \{x_i\} \) and \( \{f^*_j\} \) are biorthogonal; by Theorem 3.3, \( \{f^*_j\} \) is a basis of \( L_q(I)^* \). Corollary 12.1 of [6] tells us that under these conditions, \( \{x_i\} \) is a basis of \( L_q(I) \). ■

**References**


